

Probability Estimates for Continuous Spin Systems

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Abstract. Probability estimates for classical systems of particles with superstable interactions [1] are extended to continuous spin systems.

1. Notation and Assumptions

On a lattice \mathbb{Z}^v we consider continuous d -dimensional spins. A *spin configuration* in $A \subset \mathbb{Z}^v$ is thus a function $s_A: A \rightarrow \mathbb{R}^d$; its value at $x \in A$ will be denoted by s_x .

If $x = (x^1, \dots, x^v) \in \mathbb{Z}^v$, we write $|x| = \max_i |x^i|$. If $s = (s^1, \dots, s^d) \in \mathbb{R}^d$, we write $|s| = \left(\sum_i (s^i)^2 \right)^{1/2} = \sqrt{s^2}$.

A measure $\mu \geq 0$ on \mathbb{R}^d is given such that

$$\int \mu(ds) e^{-\alpha s^2} < +\infty$$

if $\alpha > 0$, and μ is not identically 0.

We shall call *interaction* a real function U on all configurations in all finite $A \subset \mathbb{Z}^v$ satisfying the following conditions.

(a) U is $\otimes^A \mu$ -measurable on $(\mathbb{R}^d)^A$ and invariant under translations of \mathbb{Z}^v .

(b) *Superstability*. There exist $A > 0$, $C \in \mathbb{R}$ such that if $s_A \in (\mathbb{R}^d)^A$ is a configuration on any finite A , then

$$U(s_A) \geq \sum_{x \in A} [A s_x^2 - C].$$

(c) *Regularity*. There exists a decreasing positive function Ψ on the natural integers such that

$$\sum_{x \in \mathbb{Z}^v} \Psi(|x|) < +\infty.$$

Furthermore if A_1, A_2 are disjoint finite subsets of \mathbb{Z}^v and s_{A_1}, s_{A_2} the restrictions to A_1, A_2 of a configuration $s_{A_1 \cup A_2}$ on $A_1 \cup A_2$, then

$$|W(s_{A_1 \cup A_2})| \leq \sum_{x \in A_1} \sum_{y \in A_2} \Psi(|y - x|) \frac{1}{2} (s_x^2 + s_y^2)$$

where we have written

$$U(s_{A_1 \cup A_2}) = U(s_{A_1}) + U(s_{A_2}) + W(s_{A_1}, s_{A_2}).$$

Condition (c) implies the following

(d) There are $r > 0$ and $\lambda > 0$ such that for all finite $A \subset \mathbb{Z}^v$

$$\int_{\Sigma^A} \left(\prod_{x \in A} \mu(ds_x) \right) \exp[-U(s_A)] > \lambda^{-\text{card } A}$$

where $\Sigma = \{s \in \mathbb{R}^d : |s| \leq r\}$. This is because, using (c), we have

$$U(s_A) \leq \sum_{x \in A} U(s_x) + \left(\sum_{x \in A} s_x^2 \right) \sum_y \Psi(|y|)$$

and, for sufficiently large r , $\int_{|s| \leq r} \mu(ds) > 0$.

Notice also that if there are $\varepsilon > 0$, $B \in \mathbb{R}$ such that

$$U(s_A) \geq \sum_{x \in A} [(A + \varepsilon)s_x^2 - B|s_x|]$$

then (b) holds with $C = B/4\varepsilon$.

2. Probability Estimates

Let $\Delta \subset A \subset \mathbb{Z}^v$, Δ finite. We denote by s_Δ the restriction to Δ of a configuration s_A on A , and write

$$\varrho_\Delta^{(A)}(s_\Delta) = Z_A^{-1} \int \left(\prod_{x \in A/\Delta} \mu(ds_x) \right) \exp[-U(s_A)] \tag{1}$$

where

$$Z_A = \int \left(\prod_{x \in A} \mu(ds_x) \right) \exp[-U(s_A)].$$

The probability estimates of this section are bounds on $\varrho_\Delta^{(A)}$, given in Theorem 2.2. below. To obtain these bound we imitate the arguments of [1]. That paper in effect treats a special case of the problem considered here, where $d=1$ and μ is carried by the natural integers. In [1], the probability estimates are obtained on the basis of technical results, which carry over immediately to the present case if the variable n is allowed to vary in \mathbb{R}^d rather than take natural integer values. As an example we transcribe below (Proposition 2.1) the main technical estimate of [1].

Given $\alpha > 0$, we can choose an integer $P_0 > 0$ and for each $j \geq P_0$ an integer $l_j > 0$ such that

$$|l_{j+1}/l_j - (1 + 2\alpha)| < \alpha.$$

We use the notation

$$[j] = \{x \in \mathbb{Z}^v : |x| \leq l_j\}, \quad V_j = (2l_j + 1)^v$$

2.1. Proposition. *Let $\varepsilon > 0$ and $C \geq 0$ be given, and let Ψ be a decreasing positive function on the natural integers such that*

$$\sum_{x \in \mathbb{Z}^v} \Psi(|x|) < +\infty .$$

If α is sufficiently small one can choose an increasing sequence (ψ_j) such that $\psi_j \geq 1$, $\psi_j \rightarrow \infty$, and fix $P > P_0$ so that the following is true.

Let $n(\cdot)$ be a function from \mathbb{Z} to the reals ≥ 0 . Suppose that there exists q such that $q \geq P$ and q is the largest integer for which

$$\sum_{x \in [q]} n(x)^2 \geq \psi_q V_q .$$

Then

$$\sum_{x \in [q+1]} C + \sum_{x \in [q+1]} \sum_{y \notin [q+1]} \Psi(|y-x|)^{\frac{1}{2}} (n(x)^2 + n(y)^2) \leq \varepsilon \sum_{x \in [q+1]} n(x)^2 .$$

This differs from Proposition 2.1 of [1] mostly by the fact that $n(\cdot)$ has real rather than integer values. Lemmas 2.2, 2.3, 2.4, and Proposition 2.5 of [1] similarly carry over to the present case.

To adapt Proposition 2.6 of [1] to $\varrho_A^{(A)}$ some care is needed because we do not have in general $\varrho(\{0\}) > 0$. Since however we have (d) and the regularity condition (c) (rather than only lower regularity in [1]), we can write $\varrho_A^{(A)}(s_A) = \varrho' + \varrho''$ where (3.30) and (3.31) of [1] are replaced (see Appendix) by

$$\varrho' \leq C' \exp \left[\sum_{y \in \mathbb{Z}^v} \Psi(|y| - A) s_x^2 \right] \cdot \varrho_{A \setminus \{x\}}^{(A)}(s_{A \setminus \{x\}}) \tag{2}$$

$$\varrho'' \leq \sum_{q \geq P} e^{-C'' \psi_{q+1} V_{q+1} + D'' V_{q+1}} \cdot \exp \sum_{x \in [q+1] \cap A} [-(A-3\varepsilon) s_x^2] \cdot \varrho_{A \setminus [q+1]}^{(A)}(s_{A \setminus [q+1]}) \tag{3}$$

with some constants C', C'', D'' . Therefore, by induction on card A ,

$$\varrho_A^{(A)}(s_A) \leq \exp \sum_{x \in A} (E s_x^2 + F) \tag{4}$$

with some constants E, F .

We show now, following Proposition 2.7 of [1], that for any $\varepsilon > 0$ one can choose δ independent of $(A), \Delta, s_A$ such that

$$\varrho_A^{(A)}(s_A) \leq \exp \sum_{x \in A} [-(A-3\varepsilon) s_x^2 + \delta] . \tag{5}$$

We may assume $A > 3\varepsilon$. Let $\delta = (E + A - 3\varepsilon) \psi_P V_P + F$. If $|s_x| \leq (\psi_P V_P)^{1/2}$ for each $x \in A$, then (5) follows from (4). If $|s_x| > (\psi_P V_P)^{1/2}$ for some x , we put x at the origin by a translation. Then $\varrho' = 0$, and $\varrho_A^{(A)}(s_A) = \varrho''$ so that, using (3) and induction,

$$\begin{aligned} \varrho_A^{(A)}(s_A) &\leq \exp \sum_{x \in A} [-(A-3\varepsilon) s_x^2] \\ &\sum_{q \geq P} e^{-C'' \psi_{q+1} V_{q+1} + D'' V_{q+1}} \rho^{\delta \text{card}(A \setminus [q+1])} \\ &\leq \exp \sum_{x \in A} [-(A-3\varepsilon) s_x^2] \cdot e^{\delta \text{card}(A \setminus [q+1]) + F} \end{aligned}$$

and (4) follows. We have proved the following

2.2. Theorem. Let $\varrho_A^{(A)}(s_A)$ be defined by (1) for an interaction U satisfying (a), (b), (c). Given $A^* < A$, there exists δ independent of Λ, Δ, s_A such that

$$\varrho_A^{(A)}(s_A) \leq \exp \sum_{x \in \Delta} [-A^* s_x^2 + \delta].$$

2.3. Corollary. Let $\gamma \geq 2$, and suppose that the superstability condition is strengthened to

$$U(s_A) \geq \sum_{x \in \Lambda} [A|s_x|^\gamma - C].$$

Then the conclusion of Theorem 2.2 can be strengthened to

$$\varrho_A^{(A)}(s_A) \leq \exp \sum_{x \in \Delta} [-A^* |s_x|^\gamma + \delta]$$

Define $F: \mathbb{R}^d \rightarrow \mathbb{R}^d$ by

$$Fs = \begin{cases} s & \text{if } |s| \leq 1 \\ (|s|^{2/\gamma-1})s & \text{if } |s| \geq 1 \end{cases}$$

and write $F(s_x)_{x \in \Lambda} = (Fs_x)_{x \in \Lambda}$.

Let $\tilde{\mu}$ be the image by F of the measure μ , and let $\tilde{U}(s_A) = U(Fs_A)$. Then \tilde{U} is an interaction satisfying the conditions of Section 1 with respect to the measure $\tilde{\mu}$. In particular

$$\begin{aligned} \tilde{U}(s_A) &= U(Fs_A) \geq \sum_{x \in \Lambda} [A|Fs_x|^\gamma - C] \\ &\geq \sum_{x \in \Lambda} [As_x^2 - A - C] \end{aligned}$$

and

$$\begin{aligned} |\tilde{W}(s_{A_1 \cup A_2})| &\leq \sum_{x \in A_1} \sum_{y \in A_2} \Psi(|y-x|) \frac{1}{2} (|Fs_x|^2 + |Fs_y|^2) \\ &\leq \sum_{x \in A_1} \sum_{y \in A_2} \Psi(|y-x|) \frac{1}{2} (s_x^2 + s_y^2). \end{aligned}$$

Therefore

$$\begin{aligned} \varrho_A^{(A)}(s_A) &= \tilde{\varrho}_A^{(A)}(F^{-1}s_A) \leq \exp \sum_{x \in \Delta} [-A^* |F^{-1}s_x|^2 + \delta] \\ &\leq \exp \sum_{x \in \Delta} [-A^* |s_x|^\gamma + \delta]. \end{aligned}$$

2.4. Corollary. Suppose that

$$U(s_A) = \tilde{U}(s_A) + \sum_{x \in \Lambda} V(s_x)$$

and that \tilde{U} is an interaction satisfying the conditions of Section 1 with respect to the measure $\tilde{\mu} = e^{-V} \mu$. Then Theorem 2.2 can be replaced by

$$\varrho_A^{(A)}(s_A) \leq \exp \sum_{x \in \Delta} [-A^* |s_x|^\gamma + \delta - V(s_x)].$$

This is because

$$\varrho_A^{(A)}(s_A) = \exp \left[- \sum_{x \in \Delta} V(s_x) \right] \tilde{\varrho}_A^{(A)}(s_A)$$

where $\tilde{\varrho}$ is defined by (1) with μ, U replaced by $\tilde{\mu}, \tilde{U}$.

Appendix

We sketch here the proofs of (2) and (3), using notation which is either that of [1], or has obvious meaning.

Proof of (2).

$$\begin{aligned}
 Q' &= Z_A^{-1} \int_R \mu^{A \setminus A} (ds_{A \setminus A}) \exp[-U(s_x) - U(s_{A \setminus \{x\}}) - W(s_x, s_{A \setminus \{x\}})] \\
 &\leq e^{-U(s_x)} Z_A^{-1} \int_R \mu^{A \setminus A} (ds_{A \setminus A}) \exp[-U(s_{A \setminus \{x\}}) - W(s'_x, s_{A \setminus \{x\}})] \\
 &\quad \cdot \exp\left[\left(\frac{1}{2} \sum_y \Psi(|y|)\right) (s_x^2 + s_x'^2) + 2D'\right] \\
 &\leq \lambda e^{2D'} \exp\left[-As_x^2 + C + \left(\frac{1}{2} \sum_y \Psi(|y|)\right) s_x^2\right] \\
 &\quad \cdot \sup_{s'_x \in \Sigma} \exp\left[\left(\frac{1}{2} \sum_y \Psi(|y|)\right) s_x'^2\right] \\
 &\quad \cdot Z_A^{-1} \int_{\Sigma} \mu(ds'_x) \int_R \mu^{A \setminus A} (ds_{A \setminus A}) \exp[-U(s_x^*)] \\
 &\leq C' \exp\left[\left(\sum_y \Psi(|y|) - A\right) s_x^2\right] \cdot Q_{A \setminus \{x\}}^{(A)}(s_{A \setminus \{x\}}).
 \end{aligned}$$

Proof of (3).

$$\begin{aligned}
 Q'' &= \sum_{q \geq P} Z_A^{-1} \int_{R_q} \mu^{A \setminus A} (ds_{A \setminus A}) \exp(-U(s_{[q+1] \cap A})) \\
 &\quad \cdot \exp(-W(s_{[q+1] \cap A}, s_{A \cap [q+1]}) \exp(-U(s_{A \setminus [q+1]})) \\
 &\leq \sum_{q \geq P} Z_A^{-1} \int_{R_q} \mu^{A \setminus A} (ds_{A \setminus A}) \exp \sum_{x \in [q+1] \cap A} [-As_x^2 + C] \\
 &\quad \cdot \exp \sum_{x \in [q+1] \cap A} \sum_{y \in A \setminus [q+1]} \Psi(|y-x|) \frac{1}{2} (s_x^2 + s_y^2) \\
 &\quad \cdot \exp \sum_{x \in [q+1] \cap A} \sum_{y \in A \setminus [q+1]} \Psi(|y-x|) \frac{1}{2} (s_x^2 + s_y^2) \\
 &\quad \cdot \exp[-W(s'_{[q+1] \cap A}, s_{A \setminus [q+1]}) - U(s_{A \setminus [q+1]})] \\
 &\leq \sum_{q \geq P} Z_A^{-1} \int_{R_q} \mu^{A \setminus A} (ds_{A \setminus A}) \\
 &\quad \cdot \exp\left[-(A-3\varepsilon) \sum_{x \in [q+1] \cap A} s_x^2 - C'' \Psi_{q+1} V_{q+1}\right] \\
 &\quad \cdot \exp\left(\frac{1}{2} \sum_y \Psi(|y|) \sum_{x \in [q+1] \cap A} s_x^2\right) \\
 &\quad \cdot \exp[-W(s'_{[q+1] \cap A}, s_{A \setminus [q+1]}) - U(s_{A \setminus [q+1]})] \\
 &\leq \sum_{q \geq P} \exp \sum_{x \in [q+1] \cap A} [-(A-3\varepsilon)s_x^2] \\
 &\quad \cdot e^{-C'' \psi_{q+1} V_{q+1}} \int \mu(ds) e^{-(A-3\varepsilon)s^2} | [q+1] \cap A \setminus A | \\
 &\quad \cdot \left(\sup_{s' \in \Sigma} \exp\left[\left(\frac{1}{2} \sum_y \Psi(|y|)\right) s'^2\right]\right)^{|[q+1] \cap A|} \lambda^{|[q+1] \cap A|} \\
 &\quad \cdot Z_A^{-1} \int_{\Sigma^{[q+1] \cap A}} \mu^{[q+1] \cap A} (ds'_{[q+1] \cap A}) \int \mu^{A \setminus [q+1]} (ds_{A \setminus [q+1] \setminus A}) e^{-U(s_x^*)} \\
 &\leq \sum_{q \geq P} \exp \sum_{x \in [q+1] \cap A} [-(A-3\varepsilon)s_x^2] \\
 &\quad \cdot e^{-C'' \psi_{q+1} V_{q+1} + D'' V_{q+1}} Q_{A \setminus [q+1]}^{(A)}(s_{A \setminus [q+1]}).
 \end{aligned}$$

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Reference

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