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# Hyperfunction Quantum Field Theory II. Euclidean Green's Functions

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**Abstract.** The axioms for Euclidean Green's functions are extended to hyperfunction fields without being supplemented by any condition like the linear growth condition of Osterwalder and Schrader.

## § 1. Introduction

In a previous paper [1], which will be quoted as NM I, we have formulated the quantum field theory in terms of Fourier hyperfunctions, successfully in showing that Wightman's axioms for tempered fields can be extended to hyperfunction fields. In particular we have manifested that the support concept of Fourier hyperfunctions allows us to state the locality axiom, in spite of the disadvantage that the test function space for hyperfunctions contains no $C^{\infty}$ functions of compact support.

It is quite natural that questions may arise whether the hyperfunction fields work effectively in the scattering theory, dispersion relations and other provinces of the quantum field theory relating to local singularity structure or momentum space analyticity properties of fields. Another interesting question perhaps concerns the Euclidean formulation of the hyperfunction quantum field theory. In order to get a reconstruction theorem for tempered fields satisfying the usual Wightman axioms, Osterwalder and Schrader [2, 3] were compelled to introduce a technical axiom, what they called the linear growth condition, in formulating the axioms for Euclidean Green's functions. When tempered fields are replaced by hyperfunction fields, which are of a class wider than the former, can the technical axiom such as the linear growth condition be removed completely? In other words, one may ask whether we can formulate a set of axioms for Euclidean Green's functions which contains *neither* the linear growth condition *nor* something else and is *equivalent* to a set of axioms for Wightman hyperfunctions set up in NM I.

The last question is answered affirmatively in the present and a subsequent papers. To this end, however, we find it necessary to make a slight modification of the definition of the Fourier hyperfunctions, while preserving *all* the results obtained in NM I. This will be done in the following two sections. This modification is very slight indeed, but will turn out essential for our purpose. The new Fourier hyperfunction will be called the Fourier hyperfunction of type II in distinction from the old one, the Fourier hyperfunction of type I, we used in NM I. The relation between the both types of Fourier hyperfunctions is that the type I is included in the type II. The presentation is divided into two parts. In the present paper we will be concerned with the quantum field theory of *Fourier hyperfunction fields of mixed type*, a subset of type II fields which contains all of the type I fields. As for the widest class of Fourier hyperfunction fields, *the type* II *fields*, the equivalence of the relativistic and Euclidean quantum field theories will again be attained in a subsequent paper to be published elsewhere.

In the fourth section we formulate Euclidean Green's functions for hyperfunction fields by continuing the time variables to the imaginary axis and study their properties. In the fifth section we set up a set of axioms for Euclidean Green's functions, which contains no analytic properties other than a distribution property. This section is also devoted to the proof of the reconstruction theorem from Euclidean to relativistic theories, thus establishing the equivalence of the two theories for hyperfunction fields.

## §2. Test Functions

Notatins and conventions are the same as in NM I, otherwise specified explicitly. Let us first define

$$U_{1;m}=\{z\in\mathbb{C};\ |\mathrm{Im}\,z|<1/m\}\,, \tag{2.1}$$

$$U_{2;m}=\{z\in\mathbb{C};\ |\mathrm{Im}\,z|<(1+|\mathrm{Re}\,z|)/m\}\,, \tag{2.2}$$

and put $U_{k,l;m}=[U_{2;m}]^k\times[U_{1;m}]^l$ with $l=n-k$, namely

$$U_{k,l;m}=\{(z_1,\dots,z_n)\in\mathbb{C}^n;\ z_i\in U_{2;m},\ 1\leqq i\leqq k\ \text{ and }\ z_j\in U_{1;m},\ k+1\leqq j\leqq n\}\,. \tag{2.3}$$

Let further $\mathcal{O}_c^m(U_{k,l;m})$ be a Banach space of those functions $f(z)$ which are holomorphic in $U_{k,l;m}$, continuous in the closure $\bar{U}_{k,l;m}$ of $U_{k,l;m}$ and satisfy

$$\|f\|_{k,l;m}\equiv\sup_{z\in U_{k,l;m}}|f(z)|e^{|z|/m}<\infty\,. \tag{2.4}$$

$\|f\|_{k,l;m}$ is the norm of the Banach space $\mathcal{O}_c^m(U_{k,l;m})$. The space of *rapidly decreasing holomorphic functions* $\mathscr{P}_{k,l}$ is the inductive limit of the Banach spaces $\{\mathcal{O}_c^m(U_{k,l;m})\}$: $\mathscr{P}_{k,l}=\mathrm{ind}_m\lim\mathcal{O}_c^m(U_{k,l;m})$. $\mathscr{P}_{k,l}$ is a DFS-space. It is seen at once that $\mathscr{P}_{0,n}$ is nothing but $\mathscr{P}_*$ we have defined previously in NM I. We shall sometimes write $\mathscr{P}_{**}=\mathscr{P}_{n,0}$. It is also found that $\mathscr{P}_{n,0}\subset\mathscr{P}_{k,n-k}\subset\mathscr{P}_{0,n}$ for $0<k<n$.

Let $\mathscr{T}^m$ be a Banach space of those entire functions which satisfy the condition

$$|f|_m\equiv\sup_{z\in\mathbb{C}^n}|f(z)|\exp\{|\mathrm{Re}\,z|^4/m-m|\mathrm{Im}\,z|^4\}<\infty\,, \tag{2.5}$$

where

$$|\mathrm{Re}\,z|^4=\sum_{j=1}^{n}|\mathrm{Re}\,z_j|^4\quad\text{and}\quad|\mathrm{Im}\,z|^4=\sum_{j=1}^{n}|\mathrm{Im}\,z_j|^4\,.$$

$|f|_m$ is the norm of the Banach space $\mathscr{T}^m$. The inductive limit of Banach spaces $\{\mathscr{T}^m\}$ is one of the spaces $\mathscr{S}_\alpha^\beta$ of Gel'fand and Shilov having the indices $\alpha=1/4$ and $\beta=3/4$ (see [4], p. 220).

**Lemma 2.1.** *$\mathscr{S}_{1/4}^{3/4}\subset\mathscr{P}_{k,l}$ and the original topology of $\mathscr{S}_{1/4}^{3/4}$ is stronger than that induced by $\mathscr{P}_{k,l}$.*

*Proof.* If $z\in U_{k,l;m}(m\geqq 2)$ we have a series of inequalities

$$\begin{aligned}&|\mathrm{Re}\,z|^4/m-m|\mathrm{Im}\,z|^4\\&\geqq\left(\frac{1}{m}-\frac{1}{m^3}\right)|\mathrm{Re}\,z|^4-\frac{1}{m^3}(k+4|\mathrm{Re}\,z|+6|\mathrm{Re}\,z|^2+4|\mathrm{Re}\,z|^3)\\&\geqq(|\mathrm{Re}\,z|+|\mathrm{Im}\,z|)/m-c\geqq|z|/m-c\end{aligned}$$

for some constant $c$. Hence $\|f\|_{k,l;m}\leqq e^c|f|_m$ for every $f\in\mathscr{T}^m$. Therefore

$$\mathscr{T}^m\subset\mathcal{O}_c^m(U_{k,l;m})$$

and consequently $\mathscr{S}_{1/4}^{3/4}\subset\mathscr{P}_{k,l}$, and the topology of $\mathscr{S}_{1/4}^{3/4}$ induced by $\mathscr{P}_{k,l}$ is weaker than its original topology.

**Proposition 2.2.** *$\mathscr{S}_{1/4}^{3/4}$ is dense in $\mathscr{P}_{k,l}$.*

The proof of this proposition is somewhat lengthy, so that we shall leave it to the appendix. From this proposition follows immediately

**Proposition 2.3.** *$\mathscr{P}_{**}$ is dense in $\mathscr{P}_{k,l}$, in particular in $\mathscr{P}_*$.*

In the Euclidean theory we need some classes of *distributions*, whose test function spaces are related to the spaces $\mathscr{S}_{1,m}(\mathbb{R}^n)$ of $C^\infty$ functions satisfying the condition[1]

$$\|f\|_{m,p}\equiv\sup_{x\in\mathbb{R}^n,\,|l|\leqq p}|D^l f(x)|e^{(1/m)(1-1/p)|x|}<\infty\,, \tag{2.6}$$

where $p=2,3,\dots$. The topology of $\mathscr{S}_{1,m}(\mathbb{R}^n)$ is given by a countable set of norms $\{\|\cdot\|_{m,p}\}_{p=2}^\infty$.

Let $\underline{x}_n=(x_1,\dots,x_n)\in\mathbb{R}^{4n}$ and $x_j=(x_j^0,\boldsymbol{x}_j)\in\mathbb{R}^4$. We introduce subspaces of $\mathscr{S}_{1,m}(\mathbb{R}^{4n})$ as follows:

$$\begin{aligned}\mathscr{C}_0^m(\mathbb{R}^{4n})=\{f\in\mathscr{S}_{1,m}(\mathbb{R}^{4n});\ f(\underline{x}_n)=0\ \text{ if}&\\ |x_i^0-x_j^0|\leqq 1/m\ \text{ for some }\ i\neq j\}&,\end{aligned}\tag{2.7}$$

$$\begin{aligned}\mathscr{C}_<^m(\mathbb{R}^{4n})=\{f\in\mathscr{S}_{1,m}(\mathbb{R}^{4n});\ f(\underline{x}_n)=0\ \text{ unless }\ x_1^0>1/m&\\ \text{and }\ x_{j+1}^0-x_j^0>1/m\ \text{ for }\ 1\leqq j\leqq n-1\}&,\end{aligned}\tag{2.8}$$

$$\mathscr{C}_+^m(\mathbb{R}^{4n})=\{f\in\mathscr{S}_{1,m}(\mathbb{R}^{4n});\ f(\underline{x}_n)=0\ \text{ unless }\ x_j^0>1/m\ \text{ for }\ 1\leqq j\leqq n\}\,. \tag{2.9}$$

Each of these sets equipped with the induced topology of $\mathscr{S}_{1,m}$ is a closed subspace of $\mathscr{S}_{1,m}$. If $m<m'$, then $\mathscr{C}_0^m\subset\mathscr{C}_0^{m'}$, $\mathscr{C}_<^m\subset\mathscr{C}_<^{m'}$ and $\mathscr{C}_+^m\subset\mathscr{C}_+^{m'}$. We denote by $\mathscr{C}_0$, $\mathscr{C}_<$ and $\mathscr{C}_+$ the inductive limit of spaces $\{\mathscr{C}_0^m\}$, $\{\mathscr{C}_<^m\}$ and $\{\mathscr{C}_+^m\}$, respectively.

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[1] We write $\mathscr{S}_{1,m}$ instead of $\mathscr{S}_{1,A}$, $A=m/e$, the latter being the notation used in Gel'fand and Shilov [4] and also in NM I

## §3. Fourier Hyperfunctions of Mixed Type

We let $\mathbb{C}^k \times \mathbb{R}^l$, $k+l=n$, be identified with $\mathbb{R}^{n+k}$ and denote its *compactification* by $\mathbb{D}^{k,l}$, i.e., $\mathbb{D}^{k,l}=\mathbb{R}^{n+k} \sqcup \mathbb{S}_\infty^{n+k-1}$. The topology of $\mathbb{D}^{k,l}$ is given by a way similar to $\mathbb{D}^n=\mathbb{D}^{0,n}$, the compactification of $\mathbb{R}^n$ we have considered in NM I. It is evident that the closure of $\mathbb{R}^n$ in $\mathbb{D}^{k,l}$ is identical with $\mathbb{D}^n$. We write $\mathbb{Q}^{k,l}=\mathbb{D}^{k,l}\times i\mathbb{R}^l$, which is a generalization of $\mathbb{Q}^n=\mathbb{Q}^{0,n}$ we have met in NM I.

*Definition 3.1. (The sheaf of slowly increasing holomorphic functions.)* Let $\Omega$ be an open set in $\mathbb{Q}^{k,l}$. We denote by $\tilde{\mathcal{O}}_{k,l}$ the sheaf determined by a presheaf $\{\tilde{\mathcal{O}}_{k,l}(\Omega)\}$, where $\tilde{\mathcal{O}}_{k,l}(\Omega)$ is the set of all holomorphic functions $(\in \mathcal{O}(\Omega\cap\mathbb{C}^n))$ such that $\sup_{z\in K\cap\mathbb{C}^n} |f(z)|e^{-\varepsilon|z|}<\infty$ for any $\varepsilon>0$ and any compact set $K$ in $\Omega$.

*Definition 3.2. (The sheaf of rapidly decreasing holomorphic functions.)* We denote by $\underset{\sim}{\mathcal{O}}_{k,l}$ the sheaf determined by a presheaf $\{\underset{\sim}{\mathcal{O}}_{k,l}(\Omega)\}$, where $\Omega$ is an open set in $\mathbb{Q}^{k,l}$ and $\underset{\sim}{\mathcal{O}}_{k,l}(\Omega)$ is the set of all holomorphic functions $(\in\mathcal{O}(\Omega\cap\mathbb{C}^n))$ such that for any compact set $K$ in $\Omega$ there exists some positive constant $\delta_K$ and the estimate $\sup_{z\in K\cap\mathbb{C}^n} |f(z)|e^{\delta_K|z|}<\infty$ holds.

*Definition 3.3. (Topology of $\underset{\sim}{\mathcal{O}}_{k,l}(K)$.)* Let $K$ be a compact set in $\mathbb{D}^n$. We give $\underset{\sim}{\mathcal{O}}_{k,l}(K)$ the inductive limit topology $\operatorname{ind}_m\lim \mathcal{O}_c^m(V_m)$, where $\{V_m\}$ is a fundamental system of neighbourhoods of $K$ in $\mathbb{Q}^{k,l}$, satisfying $V_m \Supset V_{m+1}$, and $\mathcal{O}_c^m(V_m)$ is the Banach space of all holomorphic functions $f(z)$ $(\in\mathcal{O}(V_m\cap\mathbb{C}^n))$ that are continuous in $V_m\cap\mathbb{C}^n$ and for which $|f(z)|\leqq Ce^{-|z|/m}$ holds for some constant $C$ (depending on $f$). The norm of $\mathcal{O}_c^m(V_m)$ is defined by $\|f\|_m=\sup_{z\in V_m\cap\mathbb{C}^n}|f(z)|e^{|z|/m}$. With this topology $\underset{\sim}{\mathcal{O}}_{k,l}(K)$ is a DFS-space.

*Remark 1.* We have used the symbol $V_m\Supset V_{m+1}$ to denote that $V_{m+1}$ has a compact neighbourhood in $V_m$ with respect to the topology of $\mathbb{Q}^{k,l}$.

*Remark 2.* For $K=\mathbb{D}^n$ we may construct $V_m$ as given by $U_{k,l;m}\cup C_\infty$, where $C_\infty$ is the point at infinity of $U_{k,l;m}$. Therefore we have $\mathscr{P}_{k,l}=\underset{\sim}{\mathcal{O}}_{k,l}(\mathbb{D}^n)$.

*Remark 3.* The introduction of the new neighbourhood $U_{2;m}$, (2.2), and the replacement of $\mathbb{Q}^n=\mathbb{D}^n\times i\mathbb{R}^n$ by $\mathbb{Q}^{k,l}$ are the *only* essential alteration we have made in comparison with NM I.

*Definition 3.4.* Let $\Omega$ be an open set in $\mathbb{D}^n$. We choose an open set $V$ in $\mathbb{Q}^{k,l}$ which contains $\Omega$ as a relatively closed set and defines $\mathscr{R}_{k,l}(\Omega)$, the space of *Fourier hyperfunctions of mixed type* over $\Omega$, by the cohomology $H_\Omega^n(V,\tilde{\mathcal{O}}_{k,l})$. They are called *Fourier hyperfunctions of type* I *or* II according as $(k,l)=(0,n)$ or $(n,0)$.

**Theorem 3.5.** *A presheaf* $\{\mathscr{R}_{k,l}(\Omega)\}$ *with* $\mathscr{R}_{k,l}(\Omega)=H_\Omega^n(V,\tilde{\mathcal{O}}_{k,l})$ *is a flabby sheaf.*

**Theorem 3.6.** *When* $K$ *is a compact set in* $\mathbb{D}^n$, *we have* $H_K^n(V,\tilde{\mathcal{O}}_{k,l})\cong(\underset{\sim}{\mathcal{O}}_{k,l}(K))'$, *in particular* $\mathscr{R}_{k,l}(\mathbb{D}^n)\cong(\mathscr{P}_{k,l})'$.

The proof of these theorems is akin to that of similar theorems for Fourier hyperfunctions of type I and we have no novel remarks to add particularly (see Kawai [5]). Since *the Fourier transformation of* $\mathscr{P}_{k,l}$ *is a topological isomorphism*, its dual defines the Fourier transformation of $\mathscr{R}_{k,l}(\mathbb{D}^n)$.

*Remark.* It follows from Proposition 2.3 that the space of Fourier hyperfunctions of type I or of mixed type is included in the space of Fourier hyperfunctions of type II.

**Proposition 3.7.** *$\mathscr{P}_{k_1,l_1}\otimes\mathscr{P}_{k_2,l_2}$ is dense in $\mathscr{P}_{k_1+k_2,l_1+l_2}$.*

*Proof.* Mityagin [6] has shown that $\mathscr{S}^{3/4}_{1/4}(\mathbb{R}^{n_1})\otimes\mathscr{S}^{3/4}_{1/4}(\mathbb{R}^{n_2})$ is dense in $\mathscr{S}^{3/4}_{1/4}(\mathbb{R}^{n_1+n_2})$. By Lemma 2.1, $\mathscr{S}^{3/4}_{1/4}(\mathbb{R}^{k_1+l_1})\otimes\mathscr{S}^{3/4}_{1/4}(\mathbb{R}^{k_2+l_2})\subset\mathscr{P}_{k_1,l_1}\otimes\mathscr{P}_{k_2,l_2}$ and the topology of $\mathscr{S}^{3/4}_{1/4}(\mathbb{R}^{k+l})$ induced by $\mathscr{P}_{k,l}$ is weaker than its original topology, and $\mathscr{S}^{3/4}_{1/4}(\mathbb{R}^{k+l})$ is dense in $\mathscr{P}_{k,l}$ by Proposition 2.2. Therefore $\mathscr{P}_{k_1,l_1}\otimes\mathscr{P}_{k_2,l_2}$ is dense in $\mathscr{P}_{k_1+k_2,l_1+l_2}$.

**Lemma 3.8.** *A separately continuous multilinear form $M$ on $\prod_{\nu=1}^{n}\mathscr{P}_{k_\nu,l_\nu}$ is jointly continuous, where $(k_\nu,l_\nu)$ takes on either $(1,0)$ or $(0,1)$ for each $\nu$.*

*Proof.* Since $\mathscr{P}_{k,l}$ for every $k$ and $l$ is a DFS-space, it is a strong dual of a reflexive Fréchet space. We have the lemma by the help of a multilinear version of Theorem 41.1 of Treves [7].

**Proposition 3.9.** *Let $M$ be a continuous multilinear form on $\prod_{\nu=1}^{n}\mathscr{P}_{k_\nu,l_\nu}$. Let further for $z,t\in\mathbb{C}$ and $\varepsilon>0$*

$$h_z^\varepsilon(t)=[2\pi i(t-z)]^{-1}\cosh\varepsilon z/\cosh\varepsilon t \tag{3.1}$$

*and put*

$$\phi^\varepsilon(z_1,\dots,z_n)=M(h^\varepsilon_{z_1},\dots,h^\varepsilon_{z_n})\,. \tag{3.2}$$

*Then $\phi^\varepsilon(z_1,\dots,z_n)$ is an $\varepsilon$-increasing holomorphic function on $(\mathbb{C}-\mathbb{R})^n$, of type* I *or* II *with respect to $z_\nu$ according as $(k_\nu,l_\nu)=(0,1)$ or $(1,0)$. Here by "$\varepsilon$-increasing" it is meant that the function $\phi^\varepsilon(z_1,\dots,z_n)$ has the estimate $|\phi^\varepsilon(z_1,\dots,z_n)|\leqq Ce^{\varepsilon|z|}$.*

*Proof.* For type I variables we consider $U_{1,m}$, given by (2.1), and its complement $U^c_{1;m}=\{z\in\mathbb{C};\,|\mathrm{Im}\,z|>1/m\}$. If $z\in U^c_{1;m'}$, $m>m'>0$ and $m>1/\varepsilon$, we have

$$\sup_{t\in U_{1;m}}|\cosh\varepsilon z/(t-z)\cosh\varepsilon t|\,e^{|t|/m}\leqq C_m e^{\varepsilon|z|}$$

for all $m$. Thus it is found that $h_z^\varepsilon(t)$ belongs to $\mathscr{P}_{0,1}$. A similar estimate holds for type II variables. Therefore we have

$$\begin{aligned}|\phi^\varepsilon(z_1,\dots,z_n)|&=|M(h^\varepsilon_{z_1},\dots,h^\varepsilon_{z_n})|\\&\leqq C\prod_{\nu=1}^{n}\|h^\varepsilon_{z_\nu}\|_{k_\nu,l_\nu;m}\leqq C'e^{\varepsilon|z|}\,.\end{aligned} \tag{3.3}$$

It is easy to see that $\phi^\varepsilon(z_1,\dots,z_n)$ is separately holomorphic and consequently holomorphic by Hartogs' theorem on holomorphy [8]. Thus $\phi^\varepsilon(z_1,\dots,z_n)$ is an $\varepsilon$-increasing holomorphic function on $(\mathbb{C}-\mathbb{R})^n$.

**Proposition 3.10.** *Let $\phi^\varepsilon(z_1,\dots,z_n)$ be as in the preceding proposition. Then for any set of $g_\nu\in\mathscr{O}^m_c(U_{k_\nu,l_\nu;m})$, $m<1/\varepsilon$ and $1\leqq\nu\leqq n$, we have*

$$\int_{\Gamma_1\times\dots\times\Gamma_n}\phi^\varepsilon(z_1,\dots,z_n)g_1(z_1)\dots g_n(z_n)dz_1\dots dz_n=M(g_1,\dots,g_n)\,, \tag{3.4}$$

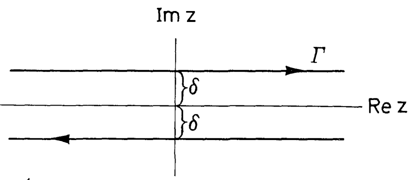


**Fig. 1**

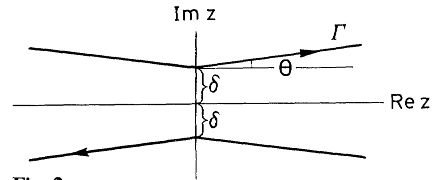


**Fig. 2**

*where $\Gamma_\nu$ is the path in the $z_\nu$ complex plane given in* Figure 1 *or* 2 *according as the $z_\nu$ is a variable of type* I *or* II. *In the figures $\delta < 1/m$ and* $\tan\theta \leqq 1/m$.

This proposition is obviously true by Cauchy's integral formula.

**Proposition 3.11.** *A separately continuous multilinear form $M$ on $\prod_{\nu=1}^{n} \mathscr{P}_{k_\nu, l_\nu}$ uniquely defines an element $F$ of $(\mathscr{P}_{k,l})'$, $k = \sum_\nu k_\nu$ and $l = \sum_\nu l_\nu$, such that $M(g_1, \ldots, g_n) = F(g_1 \times \ldots \times g_n)$ for $g_\nu \in \mathscr{P}_{k_\nu, l_\nu}$, $1 \leqq \nu \leqq n$.*

*Proof.* $M$ can be considered as a continuous multilinear form on $\prod_{\nu=1}^{n} \mathscr{P}_{k_\nu, l_\nu}$. By Proposition 3.9, the form $M$ defines an $\varepsilon$-increasing holomorphic function $\phi^\varepsilon(z_1, \ldots, z_n)$. It is clear that the integral

$$F^\varepsilon(g) \equiv \int_{\Gamma_1 \times \ldots \times \Gamma_n} \phi^\varepsilon(z_1, \ldots, z_n) g(z_1, \ldots, z_n) dz_1 \ldots dz_n \tag{3.5}$$

for every $g \in \mathcal{O}_c^m(U_{k,l;m})$ with $m < 1/\varepsilon$ defines an element of $(\mathcal{O}_c^m(U_{k,l;m}))'$. From Proposition 3.10 follows $M(g_1, \ldots, g_n) = F^\varepsilon(g_1 \times \ldots \times g_n)$. The family $\{F^\varepsilon\}_{\varepsilon > 0}$ determines an element $F$ of $(\mathscr{P}_{k,l})'$, since $\varepsilon$ can be taken as small as one likes. The uniqueness of $F$ is evident from the fact that $\bigotimes_\nu \mathscr{P}_{k_\nu, l_\nu}$ is dense in $\mathscr{P}_{k,l}$.

*Remark.* It is by this proposition that we can develop the quantum field theory in terms of hyperfunctions of type II in a way completely parallel to NM I, where it was formulated by means of hyperfunctions of type I.

**Proposition 3.12.** *If $F(z)$ is an infra-exponential holomorphic function of type* II *defined in $\{z \in \mathbb{C}^n;\ \mathrm{Im}\, z > 0\}$, then there exists a Fourier hyperfunction $\mu$ of type* II *with* $\mathrm{supp}\,\mu \subset \overline{\mathbb{R}^n_+}$ *such that*

$$\mu(\mathscr{F} f) = \int_{\gamma^n} F(z) f(z) dz\,, \qquad f \in \mathscr{P}_{**}\,, \tag{3.6}$$

*where $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n;\ x_j > 0$ for all $j = 1, \ldots, n\}$. $\mathscr{F}$ stands for the Fourier transform and $\gamma$ is the upper branch of the integration path $\Gamma$ of type* II *sketched in* Figure 2.

*Proof.* It is obvious that the integral on the right hand side of (3.6) defines a Fourier hyperfunction of type II. Since the Fourier transformation is an isomorphism in $\mathscr{P}_{**}$, there exists a Fourier hyperfunction $\mu$ of type II satisfying (3.6). We have only to show that the support of $\mu$ is contained in $\overline{\mathbb{R}^n_+}$. Without loss of generality we may assume $n = 1$. Let $h^\varepsilon_\xi(p)$ be a function given by (3.1), then it suffices to demonstrate that $\mu(h_\zeta^\varepsilon(p))$ is analytic in $\mathbb{C}-\overline{\mathbb{R}}_+$. We have

$$\begin{aligned}\mu(h_\zeta^\varepsilon(p)) &= \mu(\mathscr{F}\bar{\mathscr{F}}h_\zeta^\varepsilon(p)) \\ &= (2\pi)^{-1}\int_\gamma (F(z)\int e^{-izp}h_\zeta^\varepsilon(p)dp)dz\,. \end{aligned} \tag{3.7}$$

Now let us define two functions $f(z,\zeta)$ and $g(z,\zeta)$ by

$$f(z,\zeta)=(2\pi)^{-1}\int_{-\delta}^{\infty} e^{-izp}h_\zeta^\varepsilon(p)dp$$

and

$$g(z,\zeta)=(2\pi)^{-1}\int_{-\infty}^{-\delta} e^{-izp}h_\zeta^\varepsilon(p)dp\,,$$

respectively. If $\zeta$ belongs to $\mathbb{C}-[-\delta,\infty)$, the function $f(z,\zeta)\in\mathscr{P}_{1,0}$ and is analytic there with respect to $\zeta$. The function $g(z,\zeta)$ is analytic in $\{\operatorname{Im} z>0,\ \zeta\in\mathbb{C}-\mathbb{R}\}$ and satisfies there the estimate $|g(z,\zeta)|\leqq Ce^{-\delta(\operatorname{Im} z)}$ for some constant $C$ depending only on $\zeta$. Equation (3.7) then is expressed as

$$\begin{aligned}\mu(h_\zeta^\varepsilon(p)) &= \int_\gamma F(z)f(z,\zeta)dz+\int_\gamma F(z)g(z,\zeta)dz \\ &\equiv G(\zeta)+H(\zeta)\,.\end{aligned}$$

$G(\zeta)$ is analytic in $\mathbb{C}-[-\delta,\infty)$. By introducing the path $\gamma^\omega=\{z;z-i\omega\in\gamma\}$ we obtain

$$H(\zeta)=\lim_{\omega\to\infty}\int_{\gamma^\omega} F(z)g(z,\zeta)dz=0\,,$$

because $F(z)$ is infra-exponential and $g(z,\zeta)$ is restrained by the above estimate. Since $\delta$ can be chosen arbitrarily small, we conclude that $\mu(h_\zeta^\varepsilon(p))$ is analytic in $\mathbb{C}-[0,\infty)$. This completes the proof of the proposition.

**Proposition 3.13.** *Let $F_1(z)$ and $F_2(z)$ be two infra-exponential holomorphic functions, as given in the preceding proposition, and suppose that*

$$\int_{\gamma^n} F_1(z)f(z)dz=\int_{\gamma^n} F_2(z)f(z)dz \tag{3.8}$$

*is valid for every $f(z)\in\mathscr{P}_{**}$, then $F_1(z)=F_2(z)$ in $\{z\in\mathbb{C}^n;\operatorname{Im} z>0\}$.*

*Proof.* For simplicity we again assume $n=1$ as above. Let us take a point $\zeta=\zeta_*$ on the upper half of the complex $\zeta$-plane, then we can describe two paths $\gamma^{(1)}$ and $\gamma^{(2)}$ so that $\gamma^{(2)}$ is more distant from the real axis and not less inclined than $\gamma^{(1)}$ and the point $\zeta_*$ is found between the two paths. Consider the integrals

$$G_r^s(\zeta)=\int_{\gamma^{(s)}} F_r(z)h_\zeta^\varepsilon(z)dz\,,\qquad r,s=1 \text{ and } 2\,,$$

then $G_r^s(\zeta)$ are holomorphic at $\zeta=\zeta_*$. It follows from (3.8) that $G_1^1(\zeta_*)=G_2^1(\zeta_*)$ and $G_1^2(\zeta)=G_2^2(\zeta)$ if $\operatorname{Im}\zeta<0$. Since $G_r^2(\zeta)$, $r=1,2$, are holomorphic if $\zeta$ is not on $\gamma^{(2)}$, the equality $G_1^2(\zeta_*)=G_2^2(\zeta_*)$ holds. From $F_r(\zeta_*)=G_r^1(\zeta_*)-G_r^2(\zeta_*)$, $r=1,2$, we get at once $F_1(\zeta_*)=F_2(\zeta_*)$. Thus, since $F_r(z)$ are holomorphic in $\operatorname{Im} z>0$, the required equality holds there.

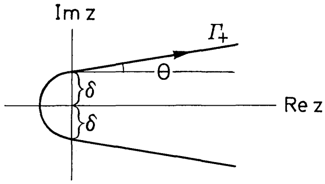


**Fig. 3**

**Proposition 3.14.** *Let $M$ be a separately continuous form on $\underset{\sim}{\mathcal{O}}_{k,0}(\overline{\mathbb{R}_+^k}) \times \underset{\sim}{\mathcal{O}}_{0,l}(\mathbb{D}^l)$, then there uniquely exists a Fourier hyperfunction $F$ of mixed type whose support is contained in $\overline{\mathbb{R}_+^k \times \mathbb{R}^l}$, such that $M(f, g) = F(f \times g)$, where $f \in \underset{\sim}{\mathcal{O}}_{k,0}(\overline{\mathbb{R}_+^k})$ and $g \in \underset{\sim}{\mathcal{O}}_{0,l}(\mathbb{D}^l)$.*

*Proof.* With the function $h_z^\varepsilon(t)$ given by (3.1) we set $\phi^\varepsilon(z_1, \ldots, z_n) = M(h_{z_1}^\varepsilon, \ldots, h_{z_n}^\varepsilon)$. It is not difficult to extend the proof of Proposition 3.9 to show that $\phi^\varepsilon(z_1, \ldots, z_n)$ is an $\varepsilon$-increasing holomorphic function in $(\mathbb{C} - \bar{\mathbb{R}}_+)^k \times (\mathbb{C} - \mathbb{R})^l$. Then, for

$$f \in \underset{\sim}{\mathcal{O}}_{k,l}(\overline{\mathbb{R}_+^k \times \mathbb{R}^l}),$$

a family of formulas

$$F^\varepsilon(f) = \int_{\Gamma_+^k \times \Gamma^l} \phi^\varepsilon(z_1, \ldots, z_n) f(z_1, \ldots, z_n) dz_1 \ldots dz_n \tag{3.9}$$

defines an element of $(\underset{\sim}{\mathcal{O}}_{k,l}(\overline{\mathbb{R}_+^k \times \mathbb{R}^l}))'$, i.e. a Fourier hyperfunction of mixed type whose support is contained in $\overline{\mathbb{R}_+^k \times \mathbb{R}^l}$. In (3.9) the path $\Gamma$ is that shown in Figure 1 and the path $\Gamma_+$ is given in Figure 3 above. $F(f \times g) = M(f, g)$ follows immediately as in Proposition 3.10.

**Proposition 3.15.** *Suppose $F(\zeta|g)$ for $g \in \mathscr{P}_{0,l}$ be a holomorphic function in $\zeta$ belonging to $\{\zeta \in \mathbb{C}^k; \operatorname{Im}\zeta > 0\}$ and satisfy*

$$|F(\zeta|g)| \leqq C_{\varepsilon,m} \|g\|_{0,l;m} e^{\varepsilon|\zeta|} \tag{3.10}$$

*for any $m$, $\varepsilon > 0$ and $\zeta$ in $\{\operatorname{Im}\zeta_j > \varepsilon(1 + |\operatorname{Re}\zeta_j|),\ 1 \leqq j \leqq k\}$. Then there exists a unique Fourier hyperfunction $\mu$ of mixed type with support $\overline{\mathbb{R}_+^k \times \mathbb{R}^l}$ such that*

$$\mu(e^{i(\cdot,\zeta)} \times g) = F(\zeta|g). \tag{3.11}$$

*Proof.* For $f \in \mathscr{P}_{k,0}$ we have

$$\begin{aligned}\int_{\gamma^k} F(\zeta|g) f(\zeta) d\zeta &= \mu(\mathscr{F} f \times g) \\ &= \int_{\gamma^k} \mu(e^{i(\cdot,\zeta)} \times g) f(\zeta) d\zeta,\end{aligned}$$

where the path $\gamma$ is an analogue of that described in Proposition 3.12. Therefore (3.11) is obtained by Propositions 3.12–14. Uniqueness follows from the fact that $\mu(e^{i(\cdot,\zeta)} \times g) \equiv 0$ for any $g \in \mathscr{P}_{0,l}$ implies $\mu = 0$.

*Remark.* This proposition becomes crucial when one comes back from the Euclidean theory to the relativistic theory.

The theory of *H-valued Fourier hyperfunctions of type* II (or *mixed type*) can be formulated without new difficulties, by starting from the counterpart of Definitions 3.1 and 3.2, where $|f(z)|$ should be replaced by $\|f(z)\|_H$, the norm in a Hilbert space, of an $H$-valued holomorphic function $f(z)$, as was done in NM I for $H$-valued Fourier hyperfunctions of type I (see also [9]). We will cease to continue such a translation word for word, but we only mention the following: Let $V_0=\mathbb{Q}^{k,l}$ and $V_j=\{z\in\mathbb{Q}^{k,l}; \operatorname{Im} z_j \neq 0\}$. Further we put $V=\cap_{j=1}^n V_j$, $\hat{V}_j=\cap_{i\neq j} V_i$, $W=\{V_j\}_{j=0}^n$ and $W'=\{V_j\}_{j=1}^n$. Then we have the isomorphisms

$$H^n_{\mathbb{D}^n}(\mathbb{Q}^{k,l}, {}^H\tilde{\mathcal{O}}_{k,l}) \cong H^n(W, W'; {}^H\tilde{\mathcal{O}}_{k,l}) \cong {}^H\tilde{\mathcal{O}}_{k,l}(V)/\sum_j {}^H\tilde{\mathcal{O}}_{k,l}(\hat{V}_j) \cong L(\underset{\sim}{\mathcal{O}}_{k,l}(\mathbb{D}^n), H),$$

where $H^n(W, W'; {}^H\tilde{\mathcal{O}}_{k,l})$ is the relative cohomology of covering and $L(\underset{\sim}{\mathcal{O}}_{k,l}(\mathbb{D}^n), H)$ is the space of all continuous linear operators from $\underset{\sim}{\mathcal{O}}_{k,l}(\mathbb{D}^n)$ to the Hilbert space $H$.

In the following we shall state proposition only in the language of scalar-valued Fourier hyperfunctions, though some of them are applied to $H$-valued cases. This is entirely for the sake of convenience, and of course all of them can be trivially extended to the case of $H$-valued Fourier hyperfunctions.

## §4. Euclidean Green's Functions for Hyperfunction Fields

First of all let us recollect the convention of variables used in NM I and introduce some new notations. As for the set of four-vectors, the difference vectors are $\xi_0=x_1$, $\xi_j=x_{j+1}-x_j$, $1\leqq j\leqq n-1$, in coordinate space and correspondingly $q_k=p_{k+1}+\ldots+p_n$, $0\leqq k\leqq n-1$, in momentum space, so that

$$\sum_{j=1}^{n} p_j\cdot x_j=\sum_{k=0}^{n-1} q_k\cdot\xi_k$$

holds, where $p_j\cdot x_j=p_j^0x_j^0-\boldsymbol{p}_j\cdot\boldsymbol{x}_j$ is the Lorentz-invariant inner product of two four-vectors $p_j=(p_j^0, \boldsymbol{p}_j)$ and $x_j=(x_j^0, \boldsymbol{x}_j)$. For any four-vector $x=(x^0, \boldsymbol{x})\in\mathbb{R}^4$ it is meant that $\theta x=(-x^0, \boldsymbol{x})$ and $\imath x=(ix^0, \boldsymbol{x})$. This convention also applies to a set of four-vectors $\underset{\sim}{x}_n=(x_1, \ldots, x_n)\in\mathbb{R}^{4n}$ by writing $\theta\underset{\sim}{x}_n$ and $\imath\underset{\sim}{x}_n$. On occasion it is more convenient to rebind $\underset{\sim}{x}_n$ in the form $\underset{\sim}{x}_n=(\underset{\sim}{x}_n^0, \underset{\sim}{\boldsymbol{x}}_n)$. Then $|\underset{\sim}{x}_n^0|$, $|\underset{\sim}{\boldsymbol{x}}_n|$ and $|\underset{\sim}{x}_n|$ stand for $\sum_{j=1}^{n}|x_j^0|$, $\sum_{j=1}^{n}\sum_{\mu=1}^{3}|x_j^\mu|$ and $\sum_{j=1}^{n}\sum_{\mu=0}^{3}|x_j^\mu|$, respectively.

We begin with stating the axioms for Wightman functions.

*Fourier Hyperfunction Property*

(R0) $\mathfrak{W}_0=1, \quad \mathfrak{W}_n(\underset{\sim}{x}_n)\in(\mathscr{P}_{4n,0})' \quad \text{for} \quad n\geqq 1$

and there is a Lorentz frame in which $\mathfrak{W}_n$ is a Fourier hyperfunction of type I for spatial variables. (The Lorentz frame may be different for different $\mathfrak{W}_n$, see the remark below.)

*Relativistic Covariance.* For each $n$, $\mathfrak{W}_n$ is Poincaré invariant:

(R1) $\mathfrak{W}_n(\underset{\sim}{x}_n)=\mathfrak{W}_n(\Lambda\underset{\sim}{x}_n+a)$,

where $\Lambda$ is a proper Lorentz transformation and $\Lambda\underset{\sim}{x}_n+a=(\Lambda x_1+a, \ldots, \Lambda x_n+a)$.

*Remark.* The axiom (R0) states that we are considering a set of those Wightman Fourier hyperfunctions of type II for which $\mathfrak{W}_n(\Lambda \underline{x}_n) \equiv \mathfrak{W}'_n(\underline{x}_n) \in (\mathscr{P}_{n,3n})'$ for some Lorentz transformation $\Lambda$, where $(\mathscr{P}_{n,3n})'$ is the collection of specified Fourier hyperfunctions of mixed type, namely, type II for $n$ time-variables and type I for $3n$ space-variables. To this set belong all of the Wightman functions that are constructed from type I Fourier hyperfunction fields considered in NM I. Owing to the axiom (R1) we are allowed to replace (R0) by the axiom (R0'): $\mathfrak{W}_0 = 1$ *and* $\mathfrak{W}_n(\underline{x}_n) \in (\mathscr{P}_{n,3n})'$, $n \geqq 1$, *for all Lorentz systems.*

*Positivity.* For any finite sequence $f_0, f_1, \ldots, f_N$ of test functions such that $f_0 \in \mathbb{C}$, $f_n \in \mathscr{P}_{n,3n}$, $1 \leqq n \leqq N$, there holds

(R2) $\sum_{n,m} \mathfrak{W}_{n+m}(f_n^* \times f_m) \geqq 0$,

where $(f_n^* \times f_m)(\underline{x}_n, \underline{y}_m) = f_n^*(\underline{x}_n) f_m(\underline{y}_m)$ and $f_n^*(\underline{x}_n) = \bar{f}_n({}_n\underline{x})$.

*Local Commutativity*

(R3) $\mathfrak{W}_n(x_1, \ldots, x_j, x_{j+1}, \ldots, x_n) = \mathfrak{W}_n(x_1, \ldots, x_{j+1}, x_j, \ldots, x_n)$ if $(x_j - x_{j+1})^2 < 0$.

*Cluster Property.* For any space-like vector $a$ and $\underline{x}_k \in \mathbb{R}^{4k}$, $\underline{y}_{n-k} \in \mathbb{R}^{4(n-k)}$, $1 \leqq k \leqq n-1$

(R4) $\lim_{\lambda \to \infty} \mathfrak{W}_n(\underline{x}_k, \underline{y}_{n-k} + \lambda a) = \mathfrak{W}_k(\underline{x}_k) \mathfrak{W}_{n-k}(\underline{y}_{n-k})$.

*Spectral Condition.* By the translation invariance there exist Fourier hyperfunctions $W_{n-1} \in (\mathscr{P}_{(n-1),3(n-1)})'$ such that $\mathfrak{W}_n(\underline{x}_n) = W_{n-1}(\underline{\xi}_{n-1})$ holds. Then

(R5) $\operatorname{supp} \tilde{W}_{n-1}(\underline{q}_{n-1}) \subset \overline{V_+^{n-1}}$,

where $\overline{V_+^{n-1}}$ is the closure of the forward light cone in $\mathbb{D}^{4(n-1)}$ and $\tilde{W}_{n-1}$ is the Fourier transform of $W_{n-1}$.

Suppose $\mathscr{P}$ be a vector space of sequences $\underline{f} = (f_0, f_1, \ldots)$, where $f_0 \in \mathbb{C}$, $f_n \in \mathscr{P}_{n,3n}$ for $1 \leqq n \leqq N$ and $f_n = 0$ if $n > N$ for some finite $N$. Let $(\underline{f}, \underline{g}) = \sum_{n,m} \mathfrak{W}_{n+m}(f_n^* \times g_m)$ with $\underline{f}, \underline{g} \in \mathscr{P}$. Owing to (R2) this serves as a semi-definite inner product and the completion of $\mathscr{P}/\mathscr{N}$, where $\mathscr{N} = \{\underline{f} \in \mathscr{P}; (\underline{f}, \underline{f}) = 0\}$, defines a Hilbert space $\mathscr{H}$. Let $\Phi$ be the natural map of $\mathscr{P}$ into $\mathscr{H}$. We set $\Phi_0 = \Phi(1, 0, 0, \ldots)$. If $\underline{f}$ has only one nonvanishing component $\underline{f} \equiv f_n \in \mathscr{P}_{n,3n}$, we write formally

$$\Phi(\underline{f}) \equiv \Phi_n(f_n) = \int \Phi_n(\underline{x}_n) f_n(\underline{x}_n) d\underline{x}_n. \tag{4.1}$$

$\Phi_n$ is a continuous linear operator from $\mathscr{P}_{n,3n}$ to $\mathscr{H}$. Upon setting $\Phi_n(\underline{x}_n) = \Psi_n(x_1, \underline{\xi}_{n-1})$ we have

$$(\Psi_n(x, \underline{\xi}_{n-1}), \Psi_m(x', \underline{\xi}'_{m-1})) = W_{n+m-1}(-{}_{n-1}\underline{\xi}, -x+x', \underline{\xi}'_{m-1}). \tag{4.2}$$

**Theorem 4.1.** *For* $n = 1, 2, \ldots$, *the support of* $\tilde{\Psi}_n(q_0, \underline{q}_{n-1})$, *the Fourier transform of* $\Psi_n$, *is contained in* $\overline{V_+^n}$, *namely,* $\tilde{\Psi}_n \in L(\underset{\sim}{\mathcal{Q}}_{n,3n}(\overline{V_+^n}), \mathscr{H})$.

*Proof.* Since $\|\tilde{\Psi}_n(f_n)\|$ is written in terms of the Fourier hyperfunction $\tilde{W}_{2n-1}$, it follows from the spectral condition (R5) that $\tilde{\Psi}_n$ is a continuous linear operator from $\underset{\sim}{\mathcal{Q}}_{n,3n}(\overline{V_+^n})$ to $\mathscr{H}$. Thus the support of $\tilde{\Psi}_n$ is contained in $\overline{V_+^n}$.

**Proposition 4.2.** *If* $\mu\in(\mathcal{O}_{n,3n}(\overline{V_+^n}))'$, *then* $F(\underline{\zeta}_n)=\mu(e^{i((\cdot,\zeta))})$ *is holomorphic in the tube* $\mathfrak{T}_+^n\equiv\mathbb{R}^{4n}\times iV^n=\{\underline{\zeta}_n\in\mathbb{C}^{4n};\ \operatorname{Im}\underline{\zeta}_n\in V_+^n\}$, *and in Euclidean points it satisfies the condition that*

$$|F(\imath\underline{\xi}_n)|\leqq C_\varepsilon e^{\varepsilon|\underline{\xi}_n|} \tag{4.3}$$

*is valid for* $\xi_j^0>2\varepsilon$, $j=1,\dots,n$, *and every* $\varepsilon>0$. *Here* $((z,\zeta))$ *is the abbreviation of* $\sum_{j=1}^{n}(z_j\cdot\zeta_j)$ *and* $\operatorname{Im}\underline{\zeta}_n\in V_+^n$ *means that* $\operatorname{Im}\zeta_j\in V_+$ *for each* $j$.

*Proof.* It is readily seen that $e^{i((\cdot,\zeta))}$ for $\underline{\zeta}_n\in\mathfrak{T}_+^n$ belongs to $\mathcal{O}_{n,3n}(\overline{V_+^n})$. Hence $\mu(e^{i((\cdot,\zeta))})$ is defined well and holomorphic there. Let $z=x+iy$ and consider

$$U_\varepsilon=\{x^0+\varepsilon>(1-\varepsilon)(\boldsymbol{x}\cdot\boldsymbol{x})^{1/2},\ |y^0|<\varepsilon(1+x^0),\ (\boldsymbol{y}\cdot\boldsymbol{y})<\varepsilon^2\},$$

then $U_\varepsilon^n$ is a neighbourhood of $V_+^n$. If $\xi_j^0>2\varepsilon$ for each $j$, we have

$$\begin{aligned}
|F(\imath\underline{\xi}_n)|&\leqq C'_\varepsilon\sup_{z\in U_\varepsilon^n}|e^{i((z,\imath\xi))+\varepsilon|z|}|\\
&\leqq C'_\varepsilon\sup_{z\in U_\varepsilon^n}\exp\{-((\boldsymbol{y},\boldsymbol{\xi}))-((x^0,\xi^0))+\varepsilon|\underline{x}_n|+\varepsilon|\underline{\boldsymbol{y}}_n|+\varepsilon|\underline{y}_n^0|\}\\
&\leqq C''_\varepsilon\sup_{z\in U_\varepsilon^n}\exp\{\varepsilon|\underline{\boldsymbol{\xi}}_n|-((x^0,\xi^0))+\varepsilon|\underline{x}_n|+\varepsilon^2(1+|\underline{x}_n^0|)\}\\
&\leqq C_\varepsilon e^{\varepsilon|\underline{\xi}_n|}
\end{aligned}$$

which completes the proof. Here we have written $((\boldsymbol{y},\boldsymbol{x}))$ for $\sum_{j=1}^{n}(\boldsymbol{y}_j\cdot\boldsymbol{x}_j)$ and $((x^0,\xi^0))$ for $\sum_{j=1}^{n}x_j^0\xi_j^0$.

This proposition can easily be extended to the case of $\mathscr{H}$-valued Fourier hyperfunctions, particularly yielding

**Theorem 4.3.**

$$\Psi_n(\zeta_0,\underline{\zeta}_{n-1})=\tilde{\Psi}_n\left(\exp\left\{i\sum_{k=0}^{n-1}\zeta_k\cdot q_k\right\}\right) \tag{4.4}$$

*is an* $\mathscr{H}$-*valued infra-exponential analytic function in* $\mathfrak{T}_+^n$ *and has* $\Psi_n(\xi_0,\underline{\xi}_{n-1})$ *as its boundary value.*

**Theorem 4.4.** *Let*

$$W_{n-1}(\underline{\zeta}_{n-1})=\tilde{W}_{n-1}\left(\exp\left\{i\sum_{k=1}^{n-1}\zeta_k\cdot q_k\right\}\right), \tag{4.5}$$

*then it is invariant under proper Lorentz transformations and we have*

$$W_{n-1}(\underline{\zeta}_{n-1})=(\Phi_0,\Psi_n(z,\underline{\zeta}_{n-1})), \tag{4.6}$$

$$W_{n+m-1}(-_{n-1}\overline{\underline{\zeta}},-\bar{z}+z',\underline{\zeta}'_{m-1})=(\Psi_n(z,\underline{\zeta}_{n-1}),\Psi_m(z',\underline{\zeta}'_{m-1})). \tag{4.7}$$

*Proof.* Since $\tilde{W}_{n-1}(\underline{q}_{n-1})$ is Lorentz invariant, so is $W_{n-1}(\underline{\zeta}_{n-1})$, too. It follows from (4.2) that

$$(\tilde{\Psi}_n(p, \underline{q}_{n-1}), \tilde{\Psi}_m(p', \underline{q}'_{m-1})) = \tilde{W}_{n+m-1}({}_{n-1}\underline{q}, p+p', \underline{q}'_{m-1})$$

and hence the theorem can be obtained by using (4.4).

By the relativistic covariance (Theorem 4.4) and the Bargmann-Hall-Wightman theorem [10], we obtain a single-valued analytic extension of the Wightman function $W_{n-1}(\underline{\zeta}_{n-1})$ into the *extended tube* $\mathfrak{T}^{n-1}_{+,\mathrm{ext}} = \{\underline{\zeta}_{n-1} \in \mathbb{C}^{4(n-1)}; \Lambda\underline{\zeta}_{n-1} \in \mathfrak{T}^{n-1}_{+}$ for some $\Lambda \in L_+(\mathbb{C})\}$, where $L_+(\mathbb{C})$ is the set of all complex proper Lorentz transformations and $\zeta_j = z_{j+1} - z_j$, $1 \leqq j \leqq n-1$. The function $\mathfrak{W}_n(\underline{z}_n)$, defined by $\mathfrak{W}_n(\underline{z}_n) = W_{n-1}(\underline{\zeta}_{n-1})$, is analytic in $\sigma_{\mathrm{ext}} = \{\underline{z}_n \in \mathbb{C}^{4n}; \underline{\zeta}_{n-1} \in \mathfrak{T}^{n-1}_{+,\mathrm{ext}}\}$ and has the Fourier hyperfunction $\mathfrak{W}_n(\underline{x}_n)$ as its boundary value. Finally using the locality (R3) we obtain a single-valued analytic extension of $\mathfrak{W}_n(\underline{z}_n)$ into the set

$$\sigma_{\mathrm{ext,perm}} = \{\underline{z}_n \in \mathbb{C}^{4n}; \underline{z}_{\pi(n)} \in \sigma^n_{\mathrm{ext}} \ \textit{for some permutation } \pi\},$$

where $\underline{z}_{\pi(n)} = (z_{\pi(1)}, \ldots, z_{\pi(n)})$ and $(\pi(1), \ldots, \pi(n))$ is a permutation of $(1, \ldots, n)$. We denote this extension again by $\mathfrak{W}_n(\underline{z}_n)$. It is invariant under the complex Poincaré group and also under permutations of the arguments $z_1, \ldots, z_n$. The set $\sigma^n_{\mathrm{ext,perm}}$ contains the set of *Euclidean points* (of noncoinciding arguments) $E^n = \{\underline{z}_n \in \mathbb{C}^{4n}; \mathrm{Re}\, z^0_k = 0, \mathrm{Im}\, z_k = 0$ *for all* $k$ *and* $z_i \neq z_j$ *if* $i \neq j\}$.

*Definition 4.5.* The restriction of the Wightman function $\mathfrak{W}_n(\underline{z}_n)$ to $E^n$ is called the *n-point Euclidean Green's function* or the *Schwinger function.*

We set $\mathfrak{S}_0 = \mathfrak{W}_0 = 1$ and

$$\mathfrak{S}_n(\underline{x}_n) = \mathfrak{W}_n(\imath\underline{x}_n), \tag{4.8}$$

$$S_{n-1}(\underline{\xi}_{n-1}) = W_{n-1}(\imath\underline{\xi}_{n-1}) = \mathfrak{S}_n(\underline{x}_n), \tag{4.9}$$

where $\underline{x}_n \in \Omega^n \equiv \{\underline{x}_n \in \mathbb{R}^{4n}; x_i \neq x_j \text{ if } i \neq j\}$. Then we have

**Theorem 4.6.** *(Distribution property)*

$$\mathfrak{S}_0 = 1 \quad \textit{and} \quad \mathfrak{S}_n \in (\mathscr{C}_0(\mathbb{R}^{4n}))' \quad \textit{for each} \quad n \geqq 1.$$

*Proof.* Let

$$Q_\varepsilon = \{\underline{\xi}_{n-1} \in \mathbb{R}^{4(n-1)}; \xi^0_j > \varepsilon, 1 \leqq j \leqq n-1\}. \tag{4.10}$$

Since $\sup_{\underline{\xi}_{n-1} \in Q_\varepsilon} |S_{n-1}(\underline{\xi}_{n-1})|\, e^{-\varepsilon|\underline{\xi}_{n-1}|} < \infty$ for all $\varepsilon > 0$ by Proposition 4.2, it is evident that $S_{n-1} \in (\mathscr{C}_+(\mathbb{R}^{4(n-1)}))'$. With the aid of a geometrical argument as made by Osterwalder and Schrader [2] we have the theorem.

From the invariance properties of Wightman functions immediately follows

**Theorem 4.7.** *(Covariance and symmetry.) The Schwinger functions* $\mathfrak{S}_n(\underline{x}_n)$ *are invariant under the inhomogeneous Euclidean group* $iSO_4$ *and under the permutation of the arguments* $x_1, \ldots, x_n$.

We define real analytic $\mathscr{H}$-valued functions $\Psi_n^E(x_1, \underline{\xi}_{n-1})$ by $\Psi_n(\imath x_1, \imath\underline{\xi}_{n-1})$ for $x_1^0>0$ and $\xi_k^0>0$, $1\leqq k\leqq n-1$, and put $\Phi_n^E(\underline{x}_n)=\Psi_n^E(x_1, \underline{\xi}_{n-1})$. Then by virtue of Theorem 4.4 it is easy to see the following lemma:

**Lemma 4.8.** *Suppose* $x_1^0, \xi_j^0, x_1'^0, \xi_j'^0$ *be all positive, then*

$$(\Phi_n^E(x_1, \underline{\xi}_{n-1}), \Phi_m^E(x_1', \underline{\xi}_{m-1}'))=\mathfrak{S}_{n+m}(\theta_n\underline{x}, \underline{x}_m').$$

This lemma yields the positivity property of the Schwinger functions.

**Proposition 4.9.** *(Positivity property.) For all finite sequences* $f_0, f_1, \ldots, f_N$, *where* $f_0\in\mathbb{C}$ *and* $f_n\in\mathscr{C}_<(\mathbb{R}^{4n})$, $1\leqq n\leqq N$, *there holds* $\sum_{n,m}\mathfrak{S}_{n+m}(\Theta f_n^*\times f_m)\geqq 0$, *in which* $\Theta f_n(\underline{x}_n)=f_n(\theta\underline{x}_n)$.

*Proof.* For $f_n\in\mathscr{C}_<(\mathbb{R}^{4n})$ the integral

$$\Phi_n^E(f_n)=\int\Phi_n^E(\underline{x}_n)f_n(\underline{x}_n)d\underline{x}_n$$

is well defined and

$$\begin{aligned}(\Phi_n^E(f_n), \Phi_m^E(g_m)) &= \int\mathfrak{S}_{n+m}(\theta({}_n\underline{x}), \underline{x}_m')\bar{f}_n(\underline{x}_n)g_m(\underline{x}_m')d\underline{x}_nd\underline{x}_m' \\ &= \int\mathfrak{S}_{n+m}(\underline{x}_n, \underline{x}_m')\bar{f}_n(\theta({}_n\underline{x}))g_m(\underline{x}_m')d\underline{x}_nd\underline{x}_m' \\ &= \mathfrak{S}_{n+m}(\Theta f_n^*\times g_m).\end{aligned}$$

Thus we obtain

$$\sum_{n,m}\mathfrak{S}_{n+m}(\Theta f_n^*\times f_m)=\left\|\sum_n\Phi_n^E(f_n)\right\|^2\geqq 0. \tag{4.11}$$

**Proposition 4.10.** *If* $f\in\mathscr{C}_<(\mathbb{R}^{4n})$, $g\in\mathscr{C}_<(\mathbb{R}^{4m})$ *and* $a=(0, \boldsymbol{a})\in\mathbb{R}^4$, *we have*

$$\lim_{\lambda\to\infty}\mathfrak{S}_{n+m}(\Theta f^*\times g_{\lambda a})=\mathfrak{S}_n(\Theta f^*)\mathfrak{S}_m(g),$$

*where* $g_{\lambda a}$ *is defined by* $g_{\lambda a}(\underline{x}_n)=g(\underline{x}_n-\lambda a)$.

*Proof.* Let $U(a, 1)$ be a unitary operator defined by $U(a, 1)\Phi_n(f)=\Phi_n(f_a)$ for $a\in\mathbb{R}^4$ and $f\in\mathscr{P}_{n, 3n}$, where $f_a(\underline{x}_n)=f(\underline{x}_n-a)$. If $a$ is a space vector, as in the proposition, it is clear that $U(a, 1)\Phi_n^E(\underline{x}_n)=\Phi_n^E(\underline{x}_n+a)$ and hence $U(a, 1)\Phi_n^E(g)=\Phi_n^E(g_a)$ for $g\in\mathscr{C}_<(\mathbb{R}^{4n})$. The cluster property (R4) implies that for any two vectors $\Phi$ and $\Psi$ in $\mathscr{H}$, and a space vector $a$, there holds $\lim_{\lambda\to\infty}(\Phi, U(\lambda a, 1)\Psi)=(\Phi, \Phi_0)(\Phi_0, \Psi)$. Upon substituting $\Phi=\Phi_n^E(f)$ and $\Psi=\Phi_m^E(g)$ we obtain the proposition.

**Proposition 4.11.** *The correspondence from Wightman functions (Fourier hyperfunctions) to Schwinger functions (distributions) is one to one.*

*Proof.* Suppose that

$$\int W_{n-1}(\imath\underline{\xi}_{n-1})g(\underline{\xi}_{n-1})d\underline{\xi}_{n-1}=0 \quad \text{for all} \quad g\in\mathscr{C}_+(\mathbb{R}^{4(n-1)}),$$

then $W_{n-1}(\imath\underline{\xi}_{n-1})=0$ for $\xi_k^0>0$, $k=1, \ldots, n-1$. Since $W_{n-1}(\imath\underline{\xi}_{n-1})$ is a real analytic function, $W_{n-1}(\underline{\zeta}_{n-1})=0$ if $\operatorname{Im}\underline{\zeta}_{n-1}\in V_+^{n-1}$. By the uniqueness of Fourier transformation of Fourier hyperfunctions we have $\tilde{W}_{n-1}=0$.

## §5. Axioms for Euclidean Field Theory and the Reconstruction of Relativistic Fields

In the preceding section, from a given set of Wightman Fourier hyperfunctions satisfying the axioms (R0)–(R5), we have constructed a set of Schwinger functions having the following properties:

*Distribution Property*

(E0) $\mathfrak{S}_0 = 1$ and $\mathfrak{S}_n(\underline{x}_n) \in (\mathscr{C}_0(\mathbb{R}^{4n}))'$ for each $n \geqq 1$.

*Euclidean Covariance*

(E1) $\mathfrak{S}_n(\underline{x}_n) = \mathfrak{S}_n(R\underline{x}_n + a)$

for each $n \geqq 1$ and all $(a, R) \in iSO_4$.

*Positivity*

(E2) $\sum_{n,m} \mathfrak{S}_{n+m}(\Theta f_n^* \times f_m) \geqq 0$

for all finite sequences $f_0, f_1, \ldots, f_N$ of test functions $f_0 \in \mathbb{C}$ and $f_n \in \mathscr{C}_<(\mathbb{R}^{4n})$, $n \geqq 1$. Here $\Theta f_n(\underline{x}_n) = f_n(\theta \underline{x}_n)$ and $\theta x = (-x^0, \boldsymbol{x})$.

*Symmetry*

(E3) $\mathfrak{S}_n(\underline{x}_n) = \mathfrak{S}_n(\underline{x}_{\pi(n)})$

for all permutations $\pi: (1, \ldots, n) \to (\pi(1), \ldots, \pi(n))$.

*Cluster Property*

(E4) $\lim_{\lambda \to \infty} \mathfrak{S}_{n+m}(\Theta f^* \times g_{\lambda a}) = \mathfrak{S}_n(\Theta f^*) \mathfrak{S}_m(g)$.

As for (E0) it is worthy to remark that $\mathscr{C}_0(\mathbb{R}^{4n})$ is *stable* against $iSO_4$ in contrast to $\mathscr{P}_{n,3n}$ which is *not stable* under Lorentz transformations.

Conversely we can prove the following theorem.

**Theorem 5.1.** *(Reconstruction of the relativistic theory.) To a given sequence of Euclidean Green's functions satisfying* (E0)–(E4) *there corresponds uniquely a sequence of Wightman Fourier hyperfunctions having the properties* (R0)–(R5) *and whose Schwinger functions coincide with the Euclidean Green's functions given initially.*

For the proof of this theorem we need some preparatory propositions. First let $\mathscr{C}_<$ be a vector space of sequences $\underline{f} = (f_0, f_1, \ldots)$, where $f_0 \in \mathbb{C}$, $f_n \in \mathscr{C}_<(\mathbb{R}^{4n})$ for $l \leqq n \leqq N$ and $f_n = 0$ if $n > N$ for some finite $N$. Let $<\underline{f}, \underline{g}> = \sum_{n,m} \mathfrak{S}_{n+m}(\Theta f_n^* \times g_m)$ with $\underline{f}, \underline{g} \in \underline{\mathscr{C}}_<$. Owing to (E2) this serves as a semi-definite inner product and the completion of $\underline{\mathscr{C}}_</\mathscr{N}$, where $\mathscr{N} = \{\underline{f} \in \underline{\mathscr{C}}_<; \langle \underline{f}, \underline{f} \rangle = 0\}$, defines a Hilbert space $\mathscr{H}$. Let $\Phi^E$ be the natural map of $\underline{\mathscr{C}}_<$ into $\mathscr{H}$. We obtain $(\Phi^E(f), \Phi^E(g)) = \langle \underline{f}, \underline{g} \rangle$. We set $\Phi_0 = \Phi^E(1, 0, 0, \ldots)$. For $\underline{f} \in \underline{\mathscr{C}}_<$ and $a = (0, \boldsymbol{a}) \in \mathbb{R}^4$ we define $\hat{U}_s(a)\underline{f}$ by $(\hat{U}_s(a) f)_n(\underline{x}_n) = f_n(\underline{x}_n - a)$. We can extend it to a unitary operator $U_s(a)$ in $\mathscr{H}$ by

(E1) (see Osterwalder-Schrader [2]). If $\underline{f}$ has only one nonvanishing component $\underline{f} \equiv f_n \in \mathscr{C}_<(\mathbb{R}^{4n})$, we write formally

$$\Phi^E(\underline{f}) \equiv \Phi_n^E(f_n) = \int \Phi_n^E(\underline{x}_n) f_n(\underline{x}_n) d\underline{x}_n \,. \tag{5.1}$$

Let us define $\Psi^E(x_1, \underline{\xi}_{n-1}) = \Phi_n^E(\underline{x}_n)$, then it is a vector-valued distribution over $\mathscr{C}_+(\mathbb{R}^{4n})$ and we have by (E2)

$$\begin{aligned}(\Psi_n^E(x_1, \underline{\xi}_{n-1}), \Psi_m^E(x_1', \underline{\xi}_{m-1}')) &= S_{n+m-1}(-\theta(_{n-1}\underline{\xi}), -\theta x_1 + x_1', \underline{\xi}_{m-1}') \\ &= \mathfrak{S}_{n+m}(\theta_n \underline{x}, \underline{x}_m') \,.\end{aligned} \tag{5.2}$$

**Lemma 5.2.** *For $t \geqq 0$ we define $\hat{T}_t : \mathscr{C}_< \to \mathscr{C}_<$ by $(\hat{T}_t f)_n(\underline{x}_n) = f_n(\underline{x}_n - t)$, where $t = (t, \mathbf{0})$. Then $\hat{T}_t$ induces a continuous one-parameter semigroup of self-adjoint contraction operators on $\mathscr{K}$.*

*Proof.* By (E1) we have $\langle f, \hat{T}_t \underline{g} \rangle = \langle \hat{T}_t \underline{f}, \underline{g} \rangle$ for $\underline{f}, \underline{g} \in \mathscr{C}_<$ and obviously $\hat{T}_s \hat{T}_t = \hat{T}_{s+t}$. Since in virtue of (2.6)

$$\begin{aligned}&\|\Theta f_n^* \times \hat{T}_t g_m\|_{1/\varepsilon, p} \\ &= \sup_{x, y, |l| \leqq p} |D^l(\Theta f_n^*(\underline{x}_n) g_m(\underline{y}_m - t)| \exp\left\{\varepsilon\left(1 - \frac{1}{p}\right)(|\underline{x}_n| + |\underline{y}_m|)\right\} \\ &\leqq \sup |D^l(\Theta f_n^*(\underline{x}_n) g(\underline{y}_m)| \exp\left\{\varepsilon\left(1 - \frac{1}{p}\right)(|\underline{x}_n| + |\underline{y}_m| + m|t|)\right\} \\ &= \|\Theta f_n^* \times g_m\|_{1/\varepsilon, p} e^{\delta|t|} \,,\end{aligned} \tag{5.3}$$

where $\delta = \varepsilon m(1 - 1/p)$, we get $|\langle \underline{f}, \hat{T}_t \underline{f} \rangle| \leqq C_\delta e^{\delta|t|}$ for any positive $\delta$ and some constant $C_\delta$. We can improve this estimate by a repeated application of the Schwarz inequality:

$$\begin{aligned}|\langle \underline{f}, \hat{T}_t \underline{f} \rangle| \leqq \|\underline{f}\| \, \|\hat{T}_t \underline{f}\| &= \|\underline{f}\| \langle \underline{f}, \hat{T}_{2t} \underline{f} \rangle^{1/2} \\ &\leqq \|\underline{f}\|^{\sum_{k=0}^{n-1} 2^{-k}} \langle \underline{f}, \hat{T}_{2^n t} \underline{f} \rangle^{2^{-n}} \\ &\leqq \|\underline{f}\|^{\sum_{k=0}^{n-1} 2^{-k}} C_\delta^{2^{-n}} e^{\delta|t|} \,,\end{aligned} \tag{5.4}$$

where we have written $\langle \underline{f}, \underline{f} \rangle^{1/2}$ as $\|\underline{f}\|$. Since the right hand side of (5.4) converges to $\|\underline{f}\|^2 e^{\delta|t|}$ as $n \to \infty$, and $\delta$ can be taken arbitrarily small, we obtain

$$|\langle \underline{f}, \hat{T}_t \underline{f} \rangle| \leqq \|\underline{f}\|^2 \,.$$

Thus $\hat{T}_t$ is a contraction operator, and it induces a continuous one-parameter semigroup of self-adjoint contraction operators $\{T_t\}$. This completes the proof.

Let $-H$ be the infinitesimal generator of $T_t$, then

$$e^{-tH} \Psi_n^E(x, \underline{\xi}_{n-1}) = \Psi_n^E(x + t, \underline{\xi}_{n-1}) \,. \tag{5.5}$$

We define for $f \in \mathscr{P}_{0, 3n}$ and $g \in \mathscr{P}_{0, 3m}$

$$\Psi_n^E(x^0, \underline{\xi}_{n-1}^0 | f) = \int \Psi_n^E(x, \underline{\xi}_{n-1}) f(\boldsymbol{x}, \underline{\boldsymbol{\xi}}_{n-1}) d\boldsymbol{x} d\underline{\boldsymbol{\xi}}_{n-1} \tag{5.6}$$

and

$$\begin{aligned}&S_{n+m-1}(_{n-1}\underline{\xi}^0, x^0 + x'^0, \underline{\xi}_{m-1}'^0 | fg) \\ &= \langle \Psi_n^E(x^0, \underline{\xi}_{n-1}^0 | f), \Psi_m^E(x'^0, \underline{\xi}_{m-1}'^0 | g) \rangle \,,\end{aligned} \tag{5.7}$$

where $\Psi_n^E(x, \underline{\xi}_{n-1})$ has already been given next to (5.1). Let $\mathbb{C}_+ = \{z \in \mathbb{C}; \operatorname{Re} z > 0\}$ and write $\mathbb{C}_+^k = (\mathbb{C}_+)^k$.

*Remark.* If one starts from Fourier hyperfunction fields of type II, the spatial integrals (5.6) and (5.7) *cannot* exist, because then one has to take the functions $f$ and $g$ from $\mathscr{P}_{3n,0}$ instead of $\mathscr{P}_{0,3n}$. This technical difficulty in the reconstruction of the relativistic theory for type II fields will be resolved in another paper.

**Proposition 5.3.** *For fixed $f$, $g$, the distribution $S_{n+m-1}(\underline{\xi}_{n+m-1}^0 | f g)$ is the restriction to the product of positive real half-axes, $\mathbb{R}_+^{n+m-1}$, of a function analytic in $\mathbb{C}_+^{n+m-1}$. There exists a vector-valued function $\Psi_n^E(z^0, \underline{\zeta}_{n-1}^0 | f)$ analytic in $\mathbb{C}_+^n$ such that*

$$S_{n+m-1}({}_{n-1}\overline{\underline{\zeta}^0}, \bar{z}^0 + z'^0, \underline{\zeta}_{m-1}'^0 | f g)$$
$$= \langle \Psi_n^E(z^0, \underline{\zeta}_{n-1}^0 | f), \Psi_m^E(z'^0, \underline{\zeta}_{m-1}'^0 | g) \rangle . \tag{5.8}$$

*Furthermore $S_{n+m-1}(\underline{\zeta}_{n+m-1}^0 | f g)$ satisfies for $\varepsilon > 0$ and $\underline{\zeta}_{n+m-1}^0 \in \Gamma$*

$$|S_{n+m-1}(\underline{\zeta}_{n+m-1}^0 | f g)| \leqq C_{\varepsilon, p, \Gamma} \|f\|_p \|g\|_p \, e^{\varepsilon |\underline{\zeta}_{n+m-1}^0|} \tag{5.9}$$

*for all p, where the norm $\|\cdot\|_p$ on $\mathscr{O}_c^p$ is given by* (2.2) *and $\Gamma$ is a closed convex cone which is strictly contained in $\mathbb{C}_+^{n+m-1}$.*

*Proof.* We use the holomorphic semigroup $e^{-\tau H}$, $\operatorname{Re} \tau > 0$, to construct the analytic continuation in the time variable of Schwinger functions. Following Osterwalder and Schrader [3, 11], $S_k(\underline{\xi}_k^0 | h)$ and $\Psi_k^E(x_1^0, \underline{\xi}_{k-1}^0 | f)$ can be analytically continued to $\mathbb{C}_+^k$. In order to get the estimate (5.9), we first derive it for the imaginary time, namely we show that if $\xi_j^0 > \varepsilon > 0$, $j = 1, \ldots, k$, then

$$|S_k(\underline{\xi}_k^0 | h)| \leqq C_{\varepsilon, p} \|h\|_p \, e^{\varepsilon |\underline{\xi}_k^0|}$$

for any $p$ and $\varepsilon > 0$ and some constant $C_{\varepsilon, p}$. This task can be carried out in the same way as Osterwalder-Schrader [3] with an obvious modification due to the difference of the seminorm of test function spaces. Then, by the use of the maximum principle, we obtain the following estimate:

If $\underline{\zeta}_k^0 \in C_k^{(N)}$, $j = 1, \ldots, k$, then

$$|S_k(\varepsilon + \underline{\zeta}_k^0 | h)| \leqq C_{\varepsilon, p}^{(N)} \|h\|_p \, e^{\varepsilon |\varepsilon + \underline{\zeta}_k^0|}$$

for any $p$ and $\varepsilon > 0$ and some constant $C_{\varepsilon, p}^{(N)}$, where $C_k^{(N)}$ is the envelope of holomorphy of

$$\hat{C}_k^{(N)} \equiv \bigcup_{j=1}^{k} \{({}_{j-1}\overline{\underline{\zeta}^0}, x^0 + x'^0 + z, \underline{\zeta}_{k-j}'^0) ; \quad (x^0, \underline{\zeta}_{j-1}^0) \in D_j^{(N-1)},$$
$$(x'^0, \underline{\zeta}_{k-j}'^0) \in D_{k-j+1}^{(N-1)}, z \in \mathbb{C}_+\}$$

and

$$D_j^{(N)} \equiv \{(x^0, \underline{\zeta}_{j-1}^0); x^0 > 0, ({}_{j-1}\overline{\underline{\zeta}^0}, 2x^0, \underline{\zeta}_{j-1}^0) \in C_{2j-1}^{(N)}\} .$$

Since $\varepsilon + C_k^{(N)}$ is a cone contained in $\mathbb{C}_+^k$ and tends increasingly to $\mathbb{C}_+^k$ as $N \to \infty$ and $\varepsilon \to 0$, the estimate (5.9) follows. For more details see Osterwalder-Schrader [3, 11].

By this proposition and Proposition 3.15, we conclude that there exist Fourier hyperfunctions $\tilde{W}_{n-1}$ with support in $\overline{\mathbb{R}_+^{n-1} \times \mathbb{R}^{3(n-1)}}$ such that

$$\int W_{n-1}(\imath\xi_{n-1})h(\underline{\xi}_{n-1})d\underline{\xi}_{n-1} = S_{n-1}(\xi^0_{n-1}|h), \tag{5.10}$$

where $W_{n-1}$ is the Fourier transform of $\tilde{W}_{n-1}$ and $h \in \mathscr{P}_{0,3(n-1)}$. From the Euclidean covariance (E1) follows the relativistic invariance of $\tilde{W}_{n-1}$ and the support of $\tilde{W}_{n-1}$ contained in $\overline{V_+^{n-1}}$.

*Remark.* In Equation (5.10) $h$ can be taken from $\mathscr{S}_1(\mathbb{R}^{3(n-1)})$ instead of $\mathscr{P}_{0,3(n-1)}$ by virtue of the distribution property of $S_{n-1}(\underline{\xi}_{n-1})$. This corresponds to (R5). In fact the Fourier transform $\tilde{f} \times \tilde{g}$ of $f \times g$, $f \in \mathscr{P}_{n-1,0}$ and $g \in \mathscr{S}_1(\mathbb{R}^{3(n-1)})$, is an element of $\underset{\sim}{\mathcal{O}}_{n-1,3(n-1)}(\overline{V_+^{n-1}})$, and hence $W_{n-1}(f \times g)$ is well defined.

*Proof of Theorem 5.1.* Define $\mathfrak{W}_n(\underline{x}_n) = W_{n-1}(\underline{\xi}_{n-1})$. The hyperfunction property (R0) is obvious. The positivity condition (R2) follows from Proposition 5.3 and the fact that the Wightman function $W_{n-1}(\xi_{n-1})$ can be obtained as a boundary value of the analytic continuation of the Schwinger function $S_{n-1}$. (E4) shows that for any vector $\Phi, \Psi \in \mathscr{H}$ we have $\lim\limits_{\lambda \to \infty} \langle \Phi, U_s(\lambda a)\Psi \rangle = \langle \Phi, \Phi_0 \rangle \langle \Phi_0, \Psi \rangle$, which implies the cluster property (R4). The relativistic covariance (R1) and the local commutativity (R3) are proved by the same arguments as used by Osterwalder and Schrader [2]. (R5) has already been mentioned after the proof of Proposition 5.3. Equation (5.10) implies that the corresponding Schwinger functions coincide with the Euclidean Green's functions given initially. Uniqueness follows from Proposition 4.11. The proof of Theorem 5.1 is thus completed.

## Appendix. Proof of Proposition 2.2

Let $\mathscr{E}$ be a set of those entire functions $f(z)$ which satisfy

$$\int_{\mathbb{C}^n} |f(z)|^2 \exp\{|\mathrm{Re}\, z|^4/3 - 8|\,\mathrm{Im}\, z|^4\} dV < \infty, \tag{A.1}$$

where $dV = dxdy$ with $z = x + iy$, and $dx = \prod_{j=1}^n dx_j$ and $dy = \prod_{j=1}^n dy_j$, then it is clear that $\mathscr{E} \subset \mathscr{S}^{3/4}_{1/4}$. Next, for positive $\varepsilon$ and an open set $\Omega \subset \mathbb{Q}^{k,l}$ we define three sets $\mathscr{A}^{2,-\varepsilon}_{\mathrm{loc}}(\Omega)$, $\mathscr{Y}(\Omega)$ and $L^{2,-\varepsilon}_{\mathrm{loc}}(\Omega)$ in succession. $\mathscr{A}^{2,-\varepsilon}_{\mathrm{loc}}(\Omega)$ is the set of all those holomorphic functions in $\mathbb{C}^n \cap \Omega$ which satisfy

$$\int_{\mathbb{C}^n \cap K} |f(z)|^2 e^{\varepsilon|z|} dV < \infty \quad \text{for every} \quad K. \tag{A.2}$$

$\mathscr{Y}(\Omega)$ is the set of all those holomorphic functions in $\mathbb{C}^n \cap \Omega$ which satisfy

$$\int_{\mathbb{C}^n \cap K} |f(z)|^2 e^{2|z|^4} dV < \infty \quad \text{for every} \quad K. \tag{A.3}$$

$L^{2,-\varepsilon}_{\mathrm{loc}}(\Omega)$ is the set of all those measurable functions in $\mathbb{C}^n \cap \Omega$ which satisfy

$$\int_{\mathbb{C}^n \cap K} |f(z)|^2 e^{\varepsilon|z|} dV < \infty \quad \text{for every} \quad K. \tag{A.4}$$

In (A.2)–(A.4), $K$ stands for a compact set such that $K \Subset \Omega$. We denote by $\mathscr{X}^{-\varepsilon}(\Omega)$ the closure of $\mathscr{Y}(\Omega)$ in $L^{2,-\varepsilon}_{\mathrm{loc}}(\Omega)$.

If we choose $\varepsilon$ so as to satisfy $\varepsilon<\delta$, then $\mathscr{A}^{2;-\delta}_{\mathrm{loc}}(\Omega)$ is contained in $\mathscr{X}^{-\varepsilon}(\Omega)$. To prove this it is sufficient to verify that $\mathscr{Y}(\Omega)$ is dense in $\mathscr{A}^{2;-\delta}_{\mathrm{loc}}(\Omega)$. Suppose $\mu$ belong to $(\mathscr{A}^{2;-\delta}_{\mathrm{loc}}(\Omega))'$ and be orthogonal to $\mathscr{Y}(\Omega)$. We are going to prove that such a $\mu$ vanishes. We use the Hahn-Banach theorem to find a certain $u$ whose support is compact in $\Omega$ and for which $\int_{\mathbb{C}^n} |u|^2 e^{-\delta|z|} dV<\infty$ and

$$(\mu, v)=\int_{\mathbb{C}^n} \bar{u}v dV \quad \text{for every} \quad v\in\mathscr{A}^{2;-\delta}_{\mathrm{loc}}(\Omega). \tag{A.5}$$

If $\phi(z)$ belongs to $\mathscr{A}^{2;-\delta}_{\mathrm{loc}}(\Omega)$, then $\phi(z)e^{-z^6/n}$ for any $n>0$ belongs to $\mathscr{Y}(\Omega)$ when the condition $\sup_{z\in\Omega\cap\mathbb{C}^n} |\mathrm{Im}\, z_j|^2/(1+|\mathrm{Re}\, z_j|^2)<1/2$ is satisfied. Therefore we have

$$0=(\mu, \phi(z)e^{-z^6/n})=\int_{\mathbb{C}^n} \bar{u}\phi(z)e^{-z^6/n}dV\to\int_{\mathbb{C}^n} \bar{u}\phi(z)dV \quad \text{as} \quad n\to\infty$$

by Lebesgue's theorem. Thus we have proved that $\mathscr{A}^{2;-\delta}_{\mathrm{loc}}(\Omega)$ is contained in $\mathscr{X}^{-\varepsilon}(\Omega)$.

Because of $\mathscr{P}_{k,l}=\operatorname{ind\,lim}_{\varepsilon,m} \mathscr{A}^{2;-\delta}_{\mathrm{loc}}(U_{k,l;m})$, in order to prove the denseness of $\mathscr{S}^{3/4}_{1/4}$ in $\mathscr{P}_{k,l}$ it suffices to ascertain the validity of the following statement: *If an element* $\mu$ *of* $(\mathscr{X}^{-\varepsilon}(U_{k,l;m}))'$ *is orthogonal to* $\mathscr{E}$, *then* $\mu$ *is zero.* From here on we fix $\varepsilon$ and $m\geqq 2$. By the Hahn-Banach theorem there exists some $u$ whose support is contained in $U_{k,l;m+\varepsilon}$ and for which $\int_{\mathbb{C}^n} |u|^2 e^{-\delta|z|} dV<\infty$ and

$$(\mu, v)=\int_{\mathbb{C}^n} \bar{u}v dV \quad \text{for every} \quad v\in\mathscr{X}^{-\varepsilon}(U_{k,l;m}).$$

We define

$$\mathscr{D}=\bigcup_{\lambda>0} \{v\in L^2(\mathbb{C}^n, \lambda\psi^+ +\phi), \bar{\partial}v=0\},$$

where $\phi$ and $\psi$ are strictly plurisubharmonic functions defined by

$$\phi(z)=\sum_j \{|\mathrm{Im}\, z_j|^4+6|\mathrm{Im}\, z_j|^2|\mathrm{Re}\, z_j|^2-|\mathrm{Re}\, z_j|^4\}+\log(1+\sum_j |z_j|^2)$$

and

$$\psi(z)=\sigma(z)*\varrho_\varepsilon+\varepsilon\log(1+\sum_j |z_j|^2),$$

where

$$\sigma(z)=\max\{(m^2+\kappa)|\mathrm{Im}\, z_j|^2-|\mathrm{Re}\, z_j|^2,\ 1\leqq j\leqq k,\ (m^2+\kappa)|\mathrm{Im}\, z_j|^2,\ k+1\leqq j\leqq k+l\}$$

and $\varrho_\varepsilon$ is a mollifier in $\mathbb{R}^{2n}$. Further $\psi^+(z)=\max\{0, \psi(z)\}$, and $\kappa$ and $\varepsilon>0$ are so chosen that $U_{k,l;m+\delta}\supset\{z; \psi(z)\leqq 0\}\supset\operatorname{supp} u$ holds for some $\delta>0$. Since

$$\begin{aligned}&-x^4/3+8y^4-\{-x^4+6x^2y^2+y^4+y^2+\log(1+x^2+y^2)\}\\&=(x^2/\sqrt{2}-\sqrt{6}y^2)^2+x^4/6+y^4-y^2-\log(1+x^2+y^2),\end{aligned}$$

which is not less than some constant depending on $\lambda$, $\mathscr{D}$ is contained in $\mathscr{E}$. Since $\mu$ is zero on $\mathscr{E}$ and $\operatorname{supp} u\subset U_{k,l;m+\varepsilon}$, $u$ belongs to $L^2(\mathbb{C}^n, -\phi)$ and $(\mu, v)=0$ for any $v$

in $\mathscr{D}$. Moreover $u=0$ wherever $\psi>0$. By the Proposition 2.3.2 of Hörmander [12] we have some $f$ which satisfies the following conditions:

(i) $\vartheta f = u$,

(ii) $\operatorname{supp} f \subset U_{k,l;m+\delta}$,

(iii) $f \in L^2_{(0,1)}(\mathbb{C}^n, -\phi)$,

where $\vartheta$ is the dual of the Cauchy-Riemann operator $\bar{\partial}$. With this $f$ we can easily verify a sequence of equalities

$$0 = \int_{\mathbb{C}^n} (\bar{\partial} v) \bar{f} dV = \int_{\mathbb{C}^n} v(\overline{\vartheta f}) dV = \int_{\mathbb{C}^n} v\bar{u} dV = (\mu, v)$$

valid for any $v \in \mathscr{Y}(U_{k,l;m})$. Thus we have proved that $\mu$ is zero on a dense subset of $\mathscr{X}^{-\varepsilon}(U_{k,l;m})$. Hence we conclude that $\mu$ is vanishing.

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*Note Added in Proof.* Results of the quantum field theory in terms of *pure* second type Fourier hyperfunctions are summarized in our short note which will soon appear in Lett. math. Phys.; a full paper will be published in Publ. RIMS Kyoto Univ. **12** Suppl.

## References

1. Nagamachi, S., Mugibayashi, N.: Commun. math. Phys. **46**, 119—134 (1976)
2. Osterwalder, K., Schrader, R.: Commun. math. Phys. **31**, 83—112 (1973)
3. Osterwalder, K., Schrader, R.: Commun. math. Phys. **42**, 281—305 (1975)
4. Gel'fand, I. M., Shilov, G. E.: Generalized functions Vol. 2. New York-London: Academic Press 1968
5. Kawai, T.: J. Fac. Sci. Univ. Tokyo IA **17**, 467—517 (1970)
6. Mityagin, B. S.: Trudy Moskov Mat. Obšč. **9**, 317—328 (1960)
7. Treves, F.: Topological vector spaces, distributions, and kernels. New York-London: Academic Press 1967
8. Hörmander, L.: Introduction to complex analysis in several variables. Amsterdam-New York: North-Holland/American Elsevier 1973
9. Ito, Y., Nagamachi, S.: J. Math. Tokushima Univ. **9**, 1—33 (1975)
10. See, e.g., Bogolubov, N. N., Logunov, A. A., Todorov, I. T.: Introduction to axiomatic quantum field theory. Reading: W. A. Benjamin, Inc. 1975
11. Osterwalder, K.: Euclidean Green's functions and Wightman distributions. In: Constructive quantum field theory, Lecture Notes in Physics No. 25, pp. 71—93. Berlin-Heidelberg-New York: Springer 1973
12. Hörmander, L.: Acta Math. **113**, 89—152 (1965)

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