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Resistance Inequalities for KMS States of the Isotropic Heisenberg Model*

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Abstract. Inequalities which show that the spin correlations between spins at two lattice sites is bounded by a constant times the inverse square root of the electrical resistance between the lattice sites is proved for KMS states of the isotropic Heisenberg model. The resistance is calculated using the inverse of the coefficients occuring in the Heisenberg Hamiltionian as the resistances between neighboring lattice sites.

Introduction

In this paper we prove inequalities which show that the spin correlation between spins at two lattice sites is bounded by a constant times the inverse square root of the electrical resistance between the lattice sites for KMS states of the isotropic Heisenberg model. The resistance is calculated using the inverse of the coefficients occuring in the Heisenberg Hamiltonian as the resistance between neighboring lattice sites.

The proof of these inequalities comes from combining the ideas of Mermin and Wagner's proof [2] of the absense of ferromagnetism for the isotropic Heisenberg model in one and two dimensions with the resistance type arguments of [3]. A key ingredient of Mermin and Wagner's argument is the use of the Bogoliubov inequality. A short proof of the Bogoliubov inequality for Gibbs states of a full $(n \times n)$-matrix algebra is given beginning on page 130 of Ruelle's book [6]. In the first section of this paper we generalize the Bogoliubov inequality to KMS states of C^*-algebras.

In the second section of this paper we prove resistance inequalities for KMS states of the isotropic Heisenberg model. These inequalities show there is no long range order for the isotropic Heisenberg model in one or two dimensions or for any graph in which the resistance between vertices grows without bound with increasing separation.

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I. Bogoliubov Inequalities for KMS States

Suppose $\{\alpha_t; -\infty < t < \infty\}$ is a strongly continuous one parameter group of *-automorphisms of a C^*-algebra $\mathfrak{A}$ with unit I. We say ω is a $\{\alpha_t, \beta\}$-KMS state of $\mathfrak{A}$ if ω is a state of $\mathfrak{A}$ so that for each $A, B \in \mathfrak{A}$ there is a complex valued function F holomorphic on the open strip $S_\beta = \{z \in \mathbb{C}; 0 < \mathrm{Im}(z) < \beta\}$ and continuous on the closed strip $\bar{S}_\beta$ so that

$$\omega(A\alpha_t(B)) = F(t) \quad \text{and} \quad \omega(\alpha_t(B)A) = F(t+i\beta)$$

for all real t. KMS states are a generalization of Gibbs states to infinite systems. We refer to Ruelle's book [6] for further discussion.

The generator of a strongly continuous one parameter group of *-automorphisms α_t is defined

$$\delta(A) = \lim_{t \to 0} (\alpha_t(A) - A)/t$$

for all A in the domain $\mathfrak{D}(\delta)$ of δ which consists of all $A \in \mathfrak{A}$ so that the above limit exists in norm. It is well known [1, 5] that $\mathfrak{D}(\delta)$ is a norm dense *-subalgebra of $\mathfrak{A}$ and δ is a closed *-derivation of $\mathfrak{D}(\delta)$ into $\mathfrak{A}$, i.e., δ is linear and $\delta(AB) = \delta(A)B + A\delta(B)$ and $\delta(A^*) = \delta(A)^*$ for $A, B \in \mathfrak{D}(\delta)$.

Theorem 1. (Bogoliubov inequality). *Suppose $t \to \alpha_t$ is a strongly continuous one parameter group of *-automorphisms of a C^*-algebra $\mathfrak{A}$ and δ is the generator of α_t and $\mathfrak{D}(\delta)$ is the domain of δ. Suppose ω is a $\{\alpha_t, \beta > 0\}$-KMS state of $\mathfrak{A}$. Then for all $A \in \mathfrak{A}$ and $C \in \mathfrak{D}(\delta)$*

$$|\omega([A, C])|^2 \leqq \tfrac{1}{2} \beta \omega(A^*A + AA^*) \omega(-i[C^*, \delta(C)]).$$

where $[A, B] = AB - BA$.

Proof. Suppose ω is a $\{\alpha_t, \beta\}$-KMS state of $\mathfrak{A}$. Let $(\pi, \mathfrak{H}, f_0)$ be a cyclic *-representation of $\mathfrak{A}$ on a Hilbert space $\mathfrak{H}$ induced by ω with cyclic vector $f_0 \in \mathfrak{H}$ so that $\omega(A) = (f_0, \pi(A) f_0)$ for all $A \in \mathfrak{A}$. Since a KMS state is α_t invariant it follows there is a self-adjoint operator H and a strongly continuous unitary group $U(t) = \exp(itH)$ so that $U(t)\pi(A) f_0 = \pi(\alpha_t(A)) f_0$ for all $A \in \mathfrak{A}$ and real t. It is known to follow from the KMS condition [7] that $\pi(A) f_0 \in \mathfrak{D}(\exp(-tH))$ for all $0 \leqq t \leqq \frac{1}{2}\beta$ and

$$(\exp(-\tfrac{1}{2}\beta H)\pi(A) f_0, \ \exp(-\tfrac{1}{2}\beta H)\pi(B) f_0) = (\pi(B^*) f_0, \pi(A^*) f_0) \tag{1.1}$$

for all $A, B \in \mathfrak{A}$. We define an inner product on $\mathfrak{A}$ by

$$(A, B)_1 = \int_0^\beta (\exp(-\tfrac{1}{2} tH)\pi(A) f_0, \ \exp(-\tfrac{1}{2} tH)\pi(B) f_0) dt \tag{1.2}$$

Since $\|\exp(-\frac{1}{2} tH)\pi(A) f_0\|^2$ is a convex function of t for $0 \leqq t \leqq \beta$ we have

$$\begin{aligned}\beta \|\exp(-\tfrac{1}{2} tH)\pi(A) f_0\|^2 &\leqq t\|\pi(A) f_0\|^2 + (\beta - t)\| \exp(-\tfrac{1}{2}\beta H)\pi(A) f_0\|^2 \\ &= t(\pi(A) f_0, \pi(A) f_0) + (\beta - t)(\pi(A^*) f_0, \pi(A^*) f_0) \\ &= t\omega(A^*A) + (\beta - t)\omega(AA^*).\end{aligned}$$

Inserting this inequality into (1.2) we find

$$(A, A)_1 \leqq \tfrac{1}{2} \beta \omega(A^*A + AA^*). \tag{1.3}$$

Next suppose $B\in\mathfrak{D}(\delta)$. Then, we have

$$\pi(\delta(B))f_0=\lim_{t\to 0}t^{-1}\pi(\alpha_t(B)-B)f_0$$
$$=\lim_{t\to 0}t^{-1}(\exp(itH)-I)\pi(B)f_0=iH\pi(B)f_0$$

where $\pi(B)f_0\in\mathfrak{D}(H)$ the domain of H. Then for $A\in\mathfrak{A}$ and $B\in\mathfrak{D}(\delta)$ we have from the above and (1.2)

$$\begin{aligned}(A,\delta(B))_1&=i\int_0^\beta(\exp(-\tfrac{1}{2}tH)\pi(A)f_0,\exp(-\tfrac{1}{2}tH)H\pi(B)f_0)dt\\&=-i\int_0^\beta(d/dt)(\exp(-\tfrac{1}{2}tH)\pi(A)f_0,\exp(-\tfrac{1}{2}tH)\pi(B)f_0)dt\\&=-i(\exp(-\tfrac{1}{2}\beta H)\pi(A)f_0,\exp(-\tfrac{1}{2}\beta H)\pi(B)f_0)+i(\pi(A)f_0,\pi(B)f_0)\end{aligned}$$

Then, it follows from Equation (1.1)

$$(A,\delta(B))_1=-i\omega(BA^*)+i\omega(A^*B)=i\omega([A^*,B]) \tag{1.4}$$

Since $(\cdot,\cdot)_1$ is a positive inner product we have from the Schwarz inequality $|(A^*,B)_1|^2\leqq(A^*,A^*)_1(B,B)_1$. Suppose $C\in\mathfrak{D}(\delta)$. We set $B=-i\delta(C)$ and obtain from (1.4) and the Schwarz inequality

$$|\omega([A,C])|^2=|(A^*,-i\delta(C))_1|^2\leqq(A^*,A^*)_1(\delta(C),\delta(C))_1$$

Combining this with (1.3) and (1.4) we have

$$|\omega([A,C])|^2\leqq\tfrac{1}{2}\beta\omega(A^*A+AA^*)\omega(i[\delta(C)^*,C])$$

Since ω is hermitian we have $\omega(i[\delta(C)^*,C])=\omega(-i[C^*,\delta(C)])$ and the inequality stated in the theorem follows.

II. Resistance Inequalities for the Isotropic Heisenberg Model

Suppose G is a finite or infinite graph consisting of vertices $\mathscr{L}$ and lines $(i,j)\in G$ connecting pairs of vertices $i,j\in\mathscr{L}$. We assume G is connected. We suppose for each line $(i,j)\in G$ there is associated a number $J(i,j)>0$. We assume there is a constant K so that

$$\sum\nolimits_{j\in G(i)}J(i,j)\leqq K \tag{2.1}$$

for all $i\in\mathscr{L}$ where $G(i)$ is the set of vertices j connected to i by a line $(i,j)\in G$.

We define the resistance between two vertices $i,j\in\mathscr{L}$ as follows. We think of the lines $(r,s)\in G$ as resistors of $J(r,s)^{-1}$ ohms. The resistance $R(i,j)$ is then the electrical resistance between i and j. We give a mathematical definition. Suppose f is a real or complex valued function on $\mathscr{L}$. We define

$$Q(f)=\sum\nolimits_{(i,j)\in G}J(i,j)|f(i)-f(j)|^2$$

where the sum is over all lines of G each line counted only once. Let $\mathfrak{D}_0$ be the set of all functions on $\mathscr{L}$ with finite or compact support. We define the resistance $R(i,j)$ between i and j as

$$R(i,j)^{-1}=\inf\{Q(f);f\in\mathfrak{D}_0,f(i)-f(j)=1\} \tag{2.2}$$

This definition of resistance is not quite equivalent to that used in [3] where the resistance was defined as follows. If f is a function on $\mathscr{L}$ we define Δf by

$$(\Delta f)(i) = \sum_{j \in G(i)} J(i,j)\,(f(j) - f(i))$$

Let $\delta_i(k) = 1$ if $i = k$ and zero otherwise. If $\mathscr{L}$ is a finite set then the equation $-\Delta V = \delta_i - \delta_j$ has a unique solution up to an additive constant. In [3] we defined the resistance $R_1(i,j) = V(i) - V(j)$. For an infinite graph we defined $R_1(i,j)$ as the limit of resistances computed for any sequence of finite subgraphs of G which converge up to G. One can show that this definition is equivalent to the definition

$$R_1(i,j)^{-1} = \inf\{Q(f);\ f(i) - f(j) = 1\} \tag{2.3}$$

In the definition of $R_1(i,j)$ there is no requirement on the support of f. Hence, $R_1(i,j) \geqq R(i,j)$.

We have $R(i,j) = R_1(i,j)$ for all finite graphs, Z^n, $n = 1, 2, \ldots$ with one ohm resistors between nearest neighbors, and for most graphs of physical interest. In fact, one can show that if $R_1(i,j) > R(i,j)$ for some $i, j \in \mathscr{L}$ then there exists a bounded harmonic function f (i.e., $\Delta f = 0$) so that $f(i) \neq f(j)$. An example of a graph for which $R_1 \neq R$ is a binary tree with one ohm resistors.

The Heisenberg model associated with G is defined as follows. We suppose that for each $i \in \mathscr{L}$ there is a full hermitian $(2j_i + 1) \times (2j_i + 1)$-matrix algebra $\mathfrak{A}_i$ which describes a particle of spin $j_i = \frac{1}{2}, 1, 1\frac{1}{2}, 2, \ldots$. The C^*-algebra $\mathfrak{A}_i$ is generated by the hermitian elements $\boldsymbol{S}_i = (S_{ix}, S_{iy}, S_{iz})$ which satisfy the relations

$$[S_{ix}, S_{iy}] = iS_{iz} \qquad [S_{iy}, S_{iz}] = iS_{ix} \qquad [S_{iz}, S_{ix}] = iS_y$$

$$\boldsymbol{S}_i^2 = S_{ix}^2 + S_{iy}^2 + S_{iz}^2 = j_i(j_i + 1)I$$

If Λ is a finite subset of $\mathscr{L}$ we define $\mathfrak{A}_\Lambda = \mathfrak{A}_{i_n} \otimes \ldots \otimes \mathfrak{A}_{i_n}$ as the tensor product of the algebras $\mathfrak{A}_{i_k}$ with $\Lambda = \{i_1, i_2, \ldots, i_n\}$. Finally, we define the Heisenberg spin algebra as the inductive limit of the $\mathfrak{A}_\Lambda$ for all finite $\Lambda \subset \mathscr{L}$.

The formal expression for the Heisenberg Hamiltonian is

$$H = \sum_{(i,j) \in G} J(i,j)\,(I - \boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j) \tag{2.4}$$

where $\boldsymbol{\mu}_i = \boldsymbol{S}_i / j_i$. We note $0 \leqq I - \boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j \leqq 4I$ for all $i, j \in \mathscr{L}$. Although the above expression for H is not well defined for infinite graphs we can use it to define a *-derivation of $\mathfrak{A}$,

$$\delta(A) = i \sum_{(i,j) \in G} J(i,j)\,[I - \boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j, A] \tag{2.5}$$

for all $A \in \mathfrak{A}_\Lambda$ for some finite $\Lambda \subset \mathscr{L}$. For such A inequality (2.1) insures the norm convergence of the above sum. Then, δ is a densely defined *-derivation of $\mathfrak{A}$. The closure of δ which we also denote by δ is the generator of a strongly continuous one parameter group of *-automorphisms α_t of $\mathfrak{A}$ (see [3] for more details).

Theorem 2. *Suppose G is a graph of vertices $\mathscr{L}$ and for each line $(i,j) \in G$ there is associated a number $J(i,j) > 0$ and these numbers satisfy* (2.1). *Suppose $\mathfrak{A}_{\mathscr{L}}$ is the Heisenberg spin algebra associated with $\mathscr{L}$ and δ is the closure of the *-derivation of defined in* (2.5) *and α_t is the one parameter group of *-automorphisms generated* *by δ. Suppose ω is a $\{\alpha_t, \beta>0\}$-KMS state of $\mathfrak{A}_{\mathscr{L}}$. Then,*

$$\omega(\boldsymbol{\mu}_k \cdot \boldsymbol{\mu}_l)^2 \leqq 6\beta R(k, l)^{-1}$$

for all $k, l \in \mathscr{L}$ and $R(k, l)$ is the resistance between k and l given by (2.2).

Proof. Suppose the hypothesis of the theorem is satisfied. Suppose $f \in \mathfrak{D}_0$ and $C = \sum_{i \in \mathscr{L}} f(i) S_{iz}$. Then $C \in \mathfrak{D}(\delta)$.

Suppose $k, l \in \mathscr{L}$ and $A = \mu_{kx}\mu_{ly} - \mu_{ky}\mu_{lx}$. Note $A = A^*$ and the norm of A satisfies $\|A\| \leqq 2$. Then we have from Theorem 1

$$\begin{aligned} |\omega([A, C])|^2 &\leqq \tfrac{1}{2} \beta\omega(A^*A + AA^*)\omega(-i[C^*, \delta(C)]) \\ &\leqq \beta\|A\|^2 \omega(-i[C^*, \delta(C)]) \leqq 4\beta\omega(-i[C^*, \delta(C)]) \end{aligned} \tag{2.6}$$

We have

$$\begin{aligned} [A, C] &= -i(f(k) - f(l))\,(\boldsymbol{\mu}_k \cdot \boldsymbol{\mu}_l - \mu_{kz}\mu_{lz}) \\ -i[C^*, \delta(C)] &= \textstyle\sum_{(i, j) \in G} J(i, j)|f(i) - f(j)|^2\,(\boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j - \mu_{iz}\mu_{jz}) \end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7), cyclicly permuting the indices x, y and z, and averaging we find using the inequality $(a+b+c)^2 \leqq 3/4((a+b)^2 + (a+c)^2 + (b+c)^2)$

$$|f(k) - f(l)|^2 \omega(\boldsymbol{\mu}_k \cdot \boldsymbol{\mu}_l)^2 \leqq 6\beta Q'(f) \leqq 6\beta Q(f)$$

with

$$Q'(f) = \textstyle\sum_{(i, j) \in G} J(i, j)\omega(\boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j)|f(i) - f(j)|^2$$

where the second inequality follows from the fact that $\omega(\boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j) \leqq 1$ for all $i, j \in \mathscr{L}$. Taking the greatest lower bound of the right hand side of the above inequality over all $f \in \mathfrak{D}_0$ with $f(k) - f(l) = 1$ and recalling (2.2) we find $\omega(\boldsymbol{\mu}_k \cdot \boldsymbol{\mu}_l)^2 \leqq 6\beta R(k, l)^{-1}$. This completes the proof of the theorem.

We prove a corollary to Theorem 2 which gives a bound on the magnetization of KMS states. We will need the notion of a mean on $\mathscr{L}$. Suppose $\mathscr{L}$ is an infinite set and $\mathfrak{B}(\mathscr{L})$ is the space of all bounded functions on $\mathscr{L}$. A mean Γ on $\mathscr{L}$ is a linear functional on $\mathfrak{B}(\mathscr{L})$ so that $\Gamma(f) \geqq 0$ if $f(i) \geqq 0$ for all $i \in \mathscr{L}$ and $\Gamma(e) = 1$ where $e(i) = 1$ for all $i \in \mathscr{L}$ and $\Gamma(f) = 0$ for all $f \in \mathfrak{D}_0$. If h is a function of two variables $i, j \in \mathscr{L}$ we denote the mean of h with respect to the second variable by $\Gamma_j(h(i, j))$.

Corollary. *Suppose the hypothesis of Theorem 2 are satisfied and Γ is a mean on $\mathscr{L}$. Suppose ω is a $\{\alpha_t, \beta>0\}$-KMS state. We define the magnetization of ω by $\boldsymbol{m} = \Gamma_i(\omega(\boldsymbol{\mu}_i))$. Then*

$$|\boldsymbol{m}|^2 \leqq (6\beta)^{+\frac{1}{2}} \Gamma_i(\Gamma_j(R(i, j)^{-\frac{1}{2}})) \tag{2.8}$$

Proof. Suppose ω is an extremal $\{\alpha_t, \beta\}$-KMS state, i.e., ω can not be written as a convex combination of distinct $\{\alpha_t, \beta\}$-KMS states. Then from a result of Takesaki [7] ω induces a factor representation of $\mathfrak{A}_{\mathscr{L}}$. Hence, ω has the cluster decomposition property (see e.g. [4]) and, thus, for each $i \in \mathscr{L}$ and $\varepsilon > 0$, $|\omega(\boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j) - \omega(\boldsymbol{\mu}_i) \cdot \omega(\boldsymbol{\mu}_j)| < \varepsilon$ except for a finite number of $j \in \mathscr{L}$. Hence, we have

$$\Gamma_i(\Gamma_j(\omega(\boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j))) = \Gamma_i(\Gamma_j(\omega(\boldsymbol{\mu}_i) \cdot \omega(\boldsymbol{\mu}_j))) = |\boldsymbol{m}|^2$$

And from Theorem 2 we have

$$|\boldsymbol{m}|^2 = \Gamma_i(\Gamma_j(\omega(\boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j))) \leqq (6\beta)^{+\frac{1}{2}} \Gamma_i(\Gamma_j(R(i,j)^{-\frac{1}{2}}))$$

From Takesaki's paper [7] it follows that each $\{\alpha_t, \beta\}$-KMS state has a unique decomposition into extremal KMS states. Since the extremal KMS states satisfy (2.8) it follows that all $\{\alpha_t, \beta\}$-KMS states satisfy (2.8). This completes the proof of the corollary.

We remark that for $\mathscr{L} = \mathbb{Z}^n$ and G consisting of lines connecting nearest neighbors of $\mathscr{L}$ with one ohm resistors then $\Gamma_i(\Gamma_j(R(i,j)^{-\frac{1}{2}})) = 0, 0, (.50546)^{-\frac{1}{2}}$ for $n = 1, 2, 3$ respectively.

We conclude by stating the following conjecture.

Conjecture. There is a constant K_0 (independent of G) so that if ω is a state satisfying the hypothesis of Theorem 2 then

$$\omega(I - \boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j) \leqq K_0 \beta^{-1} R_1'(i,j)$$

where $R_1'(i,j)$ is the resistance between $i, j \in \mathscr{L}$ given by (2.3) with $J(i,j)$ replaced by $J(i,j)\omega(\boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j)$.

The truth of this conjecture would establish the existence of long range order and spontaneous magnetization for the isotropic Heisenberg model in three dimensions. In [3] we showed that $\omega(I - \boldsymbol{\mu}_i \cdot \boldsymbol{\mu}_j) \leqq R_1(i,j)\omega(H)$ where H is given by (2.4) for all states ω. From this it follows that all states of finite energy in three dimensions have lone range order.

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