Commun. math. Phys. 52, 191—202 (1977)

Communications in Mathematical Physics

Duality for Dual Covariance Algebras

Magnus B. Landstad

Matematisk Institutt, Universitetet i Trondheim, Norges Laererhøgskole, N-7000 Trondheim, Norway

Abstract. One way of generalizing the definition of an action of the dual group of a locally compact abelian group on a von Neumann algebra to non-abelian groups is to consider $\mathscr{L}(G)$-comodules, where $\mathscr{L}(G)$ is the Hopf-von Neumann algebra generated by the left regular representation of G. To a $\mathscr{L}(G)$-comodule we shall associate a dual covariance algebra $\mathfrak{A}$ and a natural covariant system $(\mathfrak{A}, \varrho, G)$, and in Theorem 1 the covariant systems coming from $\mathscr{L}(G)$-comodules are characterized. In [2] it was shown that the covariance algebra of a covariant system in a natural way is a $\mathscr{L}(G)$-comodule. Therefore one can form the dual covariance algebra of a covariance algebra and the covariance algebra of a dual covariance algebra. Theorems 2 and 3 deal with these algebras – generalizing a result by Takesaki. As an application we give a new proof of a theorem by Digernes stating that the commutant of a covariance algebra itself is a covariance algebra and prove the similar result for dual covariance algebras.

§ 1. Introduction

If G is a locally compact group and $\varrho: G \rightarrow \mathrm{Aut}(A)$ is a continuous homomorphism of G into the group of *-automorphisms of a von Neumann algebra A, (A, ϱ, G) is called a covariant system and one can form the covariance algebra $\mathfrak{A} = W^*(A, \varrho, G)$. Takesaki showed in [8] that if G is abelian there is a natural covariant system $(\mathfrak{A}, \tau, G^\wedge)$ over the dual group $G^\wedge$ and that $W^*(\mathfrak{A}, \tau, G^\wedge) \cong A \otimes \mathscr{B}(L^2(G))$, i.e. the tensorproduct of A with the algebra of all bounded operators on $L^2(G)$.

For a non-abelian G there is no dual group to act on the covariance algebra $\mathfrak{A} = W^*(A, \varrho, G)$, but this author showed in [2] that the natural structure on $\mathfrak{A}$ corresponding to the action of a dual group is that of a $\mathscr{L}(G)$-comodule. There one used that $\mathscr{L}(G)$, the von Neumann algebra generated by the left regular representation of G is a Hopf-von Neumann algebra, cf. [7].

So if A is a von Neumann algebra, what seems to correspond to a covariant system on A over $G^\wedge$ if G is abelian is that of a $\mathscr{L}(G)$-comodule structure on A.

Given such a comodule we shall define a corresponding dual covariance algebra $\mathfrak{A}$, and it turns out that there is a natural covariant system $(\mathfrak{A}, \varrho, G)$ over $\mathfrak{A}$. In Theorem 1 we characterize those covariant systems $(\mathfrak{A}, \varrho, G)$ which come from $\mathscr{L}(G)$-comodules, this is a dual version of Theorem 1 in [2].

Since the covariance algebra $\mathfrak{A}$ of a covariant system is a $\mathscr{L}(G)$-comodule it is natural to ask what the dual covariance algebra of $\mathfrak{A}$ is, furthermore if $\mathfrak{A}$ is the dual covariance algebra of a $\mathscr{L}(G)$-comodule and $(\mathfrak{A}, \varrho, G)$ the corresponding covariant system one wants to know what the covariance algebra of $(\mathfrak{A}, \varrho, G)$ is. The answers to these two questions are given in Theorems 2 and 3 and are natural generalizations of Takesaki's result ([8, Theorem 4.5]) mentioned above.

Finally as an application of Theorem 1 in [2] and Theorem 1 in this article we prove that

(a) the commutant of a covariance algebra over G is itself a covariance algebra over G, and

(b) the commutant of a dual covariance algebra over G is itself a dual covariance algebra over G.

Digernes has proved (a) by other methods in [1].

Roberts has given a different, but related definition of an action of the dual of a group on a von Neumann algebra in [5]. His dual objects are sets of representations of G, while we in this article exploit the duality between the Hopf-von Neumann algebras $\mathscr{L}(G)$ and $L^\infty(G)$ developed in [7].

A similar notion of dual action has been given by Nakagami, [3 and 4]. His main tool is a very interesting sort of Fourier analysis on the Hopf-von Neumann algebra $\mathscr{L}(G)$ [or rather $\mathscr{L}(G)'$].

The author wants to thank Professor M. Takesaki for making him aware of Nakagami's work. In the final stage of this work the author also received a preprint [6] by S. Strătilă, D. Voiculescu and L. Zsidó announcing the same results.

§ 2. Duality for Dual Crossed Products

If G is a locally compact group we shall always equip G with a left invariant Haar-measure, usually denoted dx. Δ is the modular function on G. The spaces $L^p(G)$ with norm $\| \ \|_p$ $(1 \leqq p \leqq \infty)$ are always with respect to this measure. $C_{00}(G)$ is the space of all continuous complex valued functions on G with compact support. $\mathscr{L}(G)$ is the von Neumann algebra generated by the left regular representation of G, and for $x \in G$ we shall denote with x also the corresponding element of $\mathscr{L}(G)$, i.e.

$$(xf)(y) = f(x^{-1}y) \quad \text{for} \quad x, y \in G, f \in L^2(G). \tag{1}$$

$\mathscr{R}(G) = \mathscr{L}(G)'$ is generated by $\{v(x) \mid x \in G\}$ where

$$(v(x)f)(y) = \Delta(x)^{1/2} f(yx) \quad \text{for} \quad x, y \in G, f \in L^2(G). \tag{2}$$

If $\varphi \in L^\infty(G)$, we shall also denote with φ the corresponding operator on $L^2(G)$, i.e.

$$(\varphi f)(x) = \varphi(x) f(x) \quad \text{for} \quad \varphi \in L^\infty(G), f \in L^2(G). \tag{3}$$

If H is a Hilbert space, $\mathscr{B}(H)$ is the von Neumann algebra of all bounded operators on H.

If W_G is the unitary operator over $L^2(G\times G)$ defined by

$$W_G f(s,t)=f(s,st) \quad \text{for} \quad s,t\in G, f\in L^2(G\times G), \tag{4}$$

we can define a normal isomorphism $\delta_G:\mathscr{L}(G)\to\mathscr{L}(G)\otimes\mathscr{L}(G)$ by

$$\delta_G(a)=W_G^*(a\otimes I)W_G \quad \text{for} \quad a\in\mathscr{L}(G). \tag{5}$$

$A(G)=\mathscr{L}(G)_*=$ the predual of $\mathscr{L}(G)$ is an algebra under the multiplication defined by

$$\alpha\beta(a)=(\alpha\otimes\beta)(\delta_G(a)) \quad \text{for} \quad a\in\mathscr{L}(G), \alpha, \beta\in A(G). \tag{6}$$

In fact $A(G)$ will be a commutative semi-simple Banach *-algebra and is usually called the Fourier-algebra of G, cf. [7, § 2]. For any unexplained definitions and notations concerning von Neumann algebras and covariant systems we refer to [2].

Suppose now A is a von Neumann algebra realized over some Hilbert space H_0. If we have a normal isomorphism $\delta:A\to A\otimes\mathscr{L}(G)$ satisfying

$$(\delta\otimes i)\delta=(i\otimes\delta_G)\delta \tag{7}$$

we shall call the pair (A,δ) a *$\mathscr{L}(G)$-comodule*, and we say that δ is a *dual action* of G on A. (Note that if G is abelian (7) will in fact define a covariant system for A and $G\hat{}$, cf. [4, Theorem 2.1].) As noted in [2] a comodule (A,δ) will make the predual A_* of A into an $A(G)$-module if we define

$$\varphi\alpha(a)=(\varphi\otimes\alpha)(\delta(a)) \quad \text{for} \quad \varphi\in A_*, \alpha\in A(G), a\in A. \tag{8}$$

Definition. Let $\mathfrak{A}=W^*(A,\delta,G)$ be the von Neumann algebra generated by $\delta(A)\cup I\otimes L^\infty(G)$ over $H=L^2(G,H_0)\cong H_0\otimes L^2(G)$. $\mathfrak{A}$ is called the *dual covariance algebra of* (A,δ) and does not depend on the Hilbert space H_0 on which A is represented.

Let $\mu:L^\infty(G)\to\mathfrak{A}$ be the normal isomorphism defined by $\mu(f)=I\otimes f$. We can define a σ-continuous automorphic representation ϱ of G on $\mathfrak{A}$ by the formula

$$\varrho_x(a)=(I\otimes v(x))a(I\otimes v(x^{-1})) \quad \text{for} \quad x\in G, a\in\mathfrak{A}. \tag{9}$$

Then $\varrho_x(\delta(a))=\delta(a)$ for $a\in A$ and $\varrho_x(\mu(f))=\mu(f_x)$ for $f\in L^\infty(G)$ where

$$f_x(y)=f(yx). \tag{10}$$

So we have that $\varrho_x(\mathfrak{A})=\mathfrak{A}$, and $(\mathfrak{A},\varrho,G)$ is in fact a covariant system.

Therefore if $\mathfrak{A}=W^*(A,\delta,G)$ is the dual covariance algebra of a $\mathscr{L}(G)$-comodule (A,δ) we can define a σ-continuous automorphic representation $\varrho:G\to\mathrm{Aut}(\mathfrak{A})$ and a normal isomorphism $\mu:L^\infty(G)\to\mathfrak{A}$ such that

$$\varrho_x(\mu(f))=\mu(f_x) \quad \text{for} \quad f\in L^\infty(G), x\in G. \tag{11}$$

Our main result is that this property in fact characterizes such dual covariance algebras:

Theorem 1. *Given a von Neumann algebra $\mathfrak{A}$ and a locally compact group G, then $\mathfrak{A}$ is the dual covariance algebra of some $\mathscr{L}(G)$-comodule (A,δ) if and only if there is a σ-continuous automorphic representation ϱ of G over $\mathfrak{A}$ and a normal isomorphism*

μ of $L^\infty(G)$ into $\mathfrak{A}$ such that

$$\varrho_x(\mu(f)) = \mu(f_x) \quad \text{for} \quad f \in L^\infty(G), x \in G. \tag{11}$$

(A, δ) are uniquely determined up to isomorphisms by

$$A \cong \{a \in \mathfrak{A} | \varrho_x(a) = a \quad \text{for all} \quad x \in G\} \tag{12}$$

and

$$\langle \delta(a)(\xi \otimes f), \eta \otimes g \rangle = \int \langle a\mu(f_t)\xi, \mu(g_t)\eta \rangle dt \tag{13}$$

for $a \in A$, $\xi, \eta \in H$, $f, g \in C_{00}(G)$, if $\mathfrak{A}$ is considered as a von Neumann algebra over a Hilbert space H.

The first step is to prove the uniqueness above, in fact we shall prove:

Lemma 1. *If $\mathfrak{A} = W^*(A, \delta, G)$ and ϱ is defined by* (9), *then*

$$\delta(A) = \{a \in \mathfrak{A} | \varrho_x(a) = a \quad \text{for all} \quad x \in G\}. \tag{14}$$

Proof. Let B be the right hand side of (14), so $B = \mathfrak{A} \cap (I \otimes \mathscr{R}(G))'$. Obviously $\delta(A) \subset B$. Since $\delta(A) \subset A \otimes \mathscr{L}(G)$ we have

$$A' \otimes \mathscr{R}(G) \subset \delta(A)'. \tag{15}$$

Let $W = I \otimes W_G$, so W is a unitary operator over $H \otimes L^2(G) \cong H_0 \otimes L^2(G \times G)$. Then

$$\begin{aligned} W^*(\delta(A) \otimes I)W &= (i \otimes \delta_G)\delta(A) = (\delta \otimes i)\delta(A) \\ &\subset (\delta \otimes i)(A \otimes \mathscr{L}(G)). \end{aligned}$$

So $\delta(A)' \otimes \mathscr{R}(G) \subset W^*(\delta(A)' \otimes \mathscr{B}(L^2(G))W$. Since $W \in (I \otimes I \otimes \mathscr{R}(G))'$, this implies that $W(\delta(A)' \otimes I)W^* \subset \delta(A)' \otimes \mathscr{L}(G)$. Therefore we can define a normal isomorphism $\delta' : \delta(A)' \to \delta(A)' \otimes \mathscr{L}(G)$ by

$$\delta'(a) = W(a \otimes I)W^* \quad \text{for} \quad a \in \delta(A)'. \tag{16}$$

By (15) $I \otimes v(x) \in \delta(A)'$ for $x \in G$, and

$$\delta'(I \otimes v(x)) = I \otimes v(x) \otimes x \quad \text{for} \quad x \in G. \tag{17}$$

We next want to prove that $(\delta' \otimes i)\delta' = (i \otimes \delta_G)\delta'$. If we extend the definition of δ' to all elements in $\mathscr{B}(H)$ using the same formula (16), it will suffice to prove that

$$(\delta' \otimes i)\delta'(a \otimes f \cdot v(x)) = (i \otimes \delta_G)\delta'(a \otimes f \cdot v(x)) \tag{18}$$

for all $a \in \mathscr{B}(H_0)$, $f \in L^\infty(G)$, $x \in G$, since elements of this form is a total set in $\mathscr{B}(H)$. Now if we use (17) and that $W \in (I \otimes L^\infty(G) \otimes I)'$ we have

$$\delta'(a \otimes f \cdot v(x)) = W(a \otimes f \cdot v(x) \otimes I)W^* = a \otimes f \cdot v(x) \otimes x.$$

So the left hand side of (18) equals

$$\begin{aligned} &(\delta' \otimes i)(a \otimes f \cdot v(x) \otimes x) = a \otimes f \cdot v(x) \otimes x \otimes x = a \otimes f \cdot v(x) \otimes \delta_G(x) \\ &= (i \otimes \delta_G)(a \otimes f \cdot v(x) \otimes x) = \text{right hand side of (18).} \end{aligned}$$

This means that $(\delta(A)', \delta')$ is a $\mathscr{L}(G)$-comodule and that the map $I\otimes v: G\to\delta(A)'$ is such that the triple $(\delta(A)', I\otimes v, \delta')$ satisfies the assumptions of [2, Theorem 1] so we have in particular that

$$\delta(A)' = [C\cup I\otimes\mathscr{R}(G)]'' \tag{19}$$

where

$$C=\{c\in\delta(A)' \mid W(c\otimes I)W^* = c\otimes I\}. \tag{20}$$

Now in order to prove that $\delta(A)=B$ it suffices to show that $\delta(A)'\subset B'$, i.e. that $C\cup I\otimes\mathscr{R}(G)\subset B'$. Obviously $I\otimes\mathscr{R}(G)\subset B'$, so we must show that $C\subset B'$. Since $\mathfrak{A}'\subset B'$, it suffices to show that

$$C\subset\mathfrak{A}' = \delta(A)'\cap(I\otimes L^\infty(G))', \quad \text{i.e. that} \quad C\subset(I\otimes L^\infty(G))'.$$

This follows from observing that if $c\in C$, then

$$\begin{aligned} c\otimes I = W(c\otimes I)W^* &\in W(I\otimes I\otimes L^\infty(G))'W^*\cap(I\otimes I\otimes L^\infty(G))' \\ &= (I\otimes L^\infty(G)\otimes L^\infty(G))'. \end{aligned} \tag{21}$$

So $c\in(I\otimes L^\infty(G))'$, in fact $C=\delta(A)'\cap(I\otimes L^\infty(G))'$. This proves the lemma.

We have now proved that if we start with a $\mathscr{L}(G)$-comodule (A,δ) and $\mathfrak{A} = W^*(A,\delta,G)$ we recover (A,δ) or rather the isomorphic comodule $(\delta(A), \delta\otimes i)$ as follows: $\delta(A)=\{a\in\mathfrak{A} \mid \varrho_x(a)=a \text{ for all } x\in G\}$. That $\delta\otimes i$ then in fact is given by the formula (13) follows from:

Lemma 2. *If $a\in\mathscr{B}(H_0)\otimes\mathscr{L}(G)$, $f,g\in C_{00}(G)$, $\xi,\eta\in L^2(G,H_0)$ we have that*

$$\langle W^*(a\otimes I)W(\xi\otimes f), \eta\otimes g\rangle = \int\langle a\mu(f_t)\xi, \mu(g_t)\eta\rangle dt.$$

Proof. It suffices to prove this for $a=b\otimes x$ with $b\in\mathscr{B}(H_0)$, $x\in G$, since both sides define bounded normal linear functionals on $\mathscr{B}(H)$.

$$\begin{aligned} &\langle W^*(b\otimes x\otimes I)W(\xi\otimes f), \eta\otimes g\rangle = \langle (b\otimes x\otimes x)(\xi\otimes f), \eta\otimes g\rangle \\ &= \langle xf, g\rangle\langle (b\otimes x)\xi, \eta\rangle = \int\int f(x^{-1}t)\overline{g(t)}\langle b\xi(x^{-1}s), \eta(s)\rangle dtds \\ &= \int\int f(x^{-1}st)\overline{g(st)}\langle b\xi(x^{-1}s), \eta(s)\rangle dsdt \\ &= \int\int\langle b(\mu(f_t)\xi)(x^{-1}s), (\mu(g_t)\eta)(s)\rangle dsdt \\ &= \int\langle (b\otimes x)\mu(f_t)\xi, \mu(g_t)\eta\rangle dt. \end{aligned}$$

Let us now turn to the second part of Theorem 1, so suppose we have a von Neumann algebra $\mathfrak{A}$ over a Hilbert space H, $\varrho: G\to\mathrm{Aut}(\mathfrak{A})$ and $\mu: L^\infty(G)\to\mathfrak{A}$ as in Theorem 1. We want to prove that $\mathfrak{A}\cong W^*(A,\delta,G)$ where (A,δ) is defined by (12) and (13). Let

$$\mathfrak{A}_0 = \{a\in\mathfrak{A} \mid \exists K\geqq 0, \int\varphi\circ\varrho_x(a^*a)dx\leqq K\|\varphi\| \quad \text{for all} \quad \varphi\in\mathfrak{A}_*^+\}.$$

Then $\mathfrak{A}\mathfrak{A}_0\subset\mathfrak{A}_0$ and $\mathfrak{A}_1 =$ the $*$-algebra generated by $\mathfrak{A}_0^*\mathfrak{A}_0$ is a $*$-subalgebra of $\mathfrak{A}$.

Lemma 3. *$\mathfrak{A}_0$ and $\mathfrak{A}_1$ are both σ-dense in $\mathfrak{A}$.*

Proof. Since there is a net $f_i\in C_{00}(G)$ such that $\mu(f_i)\to I$ in the σ-topology and since $\mathfrak{A}_0$ is a left ideal in $\mathfrak{A}$, it suffices to prove that $\mu(f)\in\mathfrak{A}_0$ for all $f\in C_{00}(G)$.

If $\varphi \in \mathfrak{A}_*^+$, $\varphi \circ \mu \in L^\infty(G)_*^+ \cong L^1(G)^+$, so there is a function $h \in L^1(G)^+$ such that

$$\varphi \circ \mu(f) = \int f(x)h(x)dx \quad \text{for all} \quad f \in L^\infty(G).$$

If $f \in C_{00}(G)$ we therefore have:

$$\begin{aligned}&\int \varphi \circ \varrho_x(\mu(f^* f))dx = \int \varphi \circ \mu((f^* f)_x)dx\\ &= \int\int (f^* f)_x(y)h(y)dxdy = \int\int |f(yx)|^2 h(y)dxdy\\ &= \int |f(x)|^2 dx \int h(y)dy = \|f\|_2^2 \varphi(I) = \|f\|_2^2 \|\varphi\|,\end{aligned}$$

since there are no problems in changing the order of integration. So $\mu(f) \in \mathfrak{A}_0$.

Now let $A = \{a \in \mathfrak{A} \mid \varrho_x(a) = a \text{ for all } x \in G\}$ and define a positive linear map $p : \mathfrak{A}_1 \to A$ by requiring

$$\varphi(p(a)) = \int \varphi \circ \varrho_x(a)dx \quad \text{for all} \quad \varphi \in \mathfrak{A}_*. \tag{22}$$

Note that we in Lemma 3 in fact proved that for $f \in C_{00}(G)$

$$p(\mu(f^* f)) = \|f\|_2^2 I. \tag{23}$$

Lemma 4. *A and $\mu(L^\infty(G))$ generates $\mathfrak{A}$.*

Proof. Let $\mathscr{B} = [A \cup \mu(L^\infty(G))]''$, we shall prove that $\mathfrak{A} = \mathscr{B}$. It will by Lemma 3 suffice to show that each element of the form $b = a\mu(\alpha)$ is in $\mathscr{B}$ for all $a \in \mathfrak{A}$ and $\alpha \in C_{00}(G)$. So suppose such an element $b = a\mu(\alpha)$ is given together with a $\varphi \in \mathfrak{A}_*$ and $\varepsilon > 0$. Choose a compact neighbourhood U of e in G such that

$$|\varphi(\varrho_y(b) - b)| < \varepsilon \quad \text{for all} \quad y \in U.$$

Take another neighbourhood V of e with $V^{-1}V \subset U$ and functions $f, g \in C_{00}(G)^+$ with supports in V and such that

$$\int f(y)dy = \int g(x^{-1})dx = 1. \tag{24}$$

Take $h(y) = \int f(x^{-1}y)g(x^{-1})dx$, then h has support in U and $\int h(y)dy = 1$. Let ${}_x f(y) = f(x^{-1}y)$ and define an element c by

$$\begin{aligned}c &= \int p[b\mu({}_x f)]\mu({}_x g)dx\\ &= \int\int \varrho_y(a)\mu(\alpha_y \cdot {}_x f_y \cdot {}_x g)dydx.\end{aligned} \tag{25}$$

If $K = \text{support}(\alpha)$ we have that $(\alpha_y \cdot {}_x f_y \cdot {}_x g)(z) = \alpha(zy)f(x^{-1}zy)g(x^{-1}z) \neq 0$ only if $x \in KV^{-1}$ and $y \in V^{-1}V$. So c is well-defined as a weak Bochner integral and $c \in \mathscr{B}$.

$$\int f(x^{-1}zy)g(x^{-1}z)dx = \int f(x^{-1}y)g(x^{-1})dx = h(y)$$

for all $z \in G$, so $\int \mu({}_x f_y \cdot {}_x g)dx = h(y)I$. By changing the order of integration in (25) we see that

$$c = \int h(y)\varrho_y(b)dy. \tag{26}$$

Therefore

$$\begin{aligned}&|\varphi(c - b)| = |\int h(y)\varphi(\varrho_y(b) - b)dy|\\ &\leqq \int_U h(y)|\varphi(\varrho_y(b) - b)|dy \leqq \varepsilon \int h(y)dy = \varepsilon.\end{aligned}$$

So $\mathfrak{A}\mu(C_{00}(G))\subset\mathscr{B}$, thus $\mathscr{B}$ is σ-dense in $\mathfrak{A}$, and $\mathfrak{A}=\mathscr{B}$.

We now want to make A into a $\mathscr{L}(G)$-comodule and for $a\in\mathfrak{A}$ we shall define an element $\delta(a)$ of $\mathscr{B}(L^2(G,H))$ by

$$\langle\delta(a)(\xi\otimes f),\eta\otimes g\rangle=\int\langle\varrho_t(a)\mu(f_t)\xi,\mu(g_t)\eta\rangle dt \tag{27}$$

for $\xi,\eta\in H, f,g\in C_{00}(G)$.

Note that for $a\in A$ the definitions (27) and (13) agree. First let us check that (27) really defines a bounded operator. If $a\in\mathfrak{A}$, $\{f^i\}_{i=1}^n\subset C_{00}(G)$ and $\{\xi^i\}_{i=1}^n\subset H$ we have by (23) that

$$\begin{aligned}\int\Big\|\sum_i\varrho_t(a)\mu(f_t^i)\xi^i\Big\|^2dt&\leqq\|a\|^2\sum_{i,j}\langle\mu(f_t^i)\xi^i,\mu(f_t^j)\xi^j\rangle dt\\&=\|a\|^2\sum_{i,j}\langle f^i,f^j\rangle\langle\xi^i,\xi^j\rangle=\|a\|^2\Big\|\sum_i f^i\otimes\xi^i\Big\|^2.\end{aligned}$$

So (27) really defines $\delta(a)$ as a bounded operator over $L^2(G,H)$.

Obviously $\delta(a)\in\mathfrak{A}\otimes\mathscr{B}(L^2(G))$ for $a\in\mathfrak{A}$. Furthermore, if $a\in\mathfrak{A}$

$$\begin{aligned}&\langle(I\otimes v(x)\delta(a)(I\otimes v(x^{-1}))(\xi\otimes f),\eta\otimes g\rangle\\&=\langle\delta(a)(\xi\otimes v(x^{-1})f),\eta\otimes v(x^{-1})g\rangle\\&=\Delta(x^{-1})\int\langle\varrho_t(a)\mu(f_{tx^{-1}})\xi,\mu(g_{tx^{-1}})\eta\rangle dt\\&=\langle\delta(\varrho_x(a))(\xi\otimes f),\eta\otimes g\rangle.\end{aligned} \tag{28}$$

So $\delta(A)\subset\mathfrak{A}\otimes\mathscr{L}(G)$.

If $\varphi\in\mathfrak{A}_*$, $f,g\in C_{00}(G)$ and $h\in A(G)$ is defined by $h(x)=\langle x\cdot f,g\rangle$ for $x\in\mathscr{L}(G)$ we have from (27) that

$$\begin{aligned}&(\varphi\otimes h)\circ(\varrho_x\otimes i)(\delta(a))=(\varphi\circ\varrho_x\otimes h)(\delta(a))\\&=\int\varphi\circ\varrho_x(\mu(g_t)^*a\mu(f_t))dt\\&=\int\varphi(\mu(g_{xt})^*a\mu(f_{xt}))dt=(\varphi\otimes h)(\delta(a))\end{aligned}$$

for all $a\in A$, $x\in G$. Thus $\delta(A)\subset A\otimes\mathscr{L}(G)$.

Now note that from (23) it follows that we can define a unitary operator U over $L^2(G,H)$ such that

$$U(\xi\otimes f)(s)=\mu(f_s)\xi\quad\text{for}\quad\xi\in H, f\in C_{00}(G). \tag{29}$$

If ϱ_1 is the normal isomorphism of $\mathfrak{A}$ defined by

$$\varrho_1(a)f(s)=\varrho_s(a)f(s)\quad\text{for}\quad a\in A, f\in L^2(G,H), s\in G, \tag{30}$$

we see from (27) that we have

$$\delta(a)=U^*\varrho_1(a)U\quad\text{for}\quad a\in\mathfrak{A}, \tag{31}$$

so δ is obviously a normal isomorphism.

In order to show that (A, δ) is a $\mathscr{L}(G)$-comodule it now remains to show that $(\delta\otimes i)\delta=(i\otimes\delta_G)\delta$. If $a\in A$, $\xi, \eta\in H$, $f, g, h, k\in C_{00}(G)$ we have:

$$\langle(\delta\otimes i)\delta(a)(\xi\otimes f\otimes g), \eta\otimes h\otimes k\rangle$$
$$=\int\langle\delta(a)(\mu(f_t)\otimes I)(\xi\otimes g), (\mu(h_t)\otimes I)(\eta\otimes k)\rangle dt$$
$$=\int\langle\delta(a)(\mu(f_t)\xi\otimes g), \mu(h_t)\eta\otimes k\rangle dt$$
$$=\int\int\langle a\mu(g_s\cdot f_t)\xi, \mu(k_s\cdot h_t)\eta\rangle dtds$$
$$=\int\int\langle a\mu(g_{st}\cdot f_s)\xi, \mu(k_{st}\cdot h_s)\eta\rangle dsdt$$
$$=\int\langle\delta(a)(\xi\otimes g_t f), \eta\otimes k_t h\rangle dt$$
$$=\int\langle\delta(a)(I\otimes g_t)(\xi\otimes f), (I\otimes k_t)(\eta\otimes h)\rangle dt$$

(Lemma 2)

$$=\langle W^*(\delta(a)\otimes I)W(\xi\otimes f\otimes g), \eta\otimes h\otimes k\rangle$$
$$=\langle(i\otimes\delta_G)\delta(a)(\xi\otimes f\otimes g), \eta\otimes h\otimes k\rangle\,.$$

We have now shown that (A, δ) is a $\mathscr{L}(G)$-comodule, and in order to complete the proof of Theorem 1 we shall show that δ defined in (27) is an isomorphism between the covariant systems $(\mathfrak{A}, \varrho, G)$ and $(W^*(A, \delta, G), \varrho^\sim, G)$ where $\varrho^\sim$ is the natural automorphic representation of G on $W^*(A, \delta, G)$ defined by (9).

If we put an element $\mu(h)$ with $h\in L^\infty(G)$ into (27) we have from (23) that

$$\langle\delta(\mu(h))(\xi\otimes f), \eta\otimes g\rangle=\langle p(\mu(g)^*\mu(h)\mu(f))\xi, \eta\rangle$$
$$=\langle\xi,\eta\rangle\langle hf, g\rangle=\langle(I\otimes h)(\xi\otimes f), \eta\otimes g\rangle\,.$$

Thus $\delta(\mu(h))=I\otimes h$ and from Lemma 4 it follows that $\delta(\mathfrak{A})=W^*(A, \delta, G)$. From (28) it follows that

$$\varrho^\sim_x(\delta(a))=(I\otimes v(x))\delta(a)(I\otimes v(x^{-1}))=\delta(\varrho_x(a)) \quad \text{for} \quad x\in G, a\in\mathfrak{A}\,,$$

proving that the covariant systems $(\mathfrak{A}, \varrho, G)$ and $(W^*(A, \delta, G), \varrho^\sim, G)$ are equivalent.

§3. The Bidual of a Covariant System and of a $\mathscr{L}(G)$-Comodule

We have now seen that a $\mathscr{L}(G)$-comodule (A, δ) gives rise to a covariant system $(\mathfrak{A}, \varrho, G)$ with $\mathfrak{A}=W^*(A, \delta, G)$. In [2] it was shown that a covariant system (A, ϱ, G) gives rise to a $\mathscr{L}(G)$-comodule $(\mathfrak{A}, \delta)$ with $\mathfrak{A}=W^*(A, \varrho, G)$. It is therefore natural to ask what $W^*(W^*(A, \varrho, G), \delta, G)$ and $W^*(W^*(A, \delta, G), \varrho, G)$ are. It should come as no surprise that both are isomorphic to $A\otimes\mathscr{B}(L^2(G))$, a fact which was proved for an abelian G in [8].

Theorem 2. *Given a covariant system (A, ϱ, G) let $(\mathfrak{A}, \delta)$ be the $\mathscr{L}(G)$-comodule defined in [2, Chapter 2], i.e. $\mathfrak{A}=W^*(A, \varrho, G)$ and $\delta(a)=W^*(a\otimes I)W$. Then $W^*(\mathfrak{A}, \delta, G)\cong A\otimes\mathscr{B}(L^2(G))$.*

Proof. Suppose A acts on a Hilbert space H_0 and let $\varrho^\sim$ be the faithful representation of A on $H=L^2(G, H_0)$ given by

$$\varrho^\sim(a)f(s)=\varrho_{s^{-1}}(a)f(s) \quad \text{for} \quad a\in A, f\in H, s\in G\,. \tag{32}$$

Then $\mathfrak{A}=[\varrho\check{}(A)\cup I\otimes\mathscr{L}(G)]''$ and $\delta(\mathfrak{A})$ is generated by $\varrho\check{}(A)\otimes I\cup\{I\otimes x\otimes x\,|\,x\in G\}$. So $W^*(\mathfrak{A},\delta,G)$ is generated over $L^2(G\times G,H_0)$ by

$$\varrho\check{}(A)\otimes I\cup I\otimes\delta_G(\mathscr{L}(G))\cup I\otimes I\otimes L^\infty(G).$$

Let $\varrho^0:G\to\operatorname{Aut}\varrho\check{}(A)$ be given by

$$\varrho^0_x(a)=(I\otimes x)a(I\otimes x^{-1})\quad\text{for}\quad x\in G,\,a\in\varrho\check{}(A). \tag{33}$$

The covariant systems $(\varrho\check{}(A),\varrho^0,G)$ and (A,ϱ,G) are then equivalent. From [2, Proposition 2.2] it follows that

$$\varrho\check{}(A)\otimes\mathscr{B}(L^2(G))=[W^*(\varrho\check{}(A),\varrho^0,G)\cup I\otimes I\otimes L^\infty(G)]''. \tag{34}$$

Let U be the unitary operator over $L^2(G\times G,H_0)$ defined by

$$Uf(s,t)=f(t^{-1}s,t). \tag{35}$$

Then

$$W^*(\varrho\check{}(A),\varrho^0,G)=[U^*(\varrho\check{}(A)\otimes I)U\cup I\otimes I\otimes\mathscr{L}(G)]''.$$

So

$$\varrho\check{}(A)\otimes\mathscr{B}(L^2(G))=[U^*(\varrho\check{}(A)\otimes I)U\cup I\otimes I\otimes\mathscr{L}(G)\cup I\otimes I\otimes L^\infty(G)]''.$$

Now $U(I\otimes I\otimes x)U^*=I\otimes x\otimes x$ for $x\in G$, and $U\in(I\otimes I\otimes L^\infty(G))'$ so

$$A\otimes\mathscr{B}(L^2(G))\cong\varrho\check{}(A)\otimes\mathscr{B}(L^2(G))=U^*W^*(\mathfrak{A},\delta,G)U,$$

which proves the theorem.

Theorem 3. *Given a $\mathscr{L}(G)$-comodule (A,δ) let $(\mathfrak{A},\varrho,G)$ be the covariant system defined by $\mathfrak{A}=W^*(A,\delta,G)$ and ϱ as in* (9). *Then $W^*(\mathfrak{A},\varrho,G)\cong A\otimes\mathscr{B}(L^2(G))$.*

Proof. $\mathfrak{A}=W^*(A,\delta,G)$ is generated by $\delta(A)\cup(I\otimes L^\infty(G))$ over $L^2(G,H_0)$ if we consider A as a von Neumann algebra over H_0. $B=W^*(\mathfrak{A},\varrho,G)$ is then generated by $V^*(\mathfrak{A}\otimes I)V\cup I\otimes I\otimes\mathscr{L}(G)$ where V is the unitary operator over $L^2(G\times G,H_0)$ defined by

$$Vf(s,t)=\Delta(t)^{1/2}f(st,t). \tag{36}$$

Define another unitary operator S by

$$Sf(s,t)=\Delta(t)^{-1/2}f(s,t^{-1}) \tag{37}$$

then $S^*V^*(I\otimes\varphi\otimes I)VS=W(I\otimes I\otimes\varphi)W^*$ for $\varphi\in L^\infty(G)$, and $S^*(I\otimes I\otimes x)S=I\otimes I\otimes v(x)$ for $x\in G$. Since VS and $\delta(A)\otimes I$ commute we therefore have that

$$\begin{aligned}B&=[V^*(\delta(A)\otimes I)V\cup V^*(I\otimes L^\infty(G)\otimes I)V\cup I\otimes I\otimes\mathscr{L}(G)]''\\&=S[\delta(A)\otimes I\cup W(I\otimes I\otimes L^\infty(G))W^*\cup I\otimes I\otimes\mathscr{R}(G)]''S^*\\&=SW[W^*(\delta(A)\otimes I)W\cup I\otimes I\otimes L^\infty(G)\cup I\otimes I\otimes\mathscr{R}(G)]''W^*S^*.\end{aligned}$$

So if we can prove that

$$[W^*(\delta(A)\otimes I)W\cup I\otimes I\otimes\mathscr{B}(L^2(G))]''=\delta(A)\otimes\mathscr{B}(L^2(G)) \tag{38}$$

the theorem is proved. (38) is equivalent to

$$W^*(\delta(A)' \otimes \mathscr{B}(L^2(G)))W \cap (I \otimes I \otimes L^\infty(G))' \cap (I \otimes I \otimes \mathscr{R}(G))' = \delta(A)' \otimes I, \tag{39}$$

i.e. that

$$W^*(\delta(A)' \otimes \mathscr{L}(G))W \cap (I \otimes I \otimes L^\infty(G))' = \delta(A)' \otimes I. \tag{40}$$

Let $D = W(I \otimes I \otimes L^\infty(G))W^*$ and $E = (\delta(A)' \otimes \mathscr{L}(G)) \cap D'$. Now define a map $\delta'' : E \to E \otimes \mathscr{L}(G)$ by

$$\delta''(a) = (I \otimes I \otimes W_G^*)(a \otimes I)(I \otimes I \otimes W_G) = (i \otimes \delta_G)(a) \quad \text{for} \quad a \in E, \tag{41}$$

where i is the identity automorphism of $\mathscr{B}(L^2(G, H_0))$. Obviously $\delta''(\delta(A)' \otimes \mathscr{L}(G)) \subset \delta(A)' \otimes \mathscr{L}(G) \otimes \mathscr{L}(G)$ and $\delta''(D) = D \otimes I$, so (41) will in fact define a normal isomorphism of E into $E \otimes \mathscr{L}(G)$ which obviously satisfies $(\delta'' \otimes i)\delta'' = (i \otimes \delta_G)\delta''$. $I \otimes v(x) \otimes x = W(I \otimes v(x) \otimes I)W^* \in E$ and $\delta''(I \otimes v(x) \otimes x) = I \otimes v(x) \otimes x \otimes x$ for $x \in G$, so again we can use Theorem 1 in [2] to conclude that E is generated by

$$F = \{a \in E \mid \delta''(a) = a \otimes I\} \quad \text{and} \quad \{I \otimes v(x) \otimes x \mid x \in G\}.$$

Using the same argument as in (21) we have that

$$\begin{aligned} F &= E \cap (I \otimes I \otimes L^\infty(G))' = (\delta(A)' \otimes \mathscr{L}(G)) \cap D' \cap (I \otimes I \otimes L^\infty(G))' \\ &= (\delta(A)' \otimes \mathscr{L}(G)) \cap (I \otimes L^\infty(G) \otimes L^\infty(G))' \\ &= (\delta(A)' \cap (I \otimes L^\infty(G))') \otimes I = C \otimes I \end{aligned}$$

where C is as in (20). So the left hand side of (40) equals:

$$\begin{aligned} W^*EW &= W^*[F \cup \{I \otimes v(x) \otimes x \mid x \in G\}]''W \\ &= [C \otimes I \cup I \otimes \mathscr{R}(G) \otimes I]'' = \delta(A)' \otimes I \end{aligned}$$

according to (19). So the formula (40) holds and the theorem is proved.

§4. The Commutant of a Covariance Algebra and of a Dual Covariance Algebra

Digernes proved in [1, Theorem 3.14] the following:

Theorem 4. *Suppose A is a von Neumann algebra over a Hilbert space H and that U is a continuous unitary representation on H of the locally compact group G such that*

$$\varrho_x(a) = U_x a U_{x^{-1}} \in A \quad \textit{for all} \quad x \in G, a \in A.$$

Let the covariance algebra $\mathfrak{A} = W^*(A, \varrho, G)$ act on $L^2(G, H)$ as usual. Then $\mathfrak{A}'$ is generated by $A' \otimes I$ and $\{U_x \otimes v(x) \mid x \in G\}$, and in fact $\mathfrak{A}' \cong W^*(A', \varrho', G)$ where $\varrho' : G \to \mathrm{Aut}(A')$ is defined by

$$\varrho'_x(a) = U_x a U_{x^{-1}} \quad \textit{for} \quad a \in A'. \tag{42}$$

We shall first give an alternate proof of this theorem using [2, Theorem 1] and then state and prove a similar result for the dual covariance algebra of a $\mathscr{L}(G)$-comodule.

Proof of Theorem 4. Let $\mathfrak{A}=W^*(A,\varrho,G)$ as defined in [2], then

$$W(\mathfrak{A}'\otimes I)W^*\subset\mathfrak{A}'\otimes\mathscr{L}(G),$$

cf. the first part of the proof of Lemma 1, and it is straight forward to check that

$$U_x\otimes v(x)\in\mathfrak{A}' \quad \text{for} \quad x\in G.$$

Defining the map δ' as in (16), i.e.

$$\delta'(a)=W(a\otimes I)W^* \quad \text{for} \quad a\in\mathfrak{A}'$$

we have that $\delta'(U_x\otimes v(x))=U_x\otimes v(x)\otimes x$ and that $(\delta'\otimes i)\delta'=(i\otimes\delta_G)\delta'$, so as in Lemma 1 we can use [2, Theorem 1] to conclude that $\mathfrak{A}'$ is generated by

$$C\cup\{U_x\otimes v(x)\,|\,x\in G\}$$

where

$$C=\{c\in\mathfrak{A}'\,|\,\delta'(c)=c\otimes I\}.$$

For C we can use the same argument as in (21) to conclude that $C=\mathfrak{A}'\cap(I\otimes L^\infty(G))'$, so $C=A'\otimes I$ according to [2, Proposition 2.2].

So $\mathfrak{A}'=[A'\otimes I\cup\{U_x\otimes v(x)\,|\,x\in G\}]''$ as stated, furthermore, Theorem 1 of [2] also gives us that $\mathfrak{A}'\cong W^*(A',\varrho',G)$.

The dual version of Theorem 4 is the following:

Theorem 5. *Suppose (A,δ) is a $\mathscr{L}(G)$-comodule and let $\mathfrak{A}=W^*(\delta(A),\delta\otimes i,G)$ be the covariance algebra of the equivalent comodule $(\delta(A),\delta\otimes i)$. Then*

$$\mathfrak{A}'=[\delta(A)'\otimes I\cup W^*(I\otimes I\otimes L^\infty(G))W]'' \tag{43}$$

and $\mathfrak{A}'$ is isomorphic to the covariance algebra of the $\mathscr{L}(G)$-comodule $(\delta(A)',\delta')$ where δ' is defined by

$$\delta'(a)=W(a\otimes I)W^* \quad \textit{for} \quad a\in\delta(A)'. \tag{44}$$

Proof. $\mathfrak{A}=[W^*(\delta(A)\otimes I)W\cup I\otimes I\otimes L^\infty(G)]''$. $(I\otimes I\otimes v(x))\mathfrak{A}'(I\otimes I\otimes v(x^{-1}))=\mathfrak{A}'$ and if we define $\mu:L^\infty(G)\to\mathfrak{A}'$ by

$$\mu(f)=W^*(I\otimes I\otimes f)W \quad \text{for} \quad f\in L^\infty(G)$$

we see that with ϱ as in (9), Theorem 1 is satisfied so $\mathfrak{A}'$ is generated by $\mu(L^\infty(G))$ and $\mathfrak{A}'\cap(I\otimes I\otimes v(G))'$. From (40) we have that

$$\begin{aligned}\mathfrak{A}'\cap(I\otimes I\otimes v(G))'&=[W^*(\delta(A)\otimes I)W\cup I\otimes I\otimes L^\infty(G)\cup I\otimes I\otimes v(G)]'\\&=\delta(A)'\otimes I.\end{aligned}$$

This proves (43). Furthermore

$$\begin{aligned}W\mathfrak{A}'W^*&=[W(\delta(A)'\otimes I)W^*\cup I\otimes I\otimes L^\infty(G)]''\\&=W^*(\delta(A)',\delta',G)\end{aligned}$$

where δ' is defined by (44). It was proved in Lemma 1 that $(\delta(A)',\delta')$ really is a $\mathscr{L}(G)$-comodule.