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On the Four-Valuedness of Twistors

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Abstract. The spinors on compactified Minkowski space, in terms of which twistor theory is formulated, are really U-spinors. In this light zero-mass fields have no Grgin discontinuity.

I shall examine the spinors which are induced on compactified Minkowsky space, M^c, by twistors. The notation will follow [3], to which the reader is referred for the basic facts of twistor theory. Note in particular that I shall mainly be using *concrete* indices[1], since the abstract index notation of [4] presupposes the existence of some particular spin structure; and it is precisely this that I wish to explore.

If Z and W are two twistors with components $(Z^\alpha)=(\eta^{\mathfrak{A}}, \iota_{\mathfrak{X}'})$, $(W^\alpha)=(\xi^{\mathfrak{A}}, \sigma_{\mathfrak{X}'})$, then they determine the point $x(Z, W)$ in Minkowski space M whose components are

$$x^a = -i\sigma^{a\mathfrak{A}\mathfrak{X}'}(\eta_{\mathfrak{A}}\sigma_{\mathfrak{X}'} - \xi_{\mathfrak{A}}\iota_{\mathfrak{X}'})/\iota_{\mathfrak{Y}'}\sigma^{\mathfrak{Y}'}\,, \tag{1}$$

provided that $\iota_{\mathfrak{Y}'}\sigma^{\mathfrak{Y}'} \neq 0$. Then an element g of the twistor transformation group SU (2, 2) [5] determines a local conformal transformation $\tilde{g}$ on M by

$$\tilde{g}(x(Z, W)) = x(g(Z), g(W))\,,$$

in a domain where both sides are defined.

The two pairs of numbers which make up the components of a twistor are interpreted on M as the components of spinors with respect to a fixed coordinate basis. Not only are they related to vectors by (1), but for any Poincaré transformation $\tilde{g}$ on M one can find a g which acts on these twistor components in the way appropriate to the spinor interpretation. Moreover, this action extends to conformal transformations, under which the $\iota_{\mathfrak{X}'}$ and $\eta^{\mathfrak{A}}$ transform as the components of spinors on M of conformal weight 1 (i.e. under dilatation by a factor θ they acquire a factor θ^{-1}). Hence they are describable in terms of the conformal metric alone, and so can be defined on the image of M in M^c. However, it is well known ([3],

[1] For typographical reasons concrete twistor indices are represented by α, β etc., instead of the Hebrew of [3].

p. 258) that the extensions of these spinors to M^c, with the extension of the $\tilde{g}$ to globally defined transformations, leads to a four-valuedness.

Consider [5] the one-parameter family of transformations $g(\theta) \in SU(2,2)$ given by

$$[g(\theta)^{\alpha}{}_{\beta}] = \begin{bmatrix} e^{-i\theta}\cos 2\theta & 0 & -ie^{-i\theta}\sin 2\theta & 0 \\ 0 & e^{i\theta} & 0 & 0 \\ -ie^{-i\theta}\sin 2\theta & 0 & e^{-i\theta}\cos 2\theta & 0 \\ 0 & 0 & 0 & e^{i\theta} \end{bmatrix}$$

Then $g(\pi/2) = i \times$ identity, $\tilde{g}(\pi/2) =$ identity and the action of $\tilde{g}(\theta)$ carries the origin of coordinates in M down to $\mathscr{I}^- \equiv \mathscr{I}^+$ and back down to the origin. If we choose a pseudo-orthonormal frame at the origin it will be dragged round this path by $\tilde{g}(\theta)$, giving for each θ a conformally pseudoorthonormal frame e_θ on M^c.

On any reasonable interpretation, we should expect $g(\theta)$ to transform spinors on M^c continuously, so that, considering, say, the last two twistor components, $\lim_{\theta \nearrow \pi/2} g(\theta)_{\mathfrak{A}'}{}^{\mathfrak{B}'}\eta_{\mathfrak{B}'}$ and $\lim_{\theta \searrow \pi/2} g(\theta)_{\mathfrak{A}'}{}^{\mathfrak{B}'}\eta_{\mathfrak{B}'}$ represent the same spinor. But we cannot compare their components in the coordinate basis of M because this leads to a spinor basis in M^c which is discontinuous on $\mathscr{I}$; instead we transform to the basis e_θ.

A calculation shows that the components of the $g(\theta)$-dragged spinor $g(\theta)_{\mathfrak{A}'}{}^{\mathfrak{B}'}\eta_{\mathfrak{B}'}$ in the $g(\theta)$-frame are *constant* as θ varies from 0 to $\pi/4$, or from $\pi/4$ to $\pi/2$. Thus on return to the origin at $\theta = \pi/2$, the e_θ components are unchanged, while the coordinate-basis components (i.e. the e_0-components) have become multiplied by $+i$. If we are to extend spinors to M^c, the frame $e_{\pi/2}$ is not, as far as spinors are concerned, the same frame as e_0. Just as a frame acquires a spin-entanglement [6] on rotation through 2π, so e_θ on passing from e_0 to $e_{\pi/2}$ acquires a half-entanglement – let us call it a *spin-rotation* of π (as pointed out in [3], loc. cit.). While not allowed for spinors in the usual sense, this is permissible for U-spinors.

We recall [1] that U-spinors are defined on a space-time X by extending the bundle $L(X)$ of all pseudo-orthonormal frames to a $U\,\mathrm{Spin}_+(1,3)$-bundle $U(X)$, where $U\,\mathrm{Spin}_+(1,3) \simeq (\mathrm{Spin}_+(1,3) \times U(1))/Z_2$ (with the non-trivial factorisation); just as spinors are defined by extending $L(X)$ to a $\mathrm{Spin}_+(1,3)$-bundle $S(X)$, $\mathrm{Spin}_+(1,3) \simeq SL(2,C)$ being the covering group of the Lorentz group. If a loop in $L(X)$ is lifted to $U(X)$ it defines a transformation in $U(1)$ which, in the case of spinors on M^c belongs to the subgroup $G = \{1, i, -1, -i\}$. In this case the group of $U(X)$ reduces to $(\mathrm{Spin}_+(1,3) \times G)/Z_2$ and we have generalised spinors, as described in [2]. [Note that any generalised spinor bundle E with group

$(\mathrm{Spin}_+(1,3) \times H)/Z_2$

can be extended to a $U\,$Spin-bundle by forming $(E \times U(1))/H$. On the other hand, the two-plane bundles over S^2 with no spin structure [7] do not admit a generalised spin structure but do admit U-spinors, since their dimension-three cohomology is obviously trivial. Thus U-spinors are more general than generalised spinors.]

Finally, consider zero-mass fields on M^c. These are specified by

$$\phi_{AB\dots D}(x^a) = \oint W_A W_B \dots W_D f(W_\alpha) I^{\beta\gamma} W_\beta dW_\gamma \qquad (2)$$

where W_α is a lower-index twistor (an element of the space dual to upper-index twistors), restricted by $(W_\alpha)=(W_{\mathfrak{A}}, ix^{\mathfrak{B}\mathfrak{B}'}W_{\mathfrak{B}})$. As usual, f is of homogeneity degree $-n-2$, where n is the valence of ϕ. As before, transform to the e_θ basis by the conformal spinor transformation $S_{\theta\mathfrak{B}}{}^{\mathfrak{A}}$. Since $S_{\theta\mathfrak{B}}{}^{\mathfrak{A}}\overline{g_{\theta\mathfrak{A}}{}^{\mathfrak{C}}}=$ identity (the bar denoting complex conjugation, needed for passing from $\iota_{\mathfrak{X}'}$ to $W_{\mathfrak{A}}$), on applying $g(\theta)$ to W we have the e_θ-components given by

$$\tilde{\phi}^{\theta}_{\mathfrak{A}\mathfrak{B}\dots\mathfrak{D}}(\tilde{g}(\theta)(x^a))=\oint W_{\mathfrak{A}}W_{\mathfrak{B}}\dots W_{\mathfrak{D}}f(\overline{g(\theta)}W_\alpha)\;\times I^{\beta\alpha}\overline{g(\theta)}_\alpha{}^\gamma\overline{g(\theta)}_\beta{}^\delta W_\delta dW_\gamma$$

the contour being homologous to that in (2). Here the tilde and superscript θ on ϕ simply indicate that the components are expressed in the e_θ basis.

The discontinuity across $\mathscr{I}$ is obtained by comparing values at $\theta=-\pi/4+0$ and $\theta=\pi/4-0$ ($\mathscr{I}^+$ and $\mathscr{I}^-$, respectively). Recalling that $g(\pi/2)=ig(0)$, and hence $g(\pi/4)=ig(-\pi/4)$, and using the homogeneity of f, we find

$$\tilde{\phi}^{\pi/4}_{\mathfrak{A}\mathfrak{B}\dots\mathfrak{D}}(y^a)=i^n\tilde{\phi}^{-\pi/4}_{\mathfrak{A}\mathfrak{B}\dots\mathfrak{D}}(y^a) \tag{3}$$

where $y^a=\lim_{\theta\to\pi/4} g(\theta)x^a\in M^c$. But we already have, from considering the spin-rotation of e_θ, that

$$\tilde{\eta}^{\pi/2}_{\mathfrak{X}'}(0)=-i\tilde{\eta}^{0}_{\mathfrak{X}'}(0)\,. \tag{4}$$

Thus the spin-rotation expressed in (3) is precisely that which is expected for a continuous spinor field of the indicated type, found by conjugating (4) and taking an n-fold tensor product.

In short, there is no Grgin discontinuity.

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