Commun. math. Phys. 51, 67—84 (1976)

Communications in
Mathematical
Physics

Sommerfeld-Watson Representation for Double-Spectral Functions

III. Crossing Symmetric Pion-Pion Scattering Amplitude with Regge Poles

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Abstract. We demonstrate the existence of a solution of the nonlinear pion-pion equations that incorporate crossing symmetry, unitarity and l-plane meromorphy. In particular, we show how to guarantee the boundedness of the partial waves as $s\to\infty$, even when some Regge trajectories rise beyond unity.

1. Introduction

In previous papers of this series [1, 2], nonlinear equations for the partial-wave amplitudes were set up to guarantee unitarity and analyticity in the right half of the angular momentum plane. The treatment in Ref. [1] contained only a direct channel, as in potential theory; but that of Ref. [2] allowed for full three-channel crossing symmetry. The potential case was considered also in a later paper [3], in which improved bounds for the Legendre functions allowed contact to be made with ordinary energy-independent potential scattering. In all this work, no singularities were allowed in the right half of the angular momentum plane.

The present work is devoted to setting up the corresponding Sommerfeld-Watson equations when there are Regge poles for $\mathrm{Re}\, l>0$ [4]. For simplicity in notation, we write down only one Regge pole; but there is no difficulty in handling any finite number of them. The equations of the present paper should be regarded as a representation for a valid crossing-symmetric Regge amplitude, rather than as a dynamical scheme for its calculation. Thus, given the Regge pole position, $\alpha(s)$, and residue, $\beta(s)$, and the central inelastic double spectral function, $v(s, t)$, we write down equations that the amplitude must satisfy. No attempt is made to calculate α and β from v, in the way that the Regge parameters are calculable in a potential theory. In a relativistic theory, where one does not have the constraint implied by the existence of a given potential, one would not expect $\alpha(s)$ and $\beta(s)$ to be determined uniquely by $v(s, t)$. Nevertheless, elastic unitarity may serve to restrict the allowed functions, $\alpha(s)$ and $\beta(s)$, once $v(s, t)$ is given. Our treatment of the equations as a nonlinear mapping indicates that this may be true, since it turns out that unitarity is not satisfied by fixed points of the system, unless a certain subsidiary condition is satisfied, and this connects $\alpha(s)$ and $\beta(s)$ to $v(s, t)$. Indeed, a more thorough-going semi-dynamical treatment [5], involving the inelastic N/D equations and based on the representation of this paper, in which unitarity is built in from the start, allows one to specify only $v(s, t)$: the Regge pole then arises "dynamically" as a zero of $D_l(s)$, with a calculable residue. However, as usual one has the *CDD* ambiguity, which is not removed by the requirement of l-plane meromorphy.

Perhaps the most interesting consequence of our treatment, and actually the reason that we considered l-plane analyticity in the context of the crossing-symmetric, unitary equations in the first place, is the fact that we can demonstrate the boundedness of the physical partial-wave amplitudes, despite the bad behaviour of the integrand in the Froissart-Gribov representation. Another result, perhaps more surprising, is that we do not need any l-plane cuts, even though we have full crossing symmetry. Thus the limited inelasticity implied by elastic unitarity plus crossing symmetry does not seem to require the existence of cuts.

The general appearance of this paper is rather different from that of Refs. [1] and [2], since we are less interested here in the details of setting up a Banach space in which a fixed-point theorem can be applied, but rather in the resolution of the formal difficulties involved in writing a Sommerfeld-Watson representation when one wants to make a continuation in both Mandelstam variables. In the usual Regge theory, one replaces the partial-wave expansion

$$A(s, t) = \sum_{l=0}^{\infty} (2l+1) P_l^{ev}(z) A_l(s), \tag{1.1}$$

where

$$P_l^{ev}(z) = \tfrac{1}{2}[P_l(z) + P_l(-z)], \tag{1.2}$$

by the Sommerfeld-Watson form,

$$A(s, t) = \frac{i}{2}\int_\Gamma dl \frac{2l+1}{\sin \pi l} P_l^{ev}\left(-1 - \frac{2t}{s-4}\right) A_l(s) - \frac{\pi \bar{\beta}(s)}{\sin \pi\alpha(s)} P_{\alpha(s)}^{ev}\left(-1 - \frac{2t}{s-4}\right) \tag{1.3}$$

where

$$\Gamma = \{l : l = -\tfrac{1}{2} + \varepsilon + iy, 0 < \varepsilon < \tfrac{1}{2}, -\infty < y < \infty\}, \tag{1.4}$$

where $\alpha(s)$ is the position of the Regge pole, and where

$$\bar{\beta}(s) = [2\alpha(s) + 1]\beta(s), \tag{1.5}$$

$\beta(s)$ being the residue at the Regge pole. It should be noted that $\alpha(4)$ may not be greater than two, and that the amplitude has no pole at the point at which $\alpha(s) = 1$, because of the signature factor. However, if $\alpha(4) > 0$, we must kill the ghost pole that would occur at $s = s_0$, where $\alpha(s_0) = 0$. This is easily done by requiring that $\beta(s_0) = 0$. The representation (1.3) effects the continuation of (1.1) into the entire cut t-plane. It is however quite unsuited for direct continuation into the complex s-plane away from the real half-line $s > 4$. It is the primary purpose of Section 2 to show how we can combine unitarity and the Sommerfeld-Watson trick to find a representation that allows a simultaneous continuation in both variables.

We obtain eventually a Mandelstam representation in which all subtraction and single-spectral functions are explicitly given in terms of the Regge parameters. We note the preliminary work of Kupsch [6], in which a Khuri representation is used instead of a Regge representation. The difficulties with analyticity are less severe when one deals with Khuri poles; but unitarity is harder to implement.

In Section 3, we study the explicit continuation of the Froissart-Gribov integral, which only converges if $\mathrm{Re}\, l > \mathrm{Re}\, \alpha(s)$, to the left, since we need to be able to calculate $A_l(s)$ for $l \in \Gamma$. In order to do this, we must deform a certain contour of integration, and it is this same trick that eventually allows us to demonstrate the boundedness of $A_l(s)$ for $s \to \infty$, although $\alpha(s)$ may move as far to the right as one wishes.

In Section 4 and the appendix, we show how the equations in Section 2 and 3 can be used to define a nonlinear mapping, and we demonstrate the existence of a fixed point in a Banach space that differs only slightly from that of Ref. [2]. We observe that, although unitarity has been used in deriving the equations, the solution only satisfies unitarity if a subsidiary equation is satisfied, and this effectively forces $\alpha(s)$ and $\beta(s)$ to be functions of $v(s, t)$.

Earlier attempts to show the compatibility of Regge poles with the Mandelstam representation have been made by Chew, Jones, and Khuri, among others (Ref. [7]). We note that the Chew-Jones representation is closest in spirit to the representation of this paper: the advance we have made is to have produced a form with crossing symmetry and the correct curved support for the Mandelstam double spectral function. Moreover, we have a representation for the background term that is valid for all real or complex values of s and t; and we have shown that the expected Regge behaviour holds as one Mandelstam variable tends to infinity, while another is held fixed.

2. Crossing-symmetric Regge Representation

We have seen in the introduction that the usual Sommerfeld-Watson transformation provides a satisfactory way of making the analytic continuation in the momentum-transfer variable, t; but that it is inadequate to provide the continuation in s. Instead of trying directly to continue the background integral in s, we shall imagine writing a fixed-t dispersion relation for the amplitude:

$$A(s, t) = \frac{1}{\pi} \int_4^\infty ds' \left(\frac{1}{s'-s} + \frac{1}{s'-u} \right) A_s(s', t) \tag{2.1}$$

and then we can write a Sommerfeld-Watson representation for $A_s(s', t)$ rather than for $A(s, t)$. In this way we need only consider real values of s', the continuation in s being made explicit through (2.1).

The difficulties associated with this approach are as follows: in the first place (2.1) will in general need subtractions; and moreover, although su crossing symmetry is automatic, st crossing is not (in this connection, note that even if we ensure Bose symmetry for $A_s(s', t)$, this does not imply Bose symmetry for $A(s, t)$). In the second place, it is no longer immediate that if the background integral for $A_s(s', t)$ is "well-behaved", then $A(s, t)$ has the required Regge asymptotic behaviour, because of the smearing effect of the s'-integral in (2.1). We solve the first difficulty by writing in fact a dispersion integral only for what we shall call the "elastic" part of the amplitude, namely

$$A^{\mathrm{el}}(s,t)=\frac{1}{\pi}\int_4^\infty ds'\left(\frac{1}{s'-s}+\frac{1}{s'-u}\right)A_s^{\mathrm{el}}(s',t), \tag{2.2}$$

where

$$A_s^{\mathrm{el}}(s',t)=\sum_{l=0}^{\infty}(2l+1)P_l^{\mathrm{ev}}\left(1+\frac{2t}{s'-4}\right)q_{s'}|A_l(s')|^2, \tag{2.3}$$

with $q_s=\left(\frac{s-4}{s}\right)^{\frac{1}{2}}$. In fact, for the purposes of the fixed-point proof of Section 4, we shall redefine q_s, and hence the elastic part of the amplitude, by introducing a cut-off. However, the redefined q_s will still be exactly equal to $((s-4)/s)^{\frac{1}{2}}$ in the elastic region; and we can temporarily forget about the cut-off, until we need it in Section 4. The partial wave projections of A_s^{el} are just the imaginary parts of the partial wave amplitudes in the elastic region, because of the elastic unitarity condition. However, (2.3) remains well-defined for $s'>16$, although A_s^{el} is no longer the imaginary part of the amplitude, due to competition from inelastic channels. An advantage in considering A^{el} separately is that we can find amplitudes for which (2.2) needs no subtractions, even though the dispersion relation (2.1) for the full amplitude would need subtractions if Regge poles are present. We shall construct a suitable "inelastic" part of the amplitude, A^{in}, such that

$$A(s,t)=A^{\mathrm{el}}(s,t)+A^{\mathrm{in}}(s,t) \tag{2.4}$$

is fully crossing symmetric. In this way we solve the first problem mentioned above, at the expense of making the question of Regge behaviour more difficult to handle; we need to show not only that $A^{\mathrm{el}}(s,t)$ has Regge behaviour, but also that $A^{\mathrm{in}}(s,t)$ does not spoil this result. Fortunately we can solve this problem by an elegant method of contour deformation and demonstrate that Regge behaviour is indeed observed.

We now replace (2.3) by a Sommerfeld-Watson transform (we drop the prime on s'):

$$\begin{aligned}A_s^{\mathrm{el}}(s,t)=\frac{iq_s}{2}\int_\Gamma dl(2l+1)P_l^{\mathrm{ev}}\left(-1-\frac{2t}{s-4}\right)\frac{A_l(s_+)A_l(s_-)}{\sin\pi l}\\ -\pi q_s\left\{\frac{\bar\beta(s_+)P^{\mathrm{ev}}_{\alpha(s_+)}\left(-1-\frac{2t}{s-4}\right)}{\sin\pi\alpha(s_+)}A_{\alpha(s_+)}(s_-)\right.\\ \left.+\frac{\bar\beta(s_-)P^{\mathrm{ev}}_{\alpha(s_-)}\left(-1-\frac{2t}{s-4}\right)}{\sin\pi\alpha(s_-)}A_{\alpha(s_-)}(s_+)\right\},\end{aligned} \tag{2.5}$$

for those values of s such that $\mathrm{Re}\,\alpha(s) > -\frac{1}{2}+\varepsilon$. We may write unitarity for general l in the form

$$\frac{A_l(s_+)-A_l(s_-)}{2i} = q_s A_l(s_+)A_l(s_-)+I_l(s)\,, \tag{2.6}$$

where $I_l(s)$ is the inelastic overlap function, which must be zero for $4 \leqq s < 16$. Now $A_l(s_+)$ has a pole at $l=\alpha(s_+)$ and $A_l(s_-)$ has one at $l=\alpha(s_-)$, so if we make the assumption that $I_l(s)$ has no singularities at these points, then it follows easily from (2.6) that

$$A_{\alpha(s_-)}(s_+) = \frac{i}{2q_s} = -A_{\alpha(s_+)}(s_-)\,. \tag{2.7}$$

In fact we shall manage to construct amplitudes for which $I_l(s)$ has no singularities for $\mathrm{Re}\,l > -\frac{1}{2}+\varepsilon$ at any value of s.

We may use (2.7) to simplify the residue terms in (2.5); and moreover we may write the following dispersion relation for the Legendre function under the background integral:

$$\frac{P_l^{\mathrm{ev}}\left(-1-\dfrac{2t}{s-4}\right)}{\sin\pi l} = -\frac{1}{2\pi}\int_0^\infty dt'\left(\frac{1}{t'-t}+\frac{1}{t'-u}\right)P_l\left(1+\frac{2t'}{s-4}\right). \tag{2.8}$$

For the amplitude which we shall eventually construct, we can justify interchanging the order of the l-integration in (2.5) and the t'-integration in (2.8). Moreover, we shall be able to demonstrate the existence of the following bound:

$$|A_l(s)| \leqq \kappa[z_0+(z_0^2-1)^{\frac{1}{2}}]^{-\mathrm{Re}\,l} \tag{2.9}$$

for $\mathrm{Re}\,l$ larger than, say,

$$1+\sup_{4\leqq s<\infty}\{\mathrm{Re}\,\alpha(s)\}\,, \tag{2.10}$$

where κ is a constant; and where

$$z_0 = 1+8/(s-4)\,. \tag{2.11}$$

The bound (2.9) is proved in fact from the Froissart-Gribov representation. By considering the asymptotic behaviour of $P_l\left(1+\dfrac{2t'}{s-4}\right)$ as $\mathrm{Re}\,l \to +\infty$, we can easily show that the Γ-contour may be closed to the right in the complex l-plane, under the t'-integral, for those values of t' for which

$$z'+(z'^2-1)^{\frac{1}{2}} < \{z_0+(z_0^2-1)^{\frac{1}{2}}\}^2\,, \tag{2.12}$$

where

$$z' = 1+2t'/(s-4)\,. \tag{2.13}$$

For these values of t' we can evaluate the background integral by picking up $-2\pi i$ times the residues of the poles at $l=\alpha(s_+)$ and $l=\alpha(s_-)$. We may write then

$$A_s^{\mathrm{el}}(s,t) = B_s^{\mathrm{el}}(s,t)+R_s^{\mathrm{el}}(s,t)\,, \tag{2.14}$$

where B_s^{el} is the contribution from that part of the t'-integral for which (2.23) does *not* hold, namely

$$t' \geqq 16s/(s-4)\,, \tag{2.15}$$

and where R_s^{el} contains the remaining terms. We write explicitly

$$B_s^{\mathrm{el}}(s,t)=\frac{1}{\pi}\int\limits_{16s/(s-4)}^{\infty} dt'\left(\frac{1}{t'-t}+\frac{1}{t'-u}\right)\varrho_B^{\mathrm{el}}(s,t') \tag{2.16}$$

where

$$\varrho_B^{\mathrm{el}}(s,t)=\frac{q_s}{4i}\int\limits_{\Gamma} dl(2l+1)P_l\left(1+\frac{2t}{s-4}\right)A_l(s_+)A_l(s_-)\,, \tag{2.17}$$

and

$$\begin{aligned} R_s^{\mathrm{el}}(s,t)=\frac{i}{2}\Bigg\{&\bar{\beta}(s_+)\left[\frac{\pi}{\sin\pi\alpha(s_+)}P^{\mathrm{ev}}_{\alpha(s_+)}\left(-1-\frac{2t}{s-4}\right)\right.\\ &\left.+\frac{1}{2}\int\limits_0^{16s/(s-4)} dt'\left(\frac{1}{t'-t}+\frac{1}{t'-t}\right)P_{\alpha(s_+)}\left(1+\frac{2t'}{s-4}\right)\right]\\ &-\bar{\beta}(s_-)\left[\frac{\pi}{\sin\pi\alpha(s_-)}P^{\mathrm{ev}}_{\alpha(s_-)}\left(-1-\frac{2t}{s-4}\right)\right.\\ &\left.+\frac{1}{2}\int\limits_0^{16s/(s-4)} dt'\left(\frac{1}{t'-t}+\frac{1}{t'-u}\right)P_{\alpha(s_-)}\left(1+\frac{2t'}{s-4}\right)\right]\Bigg\}\,. \end{aligned} \tag{2.18}$$

It is important to note that (2.16) needs no subtractions, since ϱ_B^{el} has the behaviour $t'^{\varepsilon-\frac{1}{2}}$ as $t'\to\infty$, but that we could not write a similar unsubtracted relation for R_s^{el}, since this has the Regge behaviour $t^{\alpha(s)}$ for large t. However, we have the representation (2.18), which shows explicitly that R_s^{el} is analytic in the t-plane, with cuts $[16s/(s-4), \infty)$ in the variables t und u.

We may rewrite (2.2) in the form

$$A^{\mathrm{el}}(s,t)=\pi^{-1}\int\limits_4^{\infty} ds'\left(\frac{1}{s'-s}+\frac{1}{s'-u}\right)[B_s^{\mathrm{el}}(s',t)+\theta(\Lambda-s')R_s^{\mathrm{el}}(s',t)] \tag{2.19}$$

where we have supposed that $\mathrm{Re}\,\alpha(s')<-\frac{1}{2}+\varepsilon$ for $s'>\Lambda$, so that the pole term no longer contributes for these values of s'. We have now to see how to add a term A^{in}, as in (2.4), in such a way that the total amplitude has full crossing symmetry. Let us write A as a sum of background and Regge contributions

$$A(s,t)=B(s,t)+R(s,t)\,. \tag{2.20}$$

In order to make B crossing symmetric, we have clearly to symmetrize the double spectral function that occurs in (2.16). We thus obtain an unsubtracted Mandelstam representation for the background contribution:

$$B(s,t)=\bar{B}(s,t)+\bar{B}(t,u)+\bar{B}(u,s)\,, \tag{2.21}$$

where

$$\bar{B}(s,t)=\pi^{-2}\int_4^\infty ds'\int_4^\infty dt'\,\frac{\varrho_B(s',t')}{(s'-s)(t'-t)},\tag{2.22}$$

with

$$\varrho_B(s,t)=\theta\left(t-\frac{16s}{s-4}\right)\varrho_B^{\mathrm{el}}(s,t)+\theta\left(s-\frac{16t}{t-4}\right)\varrho_B^{\mathrm{el}}(t,s)+\theta(s-16)\,\theta(t-16)\,v(s,t)\,.\tag{2.23}$$

Here ϱ_B^{el} was defined in (2.17), and v may be any suitable function that is symmetrical under interchange of its arguments. It represents part of the inelastic effect that arises from other than two pions in the intermediate state in *both* s- and t-channels.

It is not possible to write an unsubtracted Mandelstam representation for the Regge term; but we must nevertheless succeed in symmetrizing the double discontinuity of the final expression. We write

$$R(s,t)=\bar{R}(s,t)+\bar{R}(t,u)+\bar{R}(u,s)\,,\tag{2.24}$$

where

$$\bar{R}(s,t)=\hat{R}(s,t)+\hat{R}(t,s)\,,\tag{2.25}$$

with

$$\hat{R}(s,t)=\pi^{-1}\int_4^A\frac{ds'}{s'-s}\,\hat{R}_s(s',t)\,,\tag{2.26}$$

in which $\hat{R}_s$ is the part of R_s^{el} with the t-channel cut, namely

$$\begin{aligned}\hat{R}_s(s,t)=\frac{i}{4}\Bigg\{&\bar{\beta}(s_+)\left[\frac{\pi}{\sin\pi\alpha(s_+)}P_{\alpha(s_+)}\left(-1-\frac{2t}{s-4}\right)\right.\\&\left.+\int_0^{16s/(s-4)}\frac{dt'}{t'-t}P_{\alpha(s_+)}\left(1+\frac{2t'}{s-4}\right)\right]\\&-\bar{\beta}(s_-)\left[\frac{\pi}{\sin\pi\alpha(s_-)}P_{\alpha(s_-)}\left(-1-\frac{2t}{s-4}\right)\right.\\&\left.+\int_0^{16s/(s-4)}\frac{dt'}{t'-t}P_{\alpha(s_-)}\left(1+\frac{2t'}{s-4}\right)\right]\Bigg\}.\end{aligned}\tag{2.27}$$

In the following sections we shall establish the existence of functions with the representation given in (2.20)–(2.27).

For partial-wave amplitudes satisfying the above equations, we shall demonstrate that the corresponding scattering amplitude, $A(s,t)$, exhibits full crossing symmetry as well as Regge asymptotic behaviour in all channels. It is rather remarkable that Regge behaviour is not spoiled by the extra terms that are required by crossing symmetry, and we have not seen such a representation in the literature.

3. Froissart-Gribov Representation for $\mathrm{Re}\, l > \varepsilon - \frac{1}{2}$

We have succeeded in finding a representation for the total amplitude that contains no unknown subtraction or single spectral-function terms. Since we wish to have $A_l(s)$ meromorphic in the right-half l-plane (excluding, in particular, Kronecker delta terms), we must still see how to continue the Froissart-Gribov integral,

$$A_l(s) = \frac{4}{\pi(s-4)} \int_4^\infty dt\, Q_l\left(1 + \frac{2t}{s-4}\right) A_t(s,t), \tag{3.1}$$

down to low values of l. We may read off A_t from the representation (2.20) et seq.:

$$A_t(s,t) = B_t(s,t) + R_t(s,t), \tag{3.2}$$

where

$$B_t(s,t) = \frac{1}{\pi} \int_4^\infty ds' \left(\frac{1}{s'-s} + \frac{1}{s'-u}\right) \varrho_B(s',t), \tag{3.3}$$

and where

$$R_t(s,t) = \hat{R}_t(s,t) + \hat{R}_t(u,t) + \tilde{R}_t(s,t) + \tilde{R}_t(u,t), \tag{3.4}$$

with

$$\hat{R}_t(s,t) = \frac{1}{4i} \theta\left(t - \frac{16\Lambda}{\Lambda - 4}\right) \int_{4t/(t-16)}^{\Lambda} \frac{ds'}{s'-s} \left[\bar{\beta}(s'_+) P_{\alpha(s'_+)}\left(1 + \frac{2t}{s'-4}\right) - \bar{\beta}(s'_-) P_{\alpha(s'_-)}\left(1 + \frac{2t}{s'-4}\right)\right] \tag{3.5}$$

and

$$\tilde{R}_t(s,t) = \theta(\Lambda - t) \hat{R}_s(t,s), \tag{3.6}$$

where $\hat{R}_s$ was defined in (2.27). The contributions of B_t and of the crossed terms, $\tilde{R}_t$, to (3.1) can be continued without change to all l-values such that $\mathrm{Re}\, l > -\frac{1}{2} + \varepsilon$. Indeed, the crossed terms only involve finite t-integrals in (3.1), so there is no question of divergence. However, the contributions of $\hat{R}_t(s,t)$ and $\hat{R}_t(u,t)$ give convergent t-integrals only if $\mathrm{Re}\, l$ is large enough. Our task in this section is to make explicit the analytic continuation of these terms to the whole of the l-plane to the right of the contour Γ.

As a preparatory step, we must study

$$\frac{\pi}{2} \bar{\beta}(s) P_{\alpha(s)}(1 + 2t/(s-4)) \tag{3.7}$$

as a function of the complex variable s, for t fixed, real, and positive. We shall require that $\alpha(s)$ be analytic, with a cut $4 \leq s < \infty$, since this is what is found in potential theory (if no Regge trajectories cross). However, $A_l(s)$ has a kinematic cut running backwards from $s = 4$, which can be removed by dividing $A_l(s)$ by $(s-4)^l$: this is a well-known result that one obtains by studying the Q_l function in (3.1). Hence we are led to suppose that

$$\tilde{\beta}(s)=(s-4)^{-\alpha(s)}\bar{\beta}(s)=(s-4)^{-\alpha(s)}[2\alpha(s)+1]\,\beta(s) \tag{3.8}$$

is free from the kinematical cut, and is in fact also analytic in the s-plane, cut $[4,\infty)$. Clearly $\bar{\beta}(s)$ has not only this cut, but also a cut $(-\infty,4]$ that originates from the factor $(s-4)^{\alpha(s)}$. Moreover, at fixed l and t, the function

$$P_l(1+2t/(s-4))$$

is analytic in the s-plane, with a cut $4-t<s<4$. It is clear then that the Function (3.7) will in general have a cut along the entire real axis from $-\infty$ to $+\infty$.

Let us define the discontinuity across the real axis,

$$\begin{aligned}&\Delta\frac{\pi}{2}\bar{\beta}(s)\,P_{\alpha(s)}\left(1+\frac{2t}{s-4}\right)\\&\equiv\frac{\pi}{4i}\left\{\bar{\beta}(s_+)\,P_{\alpha(s_+)}\left(1+\frac{2t}{s_+-4}\right)-\bar{\beta}(s_-)\,P_{\alpha(s_-)}\left(1+\frac{2t}{s_--4}\right)\right\},\end{aligned} \tag{3.9}$$

with t real and positive. For $s>4$, the discontinuity arises because of the cuts of α and $\bar{\beta}$, and we may drop the suffices $\pm$ in the arguments of the Legendre functions. However, for $s<4$ these suffices are important, although they may now be dropped in the argument of α, since this has no cut for $s<4$. There is still the kinematical cut of $\bar{\beta}$; and it is perhaps clearer to work with $\tilde{\beta}$ instead of $\bar{\beta}$ in this region:

$$\bar{\beta}(s_\pm)=e^{\pm i\pi\alpha(s)}(4-s)^{\alpha(s)}\,\tilde{\beta}(s)\,, \tag{3.10}$$

for $s<4$.

Consider now the section of the real s-axis defined by $4-t<s<4$. Here the argument of the Legendre function in (3.7) lies between -1 and $-\infty$. In this region one can use the standard formula

$$P_l\left(1+\frac{2t}{s_\pm-4}\right)=e^{\mp i\pi l}\,P_l\left(-1-\frac{2t}{s-4}\right)-\frac{2}{\pi}\sin\pi l\,Q_l\left(-1-\frac{2t}{s-4}\right) \tag{3.11}$$

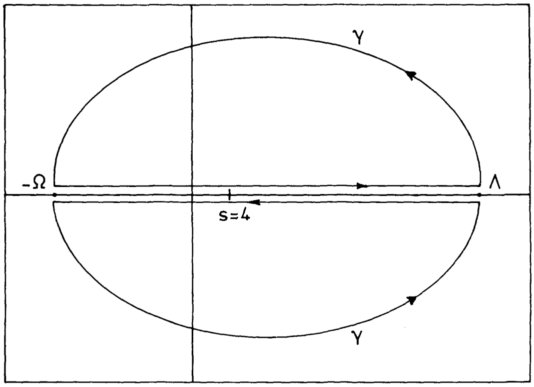
to show that

$$\Delta\frac{\pi}{2}\bar{\beta}(s)\,P_{\alpha(s)}\left(1+\frac{2t}{s-4}\right)=-\tilde{\beta}(s)\sin^2\pi\alpha(s)\cdot(4-s)^{\alpha(s)}\,Q_{\alpha(s)}\left(-1-\frac{2t}{s-4}\right), \tag{3.12}$$

when $4-t<s<4$, where we have used (3.10). For $s<4-t$, the argument of the Legendre function lies between -1 and $+1$, and there is no cut here. The only discontinuity comes from the kinematical cut of $\bar{\beta}$, so we find that

$$\Delta\frac{\pi}{2}\bar{\beta}(s)\,P_{\alpha(s)}\left(1+\frac{2t}{s-4}\right)=\frac{\pi}{2}\beta(s)\sin\pi\alpha(s)\cdot(4-s)^{\alpha(s)}\,P_{\alpha(s)}\left(1+\frac{2t}{s-4}\right), \tag{3.13}$$

for $s<4-t$.



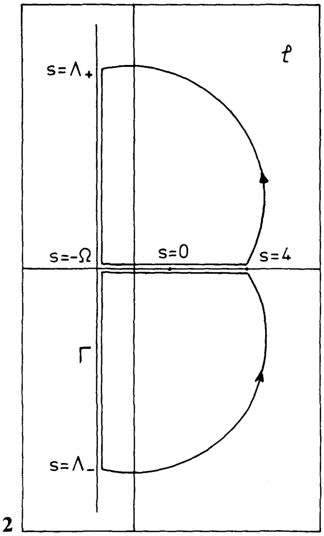


Fig. 1. The integration contour of Equation (3.14) in the variable s'. $\operatorname{Re}\alpha(s) = -\frac{1}{2} + \varepsilon$ on the complex part of the contour

Fig. 2. The image of the contour of Figure 1 in the variable $l = \alpha(s)$

We shall now write a Cauchy integral for the analytic Function (3.7), and for definiteness we set $s = s_+ \equiv x + i\eta$, $4 < x < \Lambda$, $\eta > 0$:

$$\frac{\pi}{2} \bar{\beta}(s_+) P_{\alpha(s_+)}\left(1 + \frac{2t}{s-4}\right) = \frac{1}{4i} \oint \frac{ds'}{s' - s_+} \bar{\beta}(s') P_{\alpha(s')}\left(1 + \frac{2t}{s'-4}\right), \tag{3.14}$$

where the integration contour consists of two pieces, as shown in Figure 1. We have already supposed that $\operatorname{Re}\alpha(\Lambda) = -\frac{1}{2} + \varepsilon$, and we now further assume that a positive number, Ω, exists such that also $\alpha(-\Omega) = -\frac{1}{2} + \varepsilon$. Moreover, we require that

$$\operatorname{Re}\alpha(s) < -\tfrac{1}{2} + \varepsilon\,, \tag{3.15}$$

for $s < -\Omega$ and for $s > \Lambda$. In other words, we suppose that the Regge pole is to the right of the background contour, Γ, only for $-\Omega < s < \Lambda$. The complex parts of the contour in Figure 1, which we designate by γ, are defined by the requirement $s' \in \gamma \Rightarrow \operatorname{Re}\alpha(s') = -\frac{1}{2} + \varepsilon$. In fact the integration contour in the s'-plane is an image of a contour in the plane of $l' = \alpha(s')$, as shown in Figure 2. The contour follows the Regge trajectory $\alpha(s_\pm)$ for real s when the pole is to the right of Γ and it is closed along a section of Γ itself. Notice that the complex part of the contour in the s'-plane is uniquely defined if the function $\alpha(s)$ is schlicht; but in fact we do not need to impose this restriction, but only that there does exist a bounded, continuous contour from $s = \Lambda$ to $s = -\Omega$ on which $\operatorname{Re}\alpha(s) = -\frac{1}{2} + \varepsilon$. This contour need not be unique, but in the event of bifurcations we may for definiteness always choose the contour that encloses the least area. The analytic and asymptotic properties that we impose on $\alpha(s)$ are in fact enough to guarantee the existence of such a contour.

We have now completed our study of the Function (3.7), and we can use the Cauchy integral (3.14) to rewrite (3.5) in the form

$$\hat{R}_l(s_+,t)=\theta(\Lambda-s)\frac{\pi}{2}\bar{\beta}(s_+)P_{\alpha(s_+)}\left(1+\frac{2t}{s-4}\right)-\hat{R}_l'(s_+,t)\,, \tag{3.16}$$

where

$$\hat{R}_l'(s_+,t)=\tfrac{1}{2}\int\limits_{-\Omega}^{4t/(t-16)}\frac{ds'}{s'-s_+}\Delta\bar{\beta}(s')P_{\alpha(s')}\left(1+\frac{2t}{s'-4}\right)$$
$$+\frac{1}{4i}\int\limits_{\gamma}\frac{ds'}{s'-s_+}\bar{\beta}(s')P_{\alpha(s')}\left(1+\frac{2t}{s'-4}\right), \tag{3.17}$$

where the sense of the integration direction on the two pieces of the complex contour γ is that given in Figure 1. The purpose of the above manipulations should be clear: we have isolated the Regge behaviour in the first term on the right-hand side of (3.16), and in fact we shall show later that the expression (3.17) can be constrained to be of order $t^{-\frac{1}{2}+\varepsilon}$ for large t. Hence these terms cause no trouble in the Froissart-Gribov integral. On the other hand, for $\mathrm{Re}\,l>\mathrm{Re}\,\alpha$, we have the standard formula

$$\int\limits_{z_0}^{\infty}dz\,Q_l(z)P_\alpha(z)=\frac{\Xi_l(z_0;\alpha)}{[l-\alpha]\,[l+\alpha+1]}, \tag{3.18}$$

where

$$\Xi_l(z_0;\alpha)=(z_0^2-1)\,[Q_l(z_0)P_\alpha'(z_0)-Q_l'(z_0)P_\alpha(z_0)]\,. \tag{3.19}$$

The right-hand side of (3.18) is well-defined in the entire l-plane, as a meromorphic function, and we have thus found the analytic continuation of the integral formula to the left of the line $\mathrm{Re}\,l=\mathrm{Re}\,\alpha$.

By using the above results, we can replace (3.1) by the following expression, which constitutes the analytic continuation of (3.1) to the line $\mathrm{Re}\,l=-\frac{1}{2}+\varepsilon$:

$$A_l(s_+)=\frac{\bar{\beta}(s_+)\Xi_l\left(1+\dfrac{32\Lambda}{(\Lambda-4)(s-4)};\alpha(s_+)\right)}{[l-\alpha(s_+)]\,[l+\alpha(s_+)+1]}\theta(\Lambda-s)$$
$$+\frac{2}{s-4}\theta\left(\Omega+4-\frac{16\Lambda}{\Lambda-4}-s\right)\int\limits_{16\Lambda/(\Lambda-4)}^{4-s+\Omega}dt\,Q_l\left(1+\frac{2t}{s-4}\right)$$
$$\cdot\bar{\beta}(4-s_+-t)P_{\alpha(4-s-t)}\left(1-\frac{2t}{s+t}\right)$$
$$+\frac{4}{\pi(s-4)}\int\limits_4^{\infty}dt\,Q_l\left(1+\frac{2t}{s-4}\right)[B_l(s_+,t)-\hat{R}_l'(s_+,t)-\hat{R}_l'(4-s_+-t,t)$$
$$+\tilde{R}_l(s_+,t)+\tilde{R}_l(4-s_+-t,t)]\,. \tag{3.20}$$

The second term here is part of the contribution from $\hat{R}_t(u, t)$: it comes from the expression corresponding to the first term on the right-hand side of (3.16), and it gives no trouble in (3.20), since the t-integral is never infinite.

4. Existence of Iterative Solutions

In this section we shall show that the equations of the previous two sections can be used to define a contraction mapping, if α, β, and v are specified functions which satisfy suitable conditions. This implies the existence of a solution, and moreover one that can be obtained by a convergent iterative process. The contractive existence proof involves essentially the same techniques as those described in Refs. [1] and [2], as we shall show in this section.

The sequence of equations that we use to specify the mapping,

$$A_l'(s) = P[A; s, l], \tag{4.1}$$

is defined by the following: (3.20), (3.3), (2.23), (2.17), together with the formulae that define the various functions in these equations. The next step then is to define a suitable Banach space, but before we do that, one remark is in order: although we have used the unitarity Equation (2.6) in deriving the representation (3.20), this does not guarantee that a fixed point of the mapping (4.1) will satisfy (2.7). Consequently, the amplitude would generally not satisfy elastic unitarity for $4 \leqq s \leqq 16$, and moreover the discontinuities of $B_s^{\mathrm{el}}(s', t)$ and $R_s^{\mathrm{el}}(s', t)$ might not cancel, so that $A^{\mathrm{el}}(s, t)$, defined by (2.19), would have a logarithmic infinity at $s = \Lambda$. This is unwanted, and in fact such an infinity would spoil the fixed-point proof. We shall, for the purposes of this paper, make the artificial assumption, $\beta(\Lambda) = 0$, i.e. that the residue function vanishes at the point at which the Regge pole crosses the background contour. One might hope that this ad hoc restriction could be removed by some modification of the mapping; and in fact the difficulty does not occur in the N/D treatment of Ref. [5], since unitarity is explicit at each stage there. However, in the N/D approach one only has an implicit definition of the Regge poles, and one does not know in advance of a numerical computation whether there are any poles in the right-hand half of the l-plane. Once we have made the constraint, $\beta(\Lambda) = 0$, the actual fixed-point proof is only a slight generalization of that of Refs. [1] and [2]. Indeed, since the Regge pole term, i.e. the first term on the right-hand side of (3.20), now vanishes at $s = \Lambda$, the complete amplitude, $A(l, s)$, can be kept bounded for all $s \geqq 4$ and all l on the background contour Furthermore the mapping, (4.1), differs from those of Refs. [1] and [2] only to the extent that Regge terms have been added to the inhomogeneous part of the mapping, $V(s, l)$. Therefore we can employ almost the same Banach space as that used in Ref. [2], the only difference being that one must allow for the large s behaviour of the Regge terms, as was anticipated in Section 5 of Ref. [2]. We give this generalized norm in the appendix: in particular an asymptotic behaviour at large s of the type s^λ is apparently allowed, where $\lambda > 0$. However, this unbounded behaviour does not occur for physical l-values: we show in the appendix that the physical partial waves remain bounded, as required by inelastic unitarity.

There is, however, one complication because of the new norm. We need to replace the phase-space factor q_s in Equations (2.3), (2.5), (2.6), (2.7) and (2.17) by

$$q_s = h(s)\left(\frac{s-4}{s}\right)^{\frac{1}{2}}, \tag{4.2}$$

where $h(s)$ is a Chew-Frautschi cut-off, as in Ref. [2], which we may choose to satisfy the constraints

$$h(s)=1, \qquad 4 \leqq s \leqq 16; \tag{4.3a}$$

$$0 \leqq h(s) \leqq 1, \qquad 16 \leqq s \leqq N; \tag{4.3b}$$

$$h(s)=0, \qquad N \leqq s; \tag{4.3c}$$

$$|h(s_1)-h(s_2)| \leqq |s_1-s_2|^{\mu}, \qquad 4 \leqq s_2 < s_1 \leqq N. \tag{4.3d}$$

This cut-off effectively civilizes the large s behaviour of the elastic part of the amplitude. Physically, one may argue that such a cut-off is reasonable, in order to prevent "double counting" in the interior part of the double spectral function, which would presumably occur if v were absent in (2.23) and if there were no cut-off. Of course, since we do not envisage setting $v=0$, the introduction of the cut-off merely implies a redefinition of $v(s,t)$, and a re-identification of $I_l(s)$ in (2.6) as a pseudo-overlap function. The true overlap function is then

$$I_l(s) - \left(\frac{s-4}{s}\right)^{\frac{1}{2}} [1-h(s)]\, A_l(s_+) A_l(s_-). \tag{4.4}$$

As already mentioned, these solutions will not in general satisfy (2.7), and so the elastic unitarity condition will not in general be satisfied in the region $4 \leqq s \leqq 16$. However, for fixed α and β, the central inelastic double spectral function, $v(s,t)$, can be varied to give different solutions of (4.1); and so we can consider the fixed-point of (4.1) to be an implicit function of v. Thus one might hope that a suitable choice for $v(s,t)$, at least for some α and β, could be made such that the corresponding contribution to the Froissart-Gribov representation, $V(s,l)$, is still in the contractive domain of the mapping, and such that the corresponding fixed point of the mapping satisfies

$$A(v; l=\alpha(s_\pm), s_\mp) = \frac{\pm 1}{2iq(s)}, \tag{4.5}$$

where we have made the dependence of A on v explicit. This then would ensure that the amplitude satisfies elastic unitarity in the elastic region.

Acknowledgements. It is a pleasure to thank R. L. Warnock for enlightening correspondence and for his continued interest in the progress of this work. M. Kaekebeke wishes to express his gratitude to the members of the Physics Department of I.I.T. for their hospitality during the summer of 1975. P. W. Johnson has carried out this work as a scientific member of the Stichting F.O.M. (Foundation for Fundamental Research on Matter), which is financially supported by the Z.W.O. (Netherlands Organization for Pure Scientific Research). J. S. Frederiksen wishes to thank Rothman's University Endowment Fund for the receipt of a fellowship. D. Atkinson would like to thank Prof. K. J. le Couteur for the hospitality extended to him at the Australian National University, where some of this work was done.

Appendix A

We work in a Banach space of doubly Hölder-continuous functions, $f(s,y)$, specified by the following norm

$$\begin{aligned}\|f\| &= \sup\{s_2^{-\lambda}|y_2+i|^{\frac{1}{2}+\nu}|f(s_2,y_2)|\}\\
&+\sup\left\{s_2^{-\lambda}|y_2+i|^{\frac{1}{2}+\nu}\frac{|f(s_2,y_1)-f(s_2,y_2)|}{\left|\frac{y_1-y_2}{y_1+i}\right|^{\varrho}}\right\}\\
&+\sup\left\{s_1^{-\lambda}|y_2+i|^{\frac{1}{2}+\nu}\frac{|f(s_1,y_1)-f(s_2,y_1)-f(s_1,y_2)+f(s_2,y_2)|}{\left|\frac{s_1-s_2}{s_1}\right|^{\mu}\left|\frac{y_1-y_2}{y_1+i}\right|^{\varrho}}\right\}\\
&+\sup\left\{s_2^{-\lambda}\left(\frac{s-4}{s}\right)|y_2+i|^{-\frac{1}{2}+\nu}|f_s(s_2,y_2)|\right\}\\
&+\sup\left\{s_2^{-\lambda}\left(\frac{s-4}{s}\right)|y_2+i|^{-\frac{1}{2}+\nu}\frac{|f_s(s_2,y_1)-f_s(s_2,y_2)|}{\left|\frac{y_1-y_2}{y_1+i}\right|^{\varrho}}\right\}\\
&+\sup\left\{s_1^{-\lambda}\left(\frac{s-4}{s}\right)^{1+\mu}|y_2+i|^{-\frac{1}{2}+\nu}\right.\\
&\left.\cdot\frac{|f_s(s_1,y_1)-f_s(s_2,y_1)-f_s(s_1,y_2)+f_s(s_2,y_2)|}{\left|\frac{s_1-s_2}{s_1}\right|^{\mu}\left|\frac{y_1-y_2}{y_1+i}\right|^{\varrho}}\right\},\end{aligned}$$

where f_s is the partial derivative of f with respect to s, and where the suprema have to be taken over all $s_1>s_2\geqq 4$ and $|y_1|>|y_2|$, y_1 and y_2 real. The constraints upon the indices μ, ν, and ϱ are those of Ref. [2], namely

$$0<\mu<\min(\varepsilon,\tfrac{1}{8}), \tag{A.1}$$

$$0<\varrho<\tfrac{1}{16}, \tag{A.2}$$

$$\tfrac{3}{4}+\tfrac{\mu}{2}<\nu<1-\mu-\varrho, \tag{A.3}$$

where $0<\varepsilon<\frac{1}{2}$ determines the position of the background contour, $l=-\frac{1}{2}+\varepsilon+iy$, $-\infty<y<\infty$. The index λ is chosen such that

$$\lambda>\sup\operatorname{Re}\alpha(s) \tag{A.4}$$

for reasons that we shall explain in a moment. Notice that we need the single Hölder differences in the variable y in the norm, just because λ is positive, but that we have dispensed with single Hölder differences in s, since these are implied by the double Hölder term. As mentioned in Section 3, we assume $\alpha(s)$ and $\tilde{\beta}(s)$, where

$$\tilde{\beta}(s)\equiv\beta(s)\,(s-4)^{-\alpha(s)} \tag{A.5}$$

to be real-analytic functions with a branch point at $s=4$ and a cut running from $s=4$ to $s=+\infty$.

Now we notice that the Regge terms involve only finite integrations with respect to the arguments of α and β, even in the case where a contour deformation has been performed. Thus we do not need to make assumptions about the asymptotic behaviours as $s\to\infty$, except that $\alpha(s)$ is such that $\mathrm{Re}\,\alpha(s)\leqq -\frac{1}{2}+\varepsilon$ for $s\geqq\Lambda$. It is necessary to impose only the following conditions, that have to do with Hölder continuity, threshold behaviour, and the artificial restrictions at the point $s=\Lambda$, to which we alluded in Section 4:

$$|\alpha^{(n)}(s_1)-\alpha^{(n)}(s_2)|<\kappa|s_1-s_2|^{\mu+\delta}(s_1-4)^{-n}, \qquad 4\leqq s_1\leqq s_2\leqq\Lambda\,; \tag{A.6}$$

$$|\beta^{(n)}(s_1)-\beta^{(n)}(s_2)|<\kappa|s_1-s_2|^{\mu+\delta}(s_1-4)^{-n}, \qquad 4\leqq s_1\leqq s_2\leqq\Lambda\,; \tag{A.7}$$

$$|\beta(s)|\leqq(s-4)^p, \qquad p>2\alpha(4)+\tfrac{1}{2}-\varepsilon+\mu+\delta\,; \tag{A.8}$$

$$|\beta^{(n)}(s)|\leqq\kappa|\alpha(s)-\alpha(\Lambda)|^{3-n}(\Lambda-s)^{\delta}, \tag{A.9}$$

where κ is a generic constant that may change from line to line, where δ is a small positive number, and where $\alpha^{(n)}(s)$ and $\beta^{(n)}(s)$, $n=0, 1, 2$, are the position and residue of the Regge pole and their first and second derivatives with respect to s. It readily follows, by the techniques of Refs. [1] and [2], that the Regge terms are elements of the Banach space specified by the norm given above. Note that Theorem III of Ref. [2] contains an error. On the right-hand side of Equation (B.11) of Ref. [2], the first term should be multiplied by $((s_2-4)/s_2)^{\mu}$ and the second by $((s_2-4)/s_2)^{\mu+\alpha}$.

To conclude this appendix, we shall consider in detail the t-channel Regge pole contribution to the Froissart-Gribov integral. In the final form (3.20), the only terms in the integrand that tend to infinity as $s\to\infty$, for fixed t, are those involving $\tilde{R}_t(s_+,t)$ and $\tilde{R}_t(4-s_+-t,t)$. It would appear that the expression

$$\frac{4}{\pi(s-4)}\int_4^{\Lambda} dt\, Q_l\left(1+\frac{2t}{s-4}\right) \cdot\frac{i\pi}{2}\left\{\frac{\bar{\beta}(t_+)}{\sin\pi\alpha(t_+)}P^{\mathrm{ev}}_{\alpha(t_+)}\left(-1-\frac{2s_+}{t-4}\right)-\frac{\bar{\beta}(t_-)}{\sin\pi\alpha(t_-)}P^{\mathrm{ev}}_{\alpha(t_-)}\left(-1-\frac{2s_+}{t-4}\right)\right\} \tag{A.10}$$

might behave like $s^{\alpha_{\max}-1}$, where $\alpha_{\max}=\sup\mathrm{Re}\,\alpha(s)$, and this is the reason why we have the inequality (A.4): the Banach space contains functions that explode like s^{λ}, but this causes no difficulty with the fixed point proof, because of the cut-off. Nevertheless, we must show that, for physical l, the term (A.10) is actually bounded as $s\to\infty$. This can be done by distorting the t-integration contour, in the complex t-plane, in precisely the same way that we distorted the s'-integration contour of (3.5) (see Fig. 1). There is no Cauchy pole in the present case, and so we can replace (A.10) by a corresponding integral around the rest of the contour of Figure 1. It will however be clearer if we write the two components of P^{ev}_{α} separately:

$$P^{\mathrm{ev}}_{\alpha(t)}\left(-1-\frac{2s_+}{t-4}\right)=\frac{1}{2}\left\{P_{\alpha(t)}\left(-1-\frac{2s_+}{t-4}\right)+P_{\alpha(t)}\left(1+\frac{2s}{t-4}\right)\right\}, \tag{A.11}$$

where we need the suffix on s_+ (which means $s+i\varepsilon$, $\varepsilon\downarrow 0$) only in the first component in (A.11), when $t\geqq 4$, as it is in (A.10). We use a formula analogous to (3.11) to re-express (A.11) as

$$P^{\mathrm{ev}}_{\alpha(t)}\left(-1-\frac{2s_+}{t-4}\right)$$
$$=\frac{1}{2}\left[1+e^{-i\pi\alpha(t)}\right]P_{\alpha(t)}\left(1+\frac{2s}{t-4}\right)-\frac{1}{\pi}\sin\pi\alpha(t)Q_{\alpha(t)}\left(1+\frac{2s}{t-4}\right), \tag{A.12}$$

still for $t>4$. Hence (A.10) is equal to

$$-\frac{2i}{s-4}\int_{\gamma}dt\,Q_l\left(1+\frac{2t}{s-4}\right)\frac{\bar{\beta}(t)}{\sin\pi\alpha(t)}P^{\mathrm{ev}}_{\alpha(t)}\left(-1-\frac{2s}{t-4}\right)$$
$$+\frac{4}{s-4}\int_{-\Omega}^{4}dt\,\Delta\left\{Q_l\left(1+\frac{2t}{s-4}\right)\frac{\bar{\beta}(t)}{\sin\pi\alpha(t)}P^{\mathrm{ev}}_{\alpha(t)}\left(-1-\frac{2s}{t-4}\right)\right\}, \tag{A.13}$$

where Δ means the discontinuity across the real t-axis, divided by $2i$, and where we shall use the identification (A.12).

It is clear that the contribution from the complex contour, γ, cannot be worse than $s^{\varepsilon-\frac{3}{2}}$ as $s\to\infty$, since $\mathrm{Re}\,\alpha(t)=-\frac{1}{2}+\varepsilon$ for $t\in\gamma$. We may concentrate then upon the integral from $-\Omega$ to 4, and so we must calculate the various discontinuities. In this region, $\alpha(t)$ has no cut, and that of $\bar{\beta}(t)$ comes solely from the kinematical factor, $(t-4)^{\alpha(t)}$, so that

$$\bar{\beta}(t_\pm)=e^{\pm i\alpha(t)}(4-t)^{\alpha(t)}\tilde{\beta}(t). \tag{A.14}$$

The function $P_{\alpha(t)}\left(1+\frac{2s}{t-4}\right)$ on the right-hand side of (A.12) has a cut $4-s<t<4$, and since we are interested here only in $s\to\infty$, we need consider only $s>\Omega+4$, in which case the above cut extends from $t=4$ backwards to beyond $t=-\Omega$, and we have

$$P_{\alpha(t)}\left(1+\frac{2s}{t_\pm-4}\right)$$
$$=e^{\mp i\alpha(t)}P_{\alpha(t)}\left(-1-\frac{2s}{t-4}\right)-2\pi^{-1}\sin\pi\alpha(t)Q_{\alpha(t)}\left(-1-\frac{2s}{t-4}\right). \tag{A.15}$$

The function $Q_{\alpha(t)}\left(1+\frac{2s}{t-4}\right)$ in (A.12) has a cut $-\infty<t<4$, and if $s>\Omega+4$, then the argument is less than -1 for $-\Omega<t<4$, and here

$$Q_{\alpha(t)}\left(1+\frac{2s}{t_\pm-4}\right)=-e^{\pm i\pi\alpha(t)}Q_{\alpha(t)}\left(-1-\frac{2s}{t-4}\right). \tag{A.16}$$

Finally, the function $Q_l\left(1+\frac{2t}{s-4}\right)$ in (A.13) has a cut $-\infty<t<0$, and in this case the argument is between -1 and $+1$ for $-\Omega<t<0$, so that

$$Q_l\left(1+\frac{2t_\pm}{s-4}\right)=\frac{\pi}{2\sin\pi l}\left\{e^{\mp i\pi l}P_l\left(1+\frac{2t}{s-4}\right)-P_l\left(-1-\frac{2t}{s-4}\right)\right\}. \tag{A.17}$$

Now we must combine (A.14)–(A.17) together to evaluate the integral from $-\Omega$ to 4 in (A.13). For $0<t<4$, $Q_l\left(1+\frac{2t}{s-4}\right)$ has no discontinuity, and so this piece of the integral is equal to

$$\frac{4}{\pi}\frac{1}{s-4}\int_0^4 dt\,Q_l\left(1+\frac{2t}{s-4}\right)(4-t)^{\alpha(t)}\beta(t)\left(e^{i\alpha(t)}-1\right)\sin\pi\alpha(t)\,Q_{\alpha(t)}\left(-1-\frac{2s}{t-4}\right), \tag{A.18}$$

which has an asymptotic behaviour $s^{-\alpha(0)-2}<s^{-\varepsilon-\frac{3}{2}}$, to within logarithms, and so is completely unimportant. The remainder of the integral, between $-\Omega$, and 0, is more complicated because here $Q_l\left(1+\frac{2t}{s-4}\right)$ has a discontinuity, as can be seen from (A.17). We find this piece of the integral to be equal to

$$\frac{4}{s-4}\int_{-\Omega}^{0} dt(4-t)^{\alpha(t)}\tilde{\beta}(t)\left[X_l(s,t)\,Q_{\alpha(t)}\left(-1-\frac{2s}{t-4}\right)-Y_l(s,t)\,P_{\alpha(t)}\left(-1-\frac{2s}{t-4}\right)\right], \tag{A.19}$$

where

$$X_l(s,t)=\pi^{-1}\sin\pi\alpha(t)\left(e^{i\pi\alpha(t)}-1\right)\operatorname{Re}Q_l\left(1+\frac{2t}{s-4}\right)$$
$$+\frac{1}{2}\left[1-\cos\pi\alpha(t)\left(e^{i\pi\alpha(t)}-1\right)\right]P_l\left(1+\frac{2t}{s-4}\right), \tag{A.20}$$

$$Y_l(s,t)=-\frac{\pi}{4\sin\pi\alpha(t)}\left(1+e^{-i\pi\alpha(t)}\right)P_l\left(1+\frac{2t}{s-4}\right). \tag{A.21}$$

The dominant asymptotic behaviour is given by the contribution from the P_α term, which yields $s^{\alpha(0)-1}$, as we promised. Thus we need only $\alpha(0)\leqq 1$ in order to keep the partial-wave amplitudes bounded, the extreme value $\alpha(0)=1$ being allowed.

We see then that the partial-wave amplitude is actually bounded as $s\to\infty$, if $\alpha(0)\leqq 1$, despite the fact that the norm tolerates a behaviour s^λ. In this conncetion, it should be noted that it is not a good idea to make the contour distortion to obtain (A.11) for complex l, for when z is not real and greater than unity, $Q_l(z)$ blows up exponentially as $|l|\to\infty$, either in the upper or the lower half-plane, but not in the special case of real, positive l. This behaviour would ruin the fixed point proof; and this is the reason that we make no distortion of the t-contour in (A.10) for the proof: we treat $A_l(s)$ as if a behaviour $s^{\alpha_{\max}-1}$ were possible. Only for physical (or more generally for real) l do we have a bound $s^{\alpha(0)-1}$ that is uniform with respect to l.

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Communicated by W. Hunziker

Received June 1, 1976