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Critical Indices for Dyson's Asymptotically-Hierarchical Models

P. M. Bleher and Ya. G. Sinai

Landau, Theoretical Physics Institute, Moscow V334, URSS

Abstract. It is known that the investigation of the critical point for models of the type of Dyson's hierarchical models is reduced to the solution of some non-linear integral equation. In our previous publication the Gaussian solution was investigated. Here we construct non-Gaussian solutions of the equation and find the expressions for critical indices connected with them. Our procedure permits us to construct meaningful ε-expansions.

§ 1. Introduction

Dyson's hierarchical models or their generalization – asymptotically-hierarchical models – (a.h.m.) are of great interest because the renormalization group method in the theory of critical points by K. Wilson [3] and M. Fisher [4] becomes rigorous for such models (see [2] and the papers by Jona-Lasinio [5] and Gallavotti-Knops [6]). The investigation of critical points for a.h.m. is reduced to the solution of the corresponding nonlinear integral equation, which can be considered as an equation for the fixed point of the corresponding renormalization group. In [2, 8] a case with the Gaussian solution was investigated. It was shown that the critical indices in that case are precisely the same as predicted by the Landau semiphenomenological theory of phase transitions of the second kind. However, the Gaussian solution is stable only when the potential of interaction decreases sufficiently slowly.

In this paper we construct non-Gaussian solutions of our main integral equation. These solutions appear as bifurcations branches from the Gaussian solutions. The total number of the branches is infinite but only one of them has the necessary properties of stability to appear in general as a limit distribution for normed mean spin at the critical temperature. In the second part of this paper we find the values for critical indices corresponding to this branch. They coincide with the values found in the general theory by Wilson [3].

From the formal point of view the non-Gaussian solutions can be represented by a series of the parameter ε where ε is the deviation of the given value of the parameter from its bifurcation value. These series are always asymptotic because they describe the functions with different asymptotics at infinity. The method we apply can be regarded as a procedure which permits to make these ε-series meaningful. Roughly speaking at a given ε the formal series gives a good approximation for the solution only in the domain depending on ε. Beginning from a family of test functions we apply the transformations of the renormalization group and construct ε-expansion near every iteration. The values of ε rapidly decrease because the iterations rapidly converge to the solution which we are seeking. Therefore, ε-expansion becomes more and more exact on the increasing sequence of domains when the number of iterations tends to infinity.

Now we want to recall the definition of Dyson's hierarchical models, asymptotically-hierarchical models and to deduce the main integral equation (see [1, 2]). Let an integer $r>1$ and a positively-defined quadratic form $Q(t_1,\dots,t_r)=g((t_1+\dots+t_r)/r)^2+h(t_1^2+\dots+t_r^2)/r$ with g, h as parameters be fixed. Assume also that for any integer $n>1$, there is given a volume V_n consisting of r^n points divided into r equal subvolumes $V_{n-1,i}$, $i=1,\dots,r$. We consider a classical spin system, configurations of which can be represented as functions $u(x)$, $x\in V_n$, taking the values of ± 1. The Hamiltonian of Dyson's hierarchical model depends on a parameter c, $1<c<r$, and is defined by the following recurrence relation:

$$H_n(u)=\sum_{i=1}^r H_{n-1}(u_i)-c^rQ(s_1^{(n-1)},\dots,s_r^{(n-1)})\,. \tag{1.1}$$

Here $s_i^{(n-1)}=(1/r^{n-1})\sum_{x\in V_{n-1,i}}u(x)$ is the mean spin in the subvolume $V_{n-1,i}$ of the configuration u and u_i is the restriction of the whole configuration $u(x)$, $x\in V_n$ in the subvolume $V_{n-1,i}$.

Let us introduce $g_n(t;\beta)=\mathrm{Prob}_n\{s^{(n)}=t;\beta\}$, where Prob_n is the probability, calculated by the Gibbs distribution in the volume V_n with β as the inverse temperature, $s^{(n)}$ is the mean spin in the volume V_n, $s^{(n)}=(1/r^n)\sum_{x\in V_n}u(x)$. Then from (1.1) easily follows the system of recurrent equations for functions g_n:

$$g_n(t;\beta)=(\Xi_{n-1}^r(\beta)/\Xi_n(\beta))\sum_{(1/r)\sum_{i=1}^r t_i=t} g_{n-1}(t_1;\beta)\dots g_{n-1}(t_r;\beta)e^{\beta c^n Q(t_1,\dots,t_r)} \tag{1.2}$$

where $\Xi_\kappa(\beta)$ is the grand partition function in the volume V_k, $k\geqq 1$. The main assumption which is made at the investigation of hierarchical models, is that for $\beta=\beta_{cr}$ the typical values of the mean spin have the order $c^{-n/2}$. Making the change of coordinates $t=c^{-n/2}z$ and putting $\Delta_n=c^{n/2}r^{-n}$, $f_n(z;\beta)=g_n(z\cdot c^{-n/2};\beta)\Delta_n^{-1}$ we obtain from (1.2) the following system of recurrent equations for functions $f_n(z;\beta)$:

$$f_n(z;\beta)=L_n(\beta)\sum_{(z_1+\dots+z_r)/r=z/\sqrt{c}} f_{n-1}(z_1;\beta)\dots f_{n-1}(z_r;\beta)e^{\beta Q(z_1,\dots,z_r)}\Delta_n^{r-1} \tag{1.2'}$$

where $L_n(\beta)$ is a normed constant. From the mathematical point of view, the previous assumption is equivalent to the assumption that the functions $f_n(z;\beta)$ converge at $n\to\infty$ to a limit and the limit function $f(z;\beta)$ of continuous argument z satisfies the equation

$$f(z;\beta)=L(\beta)\int\dots\int\prod_{i=1}^r f(z_i;\beta)e^{\beta Q(z_1\dots z_r)}\delta(\Sigma z_i-rz/\sqrt{c})\prod_{i=1}^r dz_i\,. \tag{1.3}$$

The constant $L(\beta)$ is the normalization factor. Equation (1.3) is the main integral equation in the theory of hierarchical models.

It is easy to verify that (1.3) has the Gaussian solution $f(z;\beta)=\sqrt{a_0(\beta)/\pi}e^{-a_0(\beta)z^2}$ with $a_0(\beta)=((g+h)/(r-c))\beta$. General solutions of (1.3) for different β are related to each other via the equality $f(z;\beta_1)=\sqrt{\beta_2/\beta_1}f(z\sqrt{\beta_2/\beta_1};\beta_2)$. Therefore, it is sufficient to consider (1.3) with $\beta=1$.

If the relations (1.2) are valid for $n \geqq n_0$ and the family of initial probability distributions $g_{n_0}(t;\beta)$ is arbitrary then the corresponding model is called asymptotically-hierarchical model. Here t takes the values from $-r^{n_0}$ to r^{n_0} and all probabilities $g_{n_0}(t;\beta)$ are defined for some closed interval $[\beta^-, \beta^+]$ and are C^1-functions of β. For any fixed interval $[\beta^-, \beta^+]$ there is a natural topology in the space of such distributions $\{g_{n_0}(t;\beta), \beta \in [\beta^-, \beta^+]\}$.

Definition 1. The solution $f(z;\beta)$, $0<\beta<\infty$, of (1.3) is called thermodynamically-stable if there exists an integer n_0 and a closed interval $[\beta^-, \beta^+]$ for which one can find an open set Ω in the space of families of probability distributions $\{g_{n_0}(t;\beta), \beta \in [\beta^-, \beta^+]\}$ such that for any family $\{g_{n_0}(t;\beta), \beta \in [\beta^-, \beta^+]\} \in \Omega$ there exists one and only one $\beta^* \in [\beta^-, \beta^+]$ for which $f_{n_0}(z;\beta^*)$ converge weakly to $f(z;\beta^*)$.

One of the main results of [2, 8] is that the Gaussian solution is thermodynamically stable for $\sqrt{r}<c<r$ and for $c<\sqrt{r}$ it is unstable. Therefore, for $c<\sqrt{r}$ it is necessary to construct non-Gaussian solutions of (1.3).

Let $c_k = r^{1/(k+1)}$, $k=1,2,\ldots$, $\varepsilon = c_k - c$. The following theorem is the main result of this paper.

Theorem 1. *For any $k=1,2,\ldots$ one can find $\delta_k>0$ such that for any ε, $0<\varepsilon \leqq \delta_k$ there exists a normed solution $f_\varepsilon(z)$ of the equation*

$$f_\varepsilon(z) = L_\varepsilon \int \ldots \int f_\varepsilon(z_1)\ldots f_\varepsilon(z_r) e^{Q(z_1,\ldots,z_r)} \delta\left(\textstyle\sum_{i=1}^r z_i - rz/\sqrt{c}\right) \prod_{i=1}^r dz_i\,. \tag{1.4}$$

For this solution $0<f_\varepsilon(z) \leqq 2\sqrt{a_0/\pi}\exp[-(a_0 z^2 + A_0 \varepsilon |z|^\alpha)]$, $a_0 = (h+g)/(r-c)$, $A_0 = A_0(k)$, α is the root of the equation $c^\alpha = r$. These solutions $f_\varepsilon(z)$ continuously depend on ε for any fixed z. ┘

It is possible to show that the branches f_ε for $k>1$ are thermodynamically unstable. The branch f_ε for $k=1$ is thermodynamically stable (see § 8 of this paper).

Theorem 1 gives the existence of the solution of (1.3) for c sufficiently close to $r^{1/2}$. In [10] this branch was investigated on computers for $r=2$ (see also Appendix 2 below). The results of [10] doubtlessly show that there is no other bifurcations for $1<c<\sqrt{2}$.

During the proof we discuss in detail only the case $r=2$ and $Q=(t_1+t_2)^2$ which corresponds to Dyson's hierarchical model. The general case can be treated by obvious modifications. The reader can easily notice the similarity between the methods of this paper and papers [2, 8].

§ 2. The Idea of the Proof of Theorem 1

For $r=2$ and $Q(t_1,t_2)=(t_1+t_2)^2$ Eq. (1.3) takes the form:

$$f(z;\beta) = L_c e^{\beta z^2} \int_{-\infty}^{\infty} f(z/\sqrt{c}+u;\beta) f(z/\sqrt{c}-u;\beta)\,du\,.$$

The substitution $f(z;\beta) = f_1(z;\beta)\exp(-a_0(\beta)z^2)$, $a_0(\beta) = \beta c/(2-c)$ reduces the latter equation to the equation

$$f(z;\beta) = L_c \int_{-\infty}^{\infty} e^{-2a_0(\beta)u^2} f(z/\sqrt{c}+u;\beta) f(z/\sqrt{c}-u;\beta)\,du\,.$$

The next step is to give up the normalization condition and to consider the equation

$$f(z)=(1/\sqrt{\pi})\int_{-\infty}^{\infty} e^{-u^2} f(z/\sqrt{c}+u) f(z/\sqrt{c}-u)du = Af. \tag{2.1}$$

After normalization of the solution of (2.1) we shall obtain the solution of the initial equation (1.3) for $\beta=(2-c)/(2c)$. As was mentioned above, the solution of (1.3) for any β can be obtained from this one by a simple change of variables.

A depends on c and (2.1) defines a family of non-linear transformations when c changes in the interval $1<c<2$. It has an obvious solution $f\equiv 1$ which corresponds to the Gaussian solution of (1.3). It is very essential that it does not depend on c. When one has a smooth family of non-linear transformations of the finite-dimensional space with a fixed point which does not depend on the parameter of the family, one should consider the family of linearized transformation near this point and find such values of the parameter for which the spectrum of the corresponding linear transformation contains 1. If the second derivative in the direction, according to the eigenvalue 1, enters the Taylor series with non-zero coefficient, then through the fixed point there passes a new branch of fixed points of transformations of our family. One can say that the initial fixed point generates new fixed points.

The procedure which is applied below, can be considered as an adaptation of the methods of the finite-dimensional case to our transformation A, acting in the infinite-dimensional functional space. The linearized operator L_1 corresponding to $f(z)\equiv 1$ takes the form

$$L_1 g(z)=(2/\sqrt{\pi})\int_{-\infty}^{\infty} e^{-u^2} g(z/\sqrt{c}-u)du\,.$$

This operator is known as the Gauss integral operator (see [11]). We consider its action in the space of even functions f. Its eigenvalues are equal to $2, 2c^{-1}, 2c^{-2}, \ldots, 2c^{-k}, \ldots$. The corresponding eigenvectors are the Hermite polynomials, which are orthogonal with the weight $\exp(-\gamma z^2)$, $\gamma=1-c^{-1}$. Thus, the critical values of c near which one can expect the appearance of new solutions have the form $c_k=2^{1/(k+1)}$, $k=1,2,\ldots$. For $c<c_k$ and close to c_k the point $f\equiv 1$ has $(k+1)$-dimensional unstable eigenspace. Accordingly, the new solution must have k-dimensional unstable eigenspace for these values of c.

Our method of construction of new solutions of (2.1) has much in common with the widely-known Hadamard-Perron theorem in the theory of smooth dynamical systems (see [12, 13]). The direct method of contracting mappings cannot be applied because we are looking for unstable solutions. The construction must begin with the construction of the stable separatrice of the solution which we are seeking. The next step is the proof that the induced mapping on the separatrice is a contraction. The first step, i.e. the construction of the separatrice is usually taken in the following way. One takes a k-dimensional manifold which is close in a natural sense to the unstable subspace and finds its intersection with the separatrice. This intersection lies in one point. This point is determined by the property that all its images lie in a small region of the fixed point.

Our procedure is similar to the above process. However, we do not construct the whole separatrice but take a special k-dimensional family of test functions, find one point of this family which lies on the separatrice and prove that it converges to the solution which we are seeking.

§ 3. Properties of Operators L_f

The differential L_f of the non-linear transformation A in an arbitrary point f has the form

$$L_f g(z)=(2/\sqrt{\pi})\int_{-\infty}^{\infty} e^{-u^2} f(z/\sqrt{c}-u)\, g(z/\sqrt{c}+u)\, du\,. \tag{3.1}$$

In this section we shall investigate several largest eigenvalues and eigenvectors of the linear operator L_f when the function f is sufficiently close to 1.

Roughly speaking we shall prove that in the case under consideration the formulae of the perturbation theory are applicable. One cannot hope that the series of the perturbation theory converge because the spectrum of non-perturbed operator when $f\equiv 1$ consists of numbers 2, $2c^{-1}$, $2c^{-2}, \ldots$ and tends to zero. However, we shall show that when the perturbation has the order ε in the appropriate norm the difference of n-th eigenvectors for perturbed and unperturbed operators is no more than $\varepsilon^{0,8}$ if ε is sufficiently small and n is fixed.

The consideration of this section will not be used below. The reader may acquaint himself with the formulation of Theorem 3.1. and proceed to the next section.

Now we are going to formulate the exact condition concerning a perturbation and to give the formulation of the theorem. Let ε, $0<\varepsilon<1$, be a certain number and

$$D_\varepsilon=[-d_0\sqrt{\ln(1/\varepsilon)},\, d_0\sqrt{\ln(1/\varepsilon)}],\; d_0=10/(c-1)\,.$$

Assume that there is given an even function $f(z)\in C^1(R^1)$ such that for $z\in D_\varepsilon$ the function f can be written in the form

$$f(z)=1-\varepsilon G_{2k}(\sqrt{\gamma}z)+R(z) \tag{3.2}$$

where $G_{2k}(z)$ is the $2k$-th Hermite polynomial (see [11]), $\gamma=1-c^{-1}$ and

$$|R(z)|,\, |dR(z)/dz|<\varepsilon^{5/3}\,. \tag{3.3}$$

For $z\notin D_\varepsilon$ the function f satisfies the estimates

$$f(z)<\exp(-(\varepsilon'/2)|z|^\alpha),\; \alpha=2(\log_2 c)^{-1}\,, \tag{3.4}$$

$\varepsilon'=\varepsilon\cdot p_0$, where p_0 is the $2k$-th coefficient of $G_{2k}(\sqrt{\gamma}\,z)$,

$$|df(z)/dz|<|z|^\alpha \exp(-(\varepsilon'/2)|z|^\alpha)\,. \tag{3.5}$$

Let us denote the Hilbert space of even functions on the line which have an integrable square with respect to the weight $\exp(-\gamma z^2)$ by $L^2_{\mathrm{ev}}(R^1;\exp(-\gamma z^2))$.

Theorem 3.1. *Let N be fixed. Then there exists a number $\varepsilon_0=\varepsilon_0(N)$ such that for any function f satisfying* (3.2)–(3.5) *with ε, $0<\varepsilon<\varepsilon_0$, the operator L_f has* $(N+1)$ *eigenvectors $e_0(z;f),\ldots,e_N(z;f)$ and accordingly, eigenvalues $\lambda_0,\ldots,\lambda_N$ such that*

a$_1$) $|\lambda_i-2/c^i|\leqq\varepsilon^{4/5},\, i=0,\ldots,N$;

$|\lambda_k-2/c^k-2\varepsilon\int_{-\infty}^{\infty}\exp(-\gamma z^2)G_{2k}(\sqrt{\gamma}\,z)AG_{2k}(\sqrt{\gamma}\,z)dz|\leqq\varepsilon^{\frac{3}{2}}$;

a$_2$) $\|e_i(z;f)-G_{2i}(\sqrt{\gamma}\,z)\|_{C^1(D_\varepsilon)}\leqq\varepsilon^{4/5}$; $\quad i=0,\ldots,N$;

a$_3$) $|e_i(z;f)|\leqq|z|^{2i+1}\exp(-(\varepsilon'/2)|z|^\alpha)$,

$|de_i(z;f)/dz|\leqq|z|^{2i}\exp(-(\varepsilon'/2)|z|^\alpha)$,

for $z\notin D_\varepsilon$; $i=0,\ldots N$;

a$_4$) *in the Hilbert space* $L^2_{ev}(R^1; \exp(-\gamma z^2))$ *there exists the closed subspace* $H_{f,N}$ *of the co-dimension* $(N+1)$ *invariant under* L_f *and such that*

$$\|L_f\|_{H_{f,N}} \leqq 2c^{-N-\frac{1}{2}}$$

$$\operatorname{dist}(H_{f,N}, L^2_N(R^1; \exp(-\gamma z^2))) \leqq \varepsilon^{4/5}$$

where $L^2_N(R^1; \exp(-\gamma z^2))$ *is the subspace of the Hilbert space* $L^2_{ev}(R^1; \exp(-\gamma z^2))$ *generated by the Hermite polynomials* $G_{2i}(\sqrt{\gamma}\, z)$, $i > N$. ┘

The proof of the theorem will be divided into several lemmas.

Lemma 3.1. *If the function* $f(z)$ *satisfies the condition* (3.2)–(3.5) *and* $\varepsilon > 0$ *is sufficiently small, then*

$$\|L_f - L_1\|_{L^2(R^1; \exp(-\gamma z^2))} \leqq \varepsilon^{31/32},$$

$$\|L_f - L_{1-\varepsilon G_{2k}}\|_{L^2(R^1; \exp(-\gamma z^2))} \leqq \varepsilon^{\frac{3}{2}}.$$

Proof. We have from (3.1) and (3.5)

$$(L_f - L_{1-\varepsilon G_{2k}})g(z) = (2/\sqrt{\pi})\int_{-\infty}^{\infty} \exp(-u^2) g(z/\sqrt{c} - u) R(z/\sqrt{c} + u) du$$

$$= (2/\sqrt{\pi})\int_{-\infty}^{\infty} \exp[-(u - z/\sqrt{c})^2] R(2z/\sqrt{c} - u) g(u) du = \int_{-\infty}^{\infty} K(z,u) g(u) du = Kg,$$

where

$$K(z,u) = (2/\sqrt{\pi}) \exp[-(u - z/\sqrt{c})^2]\, R(2z/\sqrt{c} - u).$$

Moreover

$$\|K\|_{L^2(R^1; \exp(-\gamma z^2))} = \|K_0\|_{L^2(R^1)}, \tag{3.6}$$

where

$$K_0(z,u) = K(z,u)\exp[-(\gamma/2)(z^2 - u^2)] = (2/\sqrt{\pi})\exp[-Q(z,u)] R(2z/\sqrt{c} - u),$$

$$Q(z,u) = (z/\sqrt{c} - u)^2 + (\gamma/2)(z^2 - u^2) = (1/\sqrt{c})(z-u)^2 + (1/2 + 1/(2c) - 1/\sqrt{c})$$

$$\cdot (z^2 + u^2) \geqq \tfrac{1}{2}(1 - 1/\sqrt{c})^2 (z^2 + u^2) = \alpha_0 (z^2 + u^2) > 0. \tag{3.7}$$

We shall show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |K_0(z,u)|^2 dz du < \varepsilon^3. \tag{3.8}$$

Let $\Omega_\varepsilon = \{\sqrt{z^2 + u^2} \leqq (d_0/3)\sqrt{\ln(1/\varepsilon)}\}$. For $(z,u) \in \Omega_\varepsilon$ the point $2z/\sqrt{c} - u \in D_\varepsilon$ and thus [see (3.3)]

$$\iint_{\Omega_\varepsilon} |K_0(z,u)|^2 dz du = \iint_{\Omega_\varepsilon} [(2/\sqrt{\pi})\exp(-Q(z,u)) R(2z/\sqrt{c} - u)]^2 dz du$$

$$\leqq (4/\pi)\iint_{\Omega_\varepsilon} \varepsilon^{10/3} \exp(-2\alpha_0(z^2 + u^2)) dz du \leqq \text{const}\,\varepsilon^{10/3} < \tfrac{1}{2}\varepsilon^3 \tag{3.9}$$

for a sufficiently small ε. From (3.2)–(3.4) it follows that the inequality

$$|R(z)| < 1 + z^{4k} \tag{3.10}$$

is valid for all $z \in R^1$. Therefore $|R(2z/\sqrt{c} - u)| \leqq 1 + 4(z^2 + u^2)^{2k}$ and for sufficiently small ε

$$\iint_{R^2 \setminus \Omega_\varepsilon} |K_0(z,u)|^2 dz du \leqq (4/\pi)\iint_{R^2 \setminus \Omega_\varepsilon} \exp[-2\alpha_0(z^2 + u^2)]$$

$$(1 + 4(z^2 + u^2)^{2k})^2 dz du$$

$$\leqq \text{const}\int_{(d_0/3)\sqrt{\ln(1/\varepsilon)}}^{\infty} \exp(-2\alpha_0 \varrho^2)(1 + 4\varrho^{4k})^2 d\varrho \leqq \text{const}\,\varepsilon^{2\alpha(d_0/3)^2}(\ln^{4k}\varepsilon)$$

$$\leqq \text{const}\,\varepsilon^4 \ln^{4k}\varepsilon < \tfrac{1}{2}\varepsilon^3.$$

Thus (3.8) is proved.

From the Schwartz inequality we have

$$\|K_0\|_{L^2(R^1)} \leqq \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |K_0(z,u)|^2 dz du}\,.$$

Then from this inequality, (3.6) and (3.8) one can easily derive

$$\|K\|_{L^2(R^1;\exp(-\gamma z^2))} = \|L_f - L_{1-\varepsilon G_{2k}}\|_{L^2(R^1;\exp(-\gamma z^2))} \leqq \varepsilon^{\frac{3}{2}}\,.$$

The second inequality of Lemma 3.1 is proved. The first one is obtained in a similar way. Thus Lemma 3.1 is proved.

Similar considerations lead to the proof of the following lemma.

Lemma 3.2. *Under the conditions of Lemma 3.1*

$$\|e^{-\gamma z^2}(L_f - L_1)g(z)\|_{C^1(R^1)} \leqq \varepsilon^{31/32}\|g(z)\|_{L^2(R^1;\exp(-\gamma z^2))}\,,$$

$$\|e^{-\gamma z^2}(L_f - L_{1-\varepsilon G_{2k}})g(z)\|_{C^1(R^1)} \leqq \varepsilon^{\frac{3}{2}}\|g(z)\|_{L^2(R^1;\exp(-\gamma z^2))}\,. \quad \lrcorner$$

We shall omit the proof of Lemma 3.2. Up to the end of this section we shall write $\|g(z)\|$ instead of $\|g(z)\|_{L^2(R^1;\exp(-\gamma z^2))}$.

Lemma 3.3. *Under the conditions of Lemma 3.1 the operator L_f has the main eigenfunction $e_0(z;f)$ with eigenvalue $\lambda_0(f)$ such that*

$$|2-\lambda_0(f)| < \varepsilon^{15/16}\,;$$

$$\|e_0(z;f) - 1\| < \varepsilon^{15/16}\,.$$

Proof. We shall use the method of the contraction mappings. Let us denote

$$S = \{f(z) : \|f(z)\| = 1\}\,,$$

$$S_\delta = \{f(z) : f(z) \in S,\ \|f(z) - 1\| < \delta\}$$

and consider the non-linear mapping $U_f : g(z) \to \|L_f g\|^{-1} L_f g(z)$, $U_f : S \to S$, $\delta = \frac{1}{2}\varepsilon^{15/16}$ and ε be sufficiently small. We shall show that $U_f S_\delta \subseteqq S_\delta$ and $U_f|S_\delta$ is a contraction mapping.

Let U_1 be the mapping U_f corresponding to the function $f \equiv 1$ and D_g be the differential of this mapping at the point g.

It is easy to see that the spectrum of the operator D_1 consists of the numbers $c^{-1}, c^{-2}, c^{-3}, \ldots$ and D_1 is selfadjoint. So $\|D_1\| = c^{-1} < 1$ and D_1 is a contraction operator. Hence, we deduce that the differentials D_g of the operator U_1 at the points g close to 1, namely at the points $g \in S_\delta$, are contraction operators and then we deduce that the differentials $D_g U_f$, $g \in S_\delta$ are also contraction operators. That means that $U_f|_{S_\delta}$ is a non-linear contraction operator. Moreover, it follows from our considerations that for $g_1, g_2 \in S_\delta$

$$\|U_f(g_1) - U_f(g_2)\| \leqq \tfrac{1}{2}(1 + c^{-1})\|g_1 - g_2\|\,,$$

if ε is sufficiently small. Furthermore, due to the evident estimate

$$\|U_f(1) - 1\| < \varepsilon^{31/32}$$

the latter inequality implies $U_f : S_\delta \to S_\delta$.

Thus, the mapping $U_f : S_\delta \to S_\delta$ is contractive and therefore there exists a fixed point $e_0(z;f)$ of this mapping. It is evident that the function $e_0(z;f)$ is the eigenfunction of the operator L_f:

$$L_f e_0 = \lambda_0 e_0\,.$$

Then from the inequalities $\|L_f - L_1\| < \varepsilon^{31/32}$, $\|e_0 - 1\| < \frac{2}{3}\varepsilon^{15/16}$ we obtain the estimate $|\lambda_0 - 2| < \varepsilon^{15/16}$. Q.E.D.

Lemma 3.4. *For $z, u \in R^1$ and $\alpha \geqq 2$*

$$|z+u|^\alpha + |z-u|^\alpha \geqq 2|z|^\alpha + u^2|z|^{\alpha-2}\alpha(\alpha-1)/2\,.$$

Proof. Let us divide the both sides of the inequality into $|z|^\alpha$ (for $z=0$ the inequality is evident) and denote $uz^{-1}=d$:

$$|1+d|^\alpha + |1-d|^\alpha \geqq 2 + (\alpha(\alpha-1)/2)d^2\,.$$

Without losing generality we can consider $d \geqq 0$, $d \neq 1$. Let $F(d)=|1+d|^\alpha+|1-d|^\alpha - 2-(\alpha(\alpha-1)/2)d^2$. Then

$$F''(d)=\alpha(\alpha-1)(|1+d|^{\alpha-2}+|1-d|^{\alpha-2}-1)>0$$

and $F'(0)=0$, therefore, $F'(d)>0$ for $d>0$. Then $F(0)=0$ and thus, the inequality $F'(d)>0$ for $d>0$ implies the inequality $F(d)>0$ for $d>0$. Q.E.D.

Proof of Theorem 3.1. We have

$$\|L_f^* - L_1\| = \|(L_f - L_1)^*\| = \|L_f - L_1\| < \varepsilon^{31/32}\,.$$

Therefore, from proof of Lemma 3.3 it follows that there exists the main eigenfunction $e_0^*(z;f)$ of the operator L_f^* and $\|e_0^*(z;f)-1\| < \varepsilon^{15/16}$. The hyperplane H_0 which is orthogonal to the function $e_0^*(z;f)$, is invariant with respect to the operator L_f. Using the method of contraction mappings (see Lemma 3.3) in the hyperplane H_0 we shall prove the existence of the eigenfunction $e_1(z;f) \in H_0$ close to $e_1(z;1)=G_2(\sqrt{\gamma}z)$. Then we shall prove the existence $e_1^*(z;f)$ etc. As a result, $N+1$ eigenfunctions $e_0(z;f), \ldots, e_N(z;f)$ and eigenvalues $\lambda_0, \ldots, \lambda_N$ of the operator L_f will be constructed. Besides, the following inequalities are true for $i=0,1,\ldots,N$

$$\|e_i(z;f) - G_{2i}(\sqrt{\gamma}z)\| < \varepsilon^{7/8}\,,$$
$$|\lambda_i - 2c^{-i}| < \varepsilon^{7/8}$$

and at the end we shall construct the subspace $H_{f,N} \subset L^2_{\mathrm{ev}}(R^1; \exp(-\gamma z^2))$, satisfying the condition a_4) of Theorem 3.1.

Let us now prove that $|\lambda_k - 2/c^k - \varepsilon(G_{2k}, L_{G_{2k}}G_{2k})| < \varepsilon^{\frac{3}{2}}$. For this we must find the eigenfunction $e_k(z;f)$ using the perturbation theory up to the terms of order ε included. We have

$$L_f G_{2k} = (L_1 + L_{-\varepsilon G_{2k}} + L_R)G_{2k} = (2/c^k)G_{2k} - \varepsilon L_{G_{2k}}G_{2k} + O(\varepsilon^{8/5})\,;$$

let $e_k = G_{2k} + \varepsilon\varphi$, $\lambda_k = 2c^{-k} + \varepsilon l$. Then, in the formula $L_f e_k = \lambda_k e_k$ equating all the terms of order ε, we obtain

$$-L_{G_{2k}}G_{2k} + L_1\varphi = 2c^{-k}\varphi + lG_{2k}\,,$$
$$\varphi(z) = \psi(z) + \alpha G_{2k}(\sqrt{\gamma}z),\ \psi(z) \perp G_{2k}(\sqrt{\gamma}z)\,,$$
$$-L_{G_{2k}}G_{2k} = (2c^{-k} - L_1)\psi + lG_{2k},\ l = -(L_{G_{2k}}G_{2k}, G_{2k})\,,$$
$$\psi = (2c^{-k} - L_1)^{-1}(-L_{G_{2k}}G_{2k} - lG_{2k})\,.$$

The function $L_{G_{2k}}G_{2k}$ is the polynomial of $4k$ degree and therefore it is easy to find ψ from the latter equality (it is also the polynomial of $4k$ degree).

Thus, the function φ is found and

$$\|L_f(G_{2k}+\varepsilon\varphi)-\lambda_k(G_{2k}+\varepsilon\varphi)\|=O(\varepsilon^{5/3}).$$

Then, using the method of contraction mappings we prove the estimates

$$\|e_k-G_{2k}-\varepsilon\varphi\|<\varepsilon^{\frac{3}{2}}$$
$$|\lambda_k-2c^{-k}-\varepsilon(G_{2k},L_{G_{2k}}G_{2k})|<\varepsilon^{\frac{3}{2}}$$

if ε is sufficiently small. Thus, the condition a_1) of Theorem 3.1 is proved.

Let us now prove a_3). It is necessary to point out that all the previous considerations were of a general character and they are applied to various problems of the perturbation theory. The proof of the conditions a_2) and a_3) is based on the nature of the perturbation of the main operator, reflected by conditions (3.2)–(3.5).

Let us consider the function $g_0(z)=\exp(-(\varepsilon'/2)|z|^\alpha)$ and the operator $T_f=\lambda_0^{-1}L_f$. The main eigenvalue of the operator T_g is equal to 1 and others do not exceed $\frac{1}{2}(c^{-1}+1)<1$, therefore, the iterations $g_n=T_f^n g_0$ tend to the function $\mathrm{const}\, e_0(z;f)$ in the space $L^2(R^1;\exp(-\gamma z^2))$ where $\mathrm{const}\approx 1$. In reality there takes place the convergence in C^1 on compacts because T_f is an integral operator with a smooth kernel.

More precisely from Lemma 3.2 it follows that

$$\|\exp(-\gamma z^2)(g_{n+1}(z)-g_n(z))\|_{C^1(R^1)}\leqq \mathrm{const}\|g_n-g_{n-1}\|.$$

The following estimate is evident from the definition of the function g_0:

$$\|g_1-g_0\|\leqq\varepsilon^{15/16}.$$

Besides, due to the inequality

$$\|g_{n+1}-g_n\|\leqq\tfrac{1}{2}(c^{-1}+1)\|g_n-g_{n-1}\|,$$

we have

$$\|g_{n+1}-g_n\|\leqq[\tfrac{1}{2}(c^{-1}+1)]^n\varepsilon^{15/16}$$

and

$$\|\exp(-\gamma z^2)(g_{n+1}-g_n)\|_{C^1(R^1)}\leqq\mathrm{const}[\tfrac{1}{2}(c^{-1}+1)]^n\varepsilon^{15/16}.$$

Let $D_\varepsilon^{(0)}=[-d_\varepsilon^{(0)},d_\varepsilon^{(0)}]$, where $d_\varepsilon^{(0)}=0{,}01(1-c^{-\frac{1}{2}})^{-1}\sqrt{\ln(1/\varepsilon)}$. It follows from the latter estimate that

$$\|g_n\|_{C^1(D_\varepsilon^{(0)})}\leqq 1+\mathrm{const}\,\varepsilon^{15/16}.$$

Therefore the inequality

$$g_n(z)\leqq(2+|z|^{\frac{1}{2}})\exp(-(\varepsilon'/2)|z|^\alpha) \tag{3.11}$$

is fulfilled for all n in any case for $z\in D_\varepsilon^{(0)}$. It is evident that the function $g_0(z)$ satisfies this inequality for all $z\in R^1$.

Now let us assume that the function $g_n(z)$ satisfies inequality (3.11) for all $z\in R^1$ and prove that the function $g_{n+1}(z)$ satisfies this inequality for $z\in R^1\setminus D_\varepsilon^{(0)}$.

The following inequality results from properties (3.2)–(3.4) of the function $f(z)$:

$$f(z)<(1+\chi_D(z))\exp(-(\varepsilon'/2)|z|^\alpha),$$

where $\chi_D(z)$ is an indicator of the interval $D=[-d,d]$, containing all zeroes of the polynomial $G_{2k}(\sqrt{\gamma}z)$. It is very important to point out that D does not depend on ε. From here

$$g_{n+1}(z) \leqq (2/\lambda_0\sqrt{\pi})\int_{-\infty}^{\infty} e^{-u^2}(1+\chi_D(z/\sqrt{c}-u))e^{-(\varepsilon'/2)|z/\sqrt{c}-u|^\alpha} \cdot (2+|z/\sqrt{c}+u|^{\frac{1}{2}})e^{-(\varepsilon'/2)|z/\sqrt{c}+u|^\alpha}du.$$

Let us now use Lemma 3.4. As $2=c^{\alpha/2}$

$$g_{n+1}(z) \leqq \exp(-(\varepsilon'/2)|z|^\alpha)(2/\lambda_0\sqrt{\pi})\int_{-\infty}^{\infty} e^{-u^2}(1+\chi_D(z/\sqrt{c}-u))\cdot(2+|z/\sqrt{c}+u|^{\frac{1}{2}})du.$$

It is easy to verify that the main contribution into the right-hand side of this inequality is made by the item

$$S_0=\exp(-(\varepsilon'/2)|z|^\alpha)(2/\lambda_0\sqrt{\pi})\int_{-\infty}^{\infty} e^{-u^2}|z/\sqrt{c}+u|^{\frac{1}{2}}du, \qquad z\notin D_\varepsilon^{(0)}.$$

From the inequality $|2-\lambda_0|<\varepsilon^{15/16}$ we have $2/(\lambda_0\sqrt{c})<1-(c-1)/8$ and thus for $z\notin D_\varepsilon^{(0)}$

$$S_0 \leqq |z|^{\frac{1}{2}}(1-(c-1)/10)\exp(-(\varepsilon'/2|z|^\alpha).$$

So we have proved inequality (3.11) for the function g_{n+1} for $z\notin D_\varepsilon^{(0)}$. As we have established it for $z\in D_\varepsilon^{(0)}$ too, inequality (3.11) is valid for all $z\in R^1$. That means that the first inequality of condition a_3) of Theorem 3.1 is proved for the eigenfunction $e_0(z;f)$. The second one is deduced similarly and its proof is omitted.

Now we shall sketch the proof of conditions a_3) for all the other eigenfunctions. Let us consider such μ,

$$\mu<\text{const}\,\varepsilon^{15/16}$$

that the function $g_0(z)=G_2(\sqrt{\gamma}z)\exp(-(\varepsilon'/2)|z^\alpha|)+\mu e_0(z;f)$ belongs to the hyperplane H_0 which is orthogonal to $e_0^*(z;f)$. Then the iterations $g_n=T_f^n g_0$, where $T_f=\lambda_1^{-1}A_f$, converge with $e_1(z;f)$ and, besides, the function $g_0(z)$ satisfies the inequality

$$|g_0(z)|<(\text{const}+|z|^{5/2})(\exp(-(\varepsilon'/2)|z|^\alpha)).$$

As above it is proved by induction that all the sequent functions g_n satisfy this inequality too, therefore it is fulfilled also for the function $e_1(z;f)$. Similar considerations are true for the sequent eigenfunctions.

Let us now prove a_2). Let $g_0(z)=\exp(-(\varepsilon'/2)|z|^\alpha)$. We have established already that the iterations $g_n=T_f^n g_0$, $T_f=\lambda_0^{-1}A_f$ converge to the eigenfunction $e_0(z;f)$ in C^1 on compacts and satisfy inequality (3.11). Let us show now that there exists a sequence of numbers $\{n_i\}_{i=0}^\infty$, $n_i\to\infty_{i\to\infty}$ such that

$$\|g_{n_i}-1\|_{C^1(D_\varepsilon)} \leqq \varepsilon^{7/8}. \tag{3.12}$$

It is evident that as a result, we shall prove a_2) for the eigenfunction $e_0(z;f)$.

Let $n_0=0$, $n_1=[0,0001\ln\varepsilon^{-1}]$. Let us expand the function $h_0=g_0-1$ in the Hermite polynomials up to the order M (the value M will be indicated below):

$$h_0(z)=\sum_{j=0}^M \delta_0^{(j)}G_{2j}(\sqrt{\gamma}z)+H_0(z),$$

where $(H_0(z), G_{2j}(\sqrt{\gamma}z))=0$, $j=0,1,\dots,M$. Let us denote

$$r(z)=-\varepsilon G_{2k}(\sqrt{\gamma}z)+R(z),\ T_f=T_1+T_r$$

and write

$$g_1 = T_f g_0 = T_1 g_0 + T_r g_0 = T_1 1 + T_1 h_0 + T_r g_0 = 2/\lambda_0 + T_1 h_0 + T_r g_0 .$$

Hence,

$$h_1(z) = g_1(z) - 1 = (2/\lambda_0 - 1) + \sum_{j=0}^{M} (2/(\lambda_0 c^j))\delta_0^{(j)} G_{2j}(\sqrt{\gamma} z)$$
$$+ T_1 H_0^{(z)} + T_r g_0(z) = \sum_{j=0}^{M} \delta_1^{(j)} G_{2j}(\sqrt{\gamma} z) + H_1(z) + S_1(z)$$

where

$$\delta_1^{(0)} = (2/\lambda_0 - 1)\pi^{\frac{1}{4}}\gamma^{-\frac{1}{2}} + (2/\lambda_0)\delta_0^{(0)} , \tag{3.13}$$

$$\delta_1^{(j)} = (2/(\lambda_0 c^j))\delta_0^{(j)}, \; j = 1, \dots, M , \tag{3.14}$$

$$H_1 = T_1 H_0 , \tag{3.15}$$

$$S_1 = T_r g_0 . \tag{3.16}$$

Analogous expansions are obtained for all the sequent functions $h_n(z) = g_n(z) - 1$ and the following estimates are true:

$$|\delta_n^{(0)}| \leqq 2^n \varepsilon^{15/16} , \tag{3.17}$$

$$|\delta_n^{(j)}| \leqq 2^n \lambda_0^{-n} c^{-jn} \varepsilon^{15/16} , \tag{3.18}$$

$$\|H_n(z)\| \leqq 2^n \lambda_0^{-n} c^{-n(M+1)} , \tag{3.19}$$

$$\|S_n(z)\| \leqq e^n \varepsilon^{31/32} . \tag{3.20}$$

It is very important that the validity of all these estimates is proved on the basis of the following properties of the function g_0:

1. g_0 satisfies the estimate (3.11),
2. $\|g_0(z) - 1\|_{C^1(D_\varepsilon)} \leqq \varepsilon^{15/16}$, (3.21)
3. $\|g_0(z) - 1\| \leqq \varepsilon^{31/32}$. (3.22)

Relations (3.17)–(3.20) are easily proved by induction. Let us now use estimates (3.17)–(3.20) for $n = n_1$. Then we receive for $z \in D_\varepsilon$ that

$$|h_{n_1}(z)| \leqq |\delta_{n_1}^{(0)}| \pi^{\frac{1}{4}}\gamma^{-\frac{1}{2}} + \sum_{j=0}^{M} |\delta_{n_1}^{(j)} G_{2j}(\sqrt{\gamma} z)| + |H_{n_1}(z)|$$
$$+ |S_{n_1}(z)| \leqq \varepsilon^{0.99} + \mathrm{const}(\ln \varepsilon^{-1})^M \varepsilon^{31/32} + \varepsilon^{31/32 - 0.001} \leqq \varepsilon^{15/16}$$

if ε is sufficiently small. We have used here the following considerations: the estimates $|\delta_{n_1}^{(0)}| \pi^{\frac{1}{4}}\gamma^{-\frac{1}{2}} \leqq \varepsilon^{0.99}$ and $\sum_{j=0}^{M} |\delta_{n_1}^{(j)} G_{2j}(\sqrt{\gamma} z)| \leqq \mathrm{const}\varepsilon^{31/32}[\ln \varepsilon^{-1})^M$ are deduced from (3.22), the estimate $|S_{n_1}(z)| < \varepsilon^{31/32 - 0.001}$ – from (3.20) and the estimate $|H_{n_1}(z)| < \varepsilon$ is proved in the following way

$$\|H_{n_1}(z)\|_{C^1(D_\varepsilon)} \leqq \varepsilon^{-100} \|\exp(-\gamma z^2) H_{n_1}\|_{C^1(R^1)}$$
$$\leqq \varepsilon^{-100} \|H_{n_1 - 1}\| \leqq \varepsilon^{-100} (2\lambda_0^{-1} c^{-M-1})^{n_1 - 1}$$

[see (3.19)]. Next $c^{-n_1} = \varepsilon^{0.0001 \ln c}$. Let us choose $M = 10^7 (\ln c)^{-1}$ (M does not depend on ε). Then it follows from the latter inequality that $\|H_{n_1}\|_{C^1(D_\varepsilon)} \leqq \varepsilon$. Q.E.D.

Thus it is proved that the function g_{n_1} satisfies the same conditions 1, 2, 3 as the function $g_0(z)$. Now in a similar way we prove that the function $g_{n_2}(z)$, $n_2 = 2n_1$ satisfies these conditions too and so on. As a result, we establish (3.12) Theorem 3.1 is proved.

§ 4. Inductive Assumptions, Formulations of Main Lemmas Proof of Theorem 1

We shall begin with some notations. Let

$$a=(G_{2k}, AG_{2k})=\int_{-\infty}^{\infty} e^{-\gamma z^2} G_{2k}(\sqrt{\gamma}z)A(G_{2k}(\sqrt{\gamma}z))dz$$

and $\varepsilon=(c_k-c)/a$. In Appendix 1 we show that $a\neq 0$. Below we consider only the case $\varepsilon>0$. All our assertions should begin with the phrase: "Let ε be sufficiently small". For this reason we shall omit it everywhere. Let us put $D_n=[-d_n, d_n]$ where $d_n=10(\ln\varepsilon^{-1}+\ln(1-\varepsilon)^{-n})^{\frac{1}{2}}/(c-1)$. We shall take $\omega>1$ which is the root of the equation $5^{\omega-1}=(1-\varepsilon)^{-1/100}$, i.e. $\omega-1\approx\varepsilon/(100\ln 5)$. Furthermore, we shall consider the sequence of integers $n_i=[\omega n_{i-1}+n_0]$, $i=1,2,\ldots$, $n_0=2\log_5\varepsilon^{-1}$. Our procedure will be slightly different for $n=n_i$ and $n_i<n<n_{i+1}$. At each step we shall deal with a family of functions $f_n(z;a)=A^n f_0(z;a)$, where a is a parameter of the family, all the values of which form the k-dimensional parallelepiped: $a=\{a_0,\ldots,a_{k-1}\}$, $|a_i|\leqq A_i^{(n)}$, $i=1,\ldots,k$. All the functions of the family are even.

Inductive Assumptions for $n=n_i$. Conditions (U_{n_i})

For $n=n_i$ the k-dimensional parallelepiped $\mathfrak{B}_{n_i}=\{a=(a_0,a_1,\ldots,a_{k-1}):|a_s|\leqq \varepsilon^{5/3}(1-\varepsilon/2)^{n_i},\ s=0,\ldots,k-1\}$, for each $a=(a_0,\ldots,a_{k-1})\in\mathfrak{B}_{n_i}$ the even function $f_{n_i}(z;a)$ is given so that

u$_1$) for some $a^{(0)}\in\mathfrak{B}_{n_i}$ the function $f_{n_i}(z;a^{(0)})=\bar{f}_i$ satisfies the conditions of Theorem 3.1; therefore the operator $L_{\bar{f}_i}$ has $N+1$ eigenfunctions $e_s(z;\bar{f}_i)=e_s^{(i)}$, $s=0,1,\ldots,N$ with eigenvalues $\lambda_s(\bar{f}_i)=\lambda_s^{(i)}$ and the invariant space $H_{\bar{f}_i,N}$; besides $|\lambda_s^{(i)}-2c^{-s}|<\varepsilon^{4/5}$, $s=0,1,\ldots,N$; the number N does not depend on ε and will be indicated below;

u$_2$) the function $g_{n_i}(z;a)=Af_{n_i}(z;a)-f_{n_i}(z;a)$, $a\in\mathfrak{B}_{n_i}$ can be represented in the form

$$g_{n_i}(z;a)=\sum_{j=0}^{k-1} a_j e_j^{(i)}(z)+\delta_{n_i}(a)e_k^{(i)}(z)+R_{n_i}(z;a)$$

here the function $R_{n_i}(z;a)$ being expanded on the subspace $H_{\bar{f}_i,K}$ and one-dimensional subspaces generated by $e_s^{(i)}$, $s=0,1,\ldots,K$ has zero projections on these one-dimensional subspaces; for $z\in D_{n_i}$

u$_{21}$) $|\delta_{n_i}(a)|<2\gamma^{-1/2}\varepsilon^{8/3}(1-2\varepsilon/3)^{n_i}$; $|\nabla_a\delta_{n_i}(a)|<2\gamma^{-1/2}\varepsilon^{5/2}(1-2\varepsilon/3)^{n_i}$;

u$_{22}$) $|R_{n_i}(z;a)|<\varepsilon^{8/3}(1-2\varepsilon/3)^{n_i}$;

$|\partial R_{n_i}(z;a)/\partial z|<\varepsilon^{5/2}(1-2\varepsilon/3)^{n_i}$;

$|\nabla_a R_{n_i}(z;a)|<\varepsilon^{5/2}(1-2\varepsilon/3)^{n_i}$;

u$_{23}$) $|\partial g_{n_i}(z;a)/\partial z|<\varepsilon^{5/2}(1-3\varepsilon/5)^{n_i}$;

$|\partial g_{n_i}(z;a)/\partial a_j|-e_j(z;\bar{f}_i)|<\varepsilon^{5/2}(1-3\varepsilon/5)^{n_i}$; $\quad j=0,\ldots,k-1$;

u$_3$) for $z\notin D_{n_i}$

$$0\leqq f_{n_i}(z;a)\leqq\exp(-(\varepsilon'/2)|z|^{\alpha}),$$

where $\alpha=2(\log_2 c)^{-1}$ and is sufficiently close to $2k$ for small ε;

$|\partial f_{n_i}(z;a)/\partial z|\leqq(1+|z|^{2k})\exp(-(\varepsilon'/2)|z|^{\alpha})$,

$|\nabla_a f_{n_i}(z;a)|\leqq(1+|z|^{2k+1})\exp(-(\varepsilon'/2)|z|^{\alpha})$,

$|\nabla_a\partial f_{n_i}(z;a)/\partial z|\leqq(1+|z|^{4k+1})\exp(-(\varepsilon'/2)|z|^{\alpha})$.

Inductive Assumptions for $n_i<n\leqq n_{i+1}$. Conditions (V_n)

Let a k-dimensional parallelepiped $\mathfrak{A}_n=\{a=(a_0,\ldots,a_{k-1})$: $|a_i|\leqq\frac{1}{2}\varepsilon^{5/3}(1-\varepsilon/2)^n\cdot c^{(n-n_i)/10}$, $i=0,1,\ldots,k-1\}$ and for each $a\in\mathfrak{A}_n$ an even function $f_n(z;a)$ be given. We put $g_n(z;a)=Af_n(z;a)-f_n(z,a)$ and denote

$$v_k=1-3\varepsilon/4\,,\qquad v_j=\lambda_j+\varepsilon^{\frac{1}{2}}<c^{-2(j-k)/3}\,,\qquad k<j\leqq N+1$$

where λ_j are eigenvalues of the operator $L_{\bar f_i}$ acting in the Hilbert space $L^2_{\mathrm{ev}}(R^1;\exp(-\gamma z^2))$ of even square-integrable functions with the weight $\exp(-\gamma z^2)$, $\gamma=1-c^{-1}$. Then the family $\{g_n(z;a),\ a\in\mathfrak{A}_n\}$ satisfies the conditions:

v_1) for $z\in D_n$

$$g_n(z;a)=\textstyle\sum_{s=0}^{k-1}a_s e_s^{(i)}(z)+\sum_{s=k}^{N}\delta_n^{(s)}(a)e_s^{(i)}(z)+h_n(z;a)+t_n(z;a)$$

where N is the same number as in the conditions (U_{n_i}) and will be indicated below and

v_{11}) $|\delta_n^{(s)}(a)|\leqq 2\gamma^{-\frac{1}{2}}\varepsilon^{8/3}(1-2\varepsilon/3)^{n_i}v_s^{n-n_i}$, $\quad s=k,\ldots,N$;

$|\nabla_a\delta_n^{(s)}(a)|\leqq 2\gamma^{-\frac{1}{2}}\varepsilon^{5/2}(1-2\varepsilon/3)^{n_i}v_s^{n-n_i}$, $\quad s=k,\ldots,N$;

v_{12}) $h_n(z;a)=0$ for $z\notin D_n$; $\quad h_n(z;a)\in H_{\bar f_i}$,

$\|h_n(z;a)\|_{L^2(R^1;\exp(-\gamma z^2))}<2\gamma^{-\frac{1}{2}}\varepsilon^{8/3}(1-2\varepsilon/3)^{n_i}v_{N+1}^{n-n_i}$

$\|\,|h_n(z;a)|+|\nabla_a h_n(z;a)|\,\|_{L^2(R^1;\exp(-\gamma z^2))}\leqq\varepsilon^{7/3}(1-2\varepsilon/3)^{n_i}v_{N+1}^{n-n_i}$,

$\|h_n(z;a)\|_{C^m(D_n)}+\|\nabla_a h_n(z;a)\|_{C^m(D_n)}<L_m^{(0)}\varepsilon^{7/3}(1-2\varepsilon/3)^{n_i}3^{n-n_i}$,

$m=0,1$, $L_m^{(0)}=\mathrm{const}$

v_{13}) $\|t_n(z;a)\|_{C(D_n)}\leqq\varepsilon^3(1-2\varepsilon/3)^{n_i+1}3^{n-n_i+1}$,

$$\left\|\,|\nabla_a t_n(z;a)|+\left|\frac{\partial}{\partial z}t_n(z;a)\right|+\left|\nabla_a\frac{\partial}{\partial z}t_n(z;a)\right|\right\|_{C(D_n)}<\varepsilon^{5/2}(1-3\varepsilon/5)^{n_i+1}3^{n-n_i}$$

v_{14}) $\|g_n(z;a)\|_{C(D_n)}<\varepsilon^{8/3}(1-2\varepsilon/3)^{n}3^{n-n_i+1}$

$$\left\|\frac{\partial}{\partial z}g_n(z;a)\right\|_{C(D_n)},\ \left\|\frac{\partial}{\partial a_i}g_n(z;a)-e_j(z;\bar f_i)\right\|_{C(D_n)}<\varepsilon^{5/2}(1-3\varepsilon/5)^{k_i}3^{n-n_i},$$

v_2) for $z\notin D_n$

v_{21}) $0\leqq f_n(z;a)<\exp(-\varepsilon'/2|z|^{\alpha})$;

v_{22}) $\left|\dfrac{\partial}{\partial z}f_n(z;a)\right|<(1+|z|^{2k})\exp(-\varepsilon'/2|z|^{\alpha})$;

v_{23}) $|\nabla_a f_n(z;a)|<(1+|z|^{2k+1})\exp(-\varepsilon'/2|z|^{\alpha})$;

v_{24}) $\left|\nabla_a\dfrac{\partial}{\partial z}f_n(z;a)\right|<(1+|z|^{4k+1})\exp(-\varepsilon'/2|z|^{\alpha})$.

Now we shall formulate three lemmas from which we shall deduce Theorem 1. In the formulations λ is a certain constant larger than 1.

Lemma 1. *Let $n=n_i$, and for $n=n_i$ the conditions (U_{n_i}) are valid for the family $\{f_{n_i}(z;a),\ a\in\mathfrak{B}_{n_i}\}$. There exists a subset $\mathfrak{B}'_{n_i}\subset\mathfrak{B}_{n_i}$ and C^1-diffeomorphism φ_n: $\mathfrak{B}'_{n_i}\to\mathfrak{A}_{n_i}$ such that for the family $\{f_{n_i}(z;\varphi_{n_i}^{-1}(a)),\ a\in\mathfrak{A}_{n_i}\}$ the conditions (V_{n_i}) are valid. Moreover $\|\varphi_{n_i}-\mathrm{Id}\|_{C^1}<\varepsilon^2$ where* Id *is the identity transformation in the k-dimensional space.*

Lemma 2. *Let for n, $n_i\leqq n<n_{i+1}$ the family $\{f_n(z;a), a\in\mathfrak{A}_n\}$ satisfies the conditions (V_n). Then there exists a subset $\mathfrak{A}'_n\subset\mathfrak{A}_n$ and C^1-diffeomorphism $\psi_n:\mathfrak{A}'_n\to\mathfrak{A}_{n+1}$ such that $d(\psi_n(a'),\ \psi_n(a''))\geqq\lambda d(a',a'')$ and the family $\{Af_n(z;\psi_n^{-1}(a)),\ a\in\mathfrak{A}_{n+1}\}$ satisfies the conditions (V_{n+1}). For $n=n_{i+1}-1$ the conditions (V_{n+1}) are valid with the functions $e_s^{(i)}$.*

Lemma 3. *Let for $n=n_{i+1}$ the family $\{f_{n_{i+1}}(z;a),\ a\in\mathfrak{A}_{n_{i+1}}\}$ satisfies the conditions $(V_{n_{i+1}})$ with the functions $e_s^{(i)}$. Then there exists a subset $\mathfrak{A}'_{n_{i+1}}\subset\mathfrak{A}_{n_{i+1}}$ and C^1-diffeomorphism $\chi_{n_{i+1}}:\mathfrak{A}'_{n_{i+1}}\to\mathfrak{B}_{n_{i+1}}$ such that for the family $\{f_{n_{i+1}}(z;\chi_{n_{i+1}}^{-1}(a)),$ $a\in\mathfrak{B}_{n_{i+1}}\}$ the conditions $(U_{n_{i+1}})$ are valid and $\|\chi_{n_{i+1}}-\mathrm{Id}\|\leqq\varepsilon^2$.*

Proof of Theorem 1. Let us take the initial family of functions

$$f(z;a)=\varphi(z)(1-\varepsilon G_{2k}+\varepsilon^2\textstyle\sum_{i=0}^{4k}b_iG_{2i}+\sum_{i=0}^{k-1}a_ie_i(z))\,,$$

where $\varphi(z)\in C_0^\infty$, $\varphi(z)=\varphi(-z)$, $\varphi(z)\equiv 1$ for $|z|<d_{n_0}(\varepsilon)$ and $\varphi(z)\equiv 0$ for $|z|>d_{n_0}(\varepsilon)+1$, coefficients b_i are found from the formulae of the perturbation theory

$$A(1-\varepsilon G_{2k}+\varepsilon^2\textstyle\sum_{i=0}^{4k}b_iG_{2i})-(1-\varepsilon G_{2k}+\varepsilon^2\sum_{i=0}^{4k}b_iG_{2i})=O(\varepsilon^3)$$

where $e_i(z)$ are the eigenfunctions of the operator $L_{f(z;0)}$. It is easy to see that this family satisfies the conditions (U_{n_0}). Now we can apply Lemmas 1, 2, 3 and construct a decreasing sequence of sets $\bar{\mathfrak{B}}_n$, $\bar{\mathfrak{B}}_{n_0}=\mathfrak{A}_{n_0}\supset\bar{\mathfrak{B}}_{n_0+1}=\psi_{n_0+1}^{-1}(\mathfrak{B}_{n_0+1})\supset$ $\bar{\mathfrak{B}}_{n_0+2}=\psi_{n_0+2}^{-1}\psi_{n_0+1}^{-1}(\mathfrak{B}_{n_0+2})$ for which $\bigcap_{n=n_0}^{\infty}\mathfrak{B}_n=\bar{a}\in\mathfrak{A}_{n_0}$.

We shall show that the limit $\lim\limits_{n\to\infty}A^nf(z;\bar{a})=h(z;\bar{a})$ exists uniformly on any finite interval and $Ah=h$. Let $f_{n+1}(z)=A^nf(z;\bar{a})$, $f_0(z)=f(z;\bar{a})$. Lemma 2 can be applied to the function $g_n(z)=f_{n+1}(z)-f_n(z)$ from which it follows that

$$|g_n(z)|<\varepsilon^{5/2}(1-2\varepsilon/3)^n\,,\qquad z\in D_n\,,$$

$$|g_n(z)|<\exp(-(\varepsilon/2)|z|^\alpha)\,,\qquad z\notin D_n\,.$$

Therefore for any fixed l the series $f_0(z)+\sum_{n=1}^{\infty}g_n(z)$ converges uniformly on D_l and for its limit $h(z)\geqq 0$ the following estimate is valid

$$h(z)\leqq\exp(-(\varepsilon'/2)|z|^\alpha)\,,\qquad z\in D_l\,.$$

From this estimate we have $Ah=h$. Theorem 1 is proved.

§ 5. Proof of Lemma 1

Let be $n=n_i$. Let us denote $h_j(z)=\chi_{D_n}(z)e_j^{(i)}(z)$, $j=0,\dots,N$, where χ_{D_n} is the indicator of the interval $D_n=[-d_n,d_n]$, $e_j^{(i)}(z)$ is the j-th eigenfunction of the operator $L_{\bar{f}_i}$.

Firstly we shall show that

$$\left|\int_{-\infty}^{\infty} h_j(z)h_k(z)e^{-\gamma z^2}dz-\delta_k^j\right|<\varepsilon^{2/3} \tag{5.1}$$

where δ_k^j is the Kronecker symbol. According to the Theorem 3.1

$$\|e_j^{(i)}(z)-G_{2j}(z\sqrt{\gamma})\|_{L^2(R^1;e^{-\gamma z^2})}<\varepsilon^{4/5}$$

and for $|z|>d_0=10(c-1)^{-1}\sqrt{\ln\varepsilon^{-1}}$ $|e_j^{(i)}(z)|<|z|^{2j+1}$. Therefore

$$\|h_j(z)-e_j^{(i)}(z)\|^2_{L^2(R^1;e^{-\gamma z^2})}=\int_{|z|>d_n}|e_j^{(i)}(z)|^2e^{-\gamma z^2}dz$$
$$<\int_{|z|>d_0}|z|^{4j+2}e^{-\gamma z^2}dz<\varepsilon\,.$$

Consequently,

$$\|h_j(z)-G_{2j}(z\sqrt{\gamma})\|_{L^2(R^1;\exp(-\gamma z^2))}\leqq 2\varepsilon^{4/5}\,. \tag{5.2}$$

The Hermite polynomials $\{G_{2j}(z\sqrt{\gamma})\}$ are orthogonal in the space $L^2(R^1;\exp(-\gamma z^2))$, therefore inequality (5.1) follows from the last inequality. Then from (5.2) we may readily obtain:

$$\left|\int_{-\infty}^{\infty} h_j(z)h(z)e^{-\gamma z^2}dz\right|<2\varepsilon^{4/5} \tag{5.3}$$

for $j=0,\ldots,N$ and $h(z)\in H_{\bar{f}_i,N}$.

Inequalities (5.1) and (5.3) allow to expand the function $R_n(z;a)$ in functions $\{h_j(z)\}$ for small ε [see the condition (U_{n_i})]

$$R_n(z;a)=\sum_{j=0}^N\bar{\delta}_n^{(j)}(a)h_j(z)+\bar{h}_n(z;a)\,,$$

where for $a\in\mathfrak{B}_n$ $\operatorname{supp}\bar{h}_n(z;a)\subset D_n$ and $\bar{h}_n(z;a)\in H_{\bar{f}_i,N}$ and obtain for any j, $0\leqq j\leqq N$, the following estimates

$$|\bar{\delta}_n^{(j)}(a)|\leqq(1+\varepsilon^{2/3})\|R_n(z;a)\|\,, \tag{5.4}$$

$$\|\bar{h}_n(z;a)\|\leqq(1+\varepsilon^{2/3})\|R_n(z;a)\|\,, \tag{5.5}$$

$$|\nabla_a\bar{\delta}_n^{(j)}(a)|\leqq(1+\varepsilon^{2/3})\|\nabla_aR_n(z;a)\|\,, \tag{5.6}$$

$$\|\nabla_a\bar{h}_n(z;a)\|\leqq(1+\varepsilon^{2/3})\|\nabla_aR_n(z;a)\|\,, \tag{5.7}$$

where $\|\cdot\|=\|\cdot\|_{L^2(R^1;\exp(-\gamma z^2))}$.

As a result, we have the expansion of the function $g_n(z;a)$ as follows

$$g_n(z;a)=\sum_{j=0}^{k-1}(a_j+\bar{\delta}_n^{(j)}(a))e_j^{(i)}(z)+(\delta_n(a)+\bar{\delta}_n^{(k)}(a))e_k^{(i)}(z)$$
$$+\sum_{j=k+1}^N\bar{\delta}_n^{(j)}(a)e_j^{(i)}(z)+\bar{h}_n(z;a)\,. \tag{5.8}$$

The estimates u_{22}) in the condition (U_{n_i}), and the estimates (5.4), (5.6) show that

$$|\bar{\delta}_n^{(j)}(a)|<2\gamma^{-\frac{1}{2}}\varepsilon^{8/3}(1-2\varepsilon/3)^{n_i}\,, \tag{5.9}$$

$$|\nabla_a\bar{\delta}_n^{(j)}(a)|<0.9\cdot2\gamma^{-\frac{1}{2}}\varepsilon^{5/2}(1-2\varepsilon/3)^{n_i}\,. \tag{5.10}$$

Let us define the mapping $\varphi_n:\mathfrak{B}_n\to R^k$ by the formula:

$$(a_0,\ldots,a_{k-1})=a\xrightarrow{\varphi_n}b=(a_0+\bar{\delta}_n^{(0)}(a),\ldots,a_{k-1}+\bar{\delta}_n^{(k-1)}(a))\,.$$

Then the estimates (5.9) and (5.10) mean that

$$\|\varphi_n - \mathrm{Id}\|_{C^1} \leqq \varepsilon^{5/2} \tag{5.11}$$

that is, φ_n is C^1-diffeomorphism which is close to the identical one. For the point $x \in \partial\mathfrak{B}_n$, the boundary of the cube $\mathfrak{B}_n$,

$$|\varphi_n(x) - x| < \varepsilon^{\frac{1}{2}}|x|.$$

Let us denote $\mathfrak{A}_n = \{|a_j| < \frac{1}{2}\varepsilon^{5/3}(1-\varepsilon/2)^n\}$. From the last inequality it follows that

$$\mathfrak{A}_n \subset \varphi_n(\mathfrak{B}_n)$$

where $\varphi_n(\mathfrak{B}_n)$ is the image of the cube $\mathfrak{B}_n$ under the mapping φ_n. Let us put $\mathfrak{B}'_n = \varphi_n^{-1}(\mathfrak{A}_n)$. We have proved that the mapping $\varphi_n : \mathfrak{B}'_n \to \mathfrak{A}_n$ satisfies the estimates formulated in the lemma. It should be verified that expansion (5.8) of the function $g_n(z; a)$ satisfies all the requirements of the condition (V_n), provided the variables $a = \varphi_n^{-1}(b)$ are substituted in this expansion. Let us denote

$$\begin{aligned}
&\delta_n^{(k)}(b) = \delta_n(\varphi_n^{-1}(b)) + \bar{\delta}_n^{(k)}(\varphi_n^{-1}(b)),\\
&\delta_n^{(j)}(b) = \bar{\delta}_n^{(j)}(\varphi_n^{-1}(b)), j = k+1, \ldots, N,\\
&h_n(z; b) = \bar{h}_n(\varphi_n^{-1}(b)),\\
&t_n(z; b) = 0.
\end{aligned}$$

We have for $k+1 \leqq j \leqq N$

$$\begin{aligned}
|\delta_n^{(j)}(b)| &= |\bar{\delta}_n^{(j)}(\varphi_n^{-1}(b))| < 2\gamma^{-\frac{1}{2}}\varepsilon^{8/3}(1-2\varepsilon/3)^{n_i},\\
|\nabla_b \delta_n^{(j)}(b)| &\leqq |\nabla_{\varphi_n^{-1}(b)} \bar{\delta}_n^{(j)}(\varphi_n^{-1}(b))| \cdot |\nabla_b \varphi_n^{-1}(b)|\\
&\leqq 2 \cdot 0.9\gamma^{-\frac{1}{2}}\varepsilon^{5/2}(1-2\varepsilon/3)^{n_i}(1+\varepsilon^{2/3}) < 2\gamma^{-\frac{1}{2}}\varepsilon^{5/2}(1-2\varepsilon/3)^{n_i}.
\end{aligned}$$

In the same way we verify the remaining parts of the condition (V_n). Lemma 1 is proved.

§ 6. Proof of Lemma 2

We have

$$\begin{aligned}
g_{n+1} &= f_{n+2} - f_{n+1} = Af_{n+1} - Af_n = L_{f_n}(f_{n+1} - f_n)\\
&\quad + A(f_{n+1} - f_n) = L_{\bar{f}_i} g_n + (L_{f_n - \bar{f}_i}) g_n + A g_n\\
&= L_{\bar{f}_i} g_n + t'_n + t''_n.
\end{aligned} \tag{6.1}$$

Firstly we show that for $z \in D_n$

$$|t'_n(z; a)| < (1/4)\varepsilon^{7/2}(1-2\varepsilon/3)^{n_i+1}3^{n-n_i+1}, \tag{6.2}$$

$$|t''_n(z; a)| < (1/4)\varepsilon^{7/2}(1-2\varepsilon/3)^{n_i+1} \cdot 3^{n-n_i+1}, \tag{6.3}$$

$$|\nabla_a t'_n(z; a)|, |\nabla_a t''_n(z; a)| < (1/4)\varepsilon^{10/3}(1-\varepsilon/2)^{n_i+1}3^{n-n_i+1}. \tag{6.4}$$

In order to derive (6.2) let us first establish, that for $z \in D_{n_i}$

$$|f_n(z;a) - \bar{f}_i(z)| < \varepsilon^{7/3}(1-\varepsilon/3)^{n_i}. \tag{6.5}$$

Let us denote $a^{(n)} = a$, $a^{(j)} = \psi_j^{-1}(a^{(j+1)})$ for $j = n-1, \ldots, n_i$, $b^{(n_i)} = \varphi_{n_i}^{-1}(a^{(n_i)})$. From v_{14})

$$|f_n(z;a) - f_{n_i}(z;b^{(n_i)})| \leqq \textstyle\sum_{j=n_i}^{n-1} |g_j(z;a^{(j)})| \leqq 2\varepsilon^{7/3}(1-2\varepsilon/3)^n \tag{6.6}$$

and from u_3) it follows

$$\begin{aligned} |f_{n_i}(z;b^{(n_i)}) - \bar{f}_i(z)| &= |f_{n_i}(z;b^{(n_i)}) - f_{n_i}(z;0)| \\ &\leqq |b^{(n_i)}| \sup_{b \in \mathfrak{B}_{n_i}} |\nabla_b f_{n_i}(z;b)| \leqq \varepsilon^{5/2}(1-\varepsilon/2)^{n_i}\varepsilon^{-1/20} d_{n_i}^{2k+1} \\ &< \varepsilon^{7/3}(1-\varepsilon/3)^{n_i}\varepsilon^{1/10}(1-\varepsilon/2)^{n_i}(1-\varepsilon/3)^{-n_i} d_{n_i}^{2k+1}. \end{aligned} \tag{6.7}$$

Inequality (6.5) will result from (6.6), (6.7), if we show that

$$\varepsilon^{1/10}(1-\varepsilon/6)^{n_i} d_{n_i}^{2k+1} < 1. \tag{6.8}$$

From the form of d_{n_i}, denoting $x = \varepsilon(1-\varepsilon)^{n_i}$, we have

$$\varepsilon^{1/10}(1-\varepsilon/6)^{n_i} d_{n_i}^{2k+1} < (\varepsilon(1-\varepsilon)^{n_i})^{1/10} d_{n_i}^{2k+1} = L x^{1/10} \ln^{k+\frac{1}{2}} x^{-1}$$

where L is limited, and $x \to 0$ for $\varepsilon \to 0$. Thus, (6.8) and therefore (6.5) are proved.

Let us consider now (6.2). Suppose $z \in D_{n+1}$. Then

$$\begin{aligned} |(L_{f_n} - L_{\bar{f}_i}) g_n(z;a)| &= |2/\sqrt{\pi} \textstyle\int_{-\infty}^{\infty} e^{-u^2}(f_n(z/\sqrt{c} - u; a) \\ &\quad - \bar{f}_i(z/\sqrt{c} - u)) g_n(z/\sqrt{c} + u; a) du| \\ &\leqq (2/\sqrt{\pi}) |\textstyle\int_{|u| < 0.9(1-c^{-\frac{1}{2}}) d_{n+1}}| + (2/\sqrt{\pi}) |\textstyle\int_{|u| > 0.9(1-c^{-\frac{1}{2}}) d_{n+1}}| \\ &= I_1 + I_2. \end{aligned}$$

From u_3) it obviously follows that $f_n \leqq 2$, and, putting $u_n = 0.9(1-c^{-\frac{1}{2}}) d_{n+1}$, we have

$$I_2 < L \exp(-u_n^2) < (\varepsilon(1-\varepsilon)^{n+1})^{12}.$$

In case $|u| < u_n$, and $z \in D_{n+1}$, we have

$$\begin{aligned} |z/\sqrt{c} \pm u| &\leqq |z/\sqrt{c}| + |u| \leqq d_{n+1}/\sqrt{c} + u_n = d_{n+1}(1/\sqrt{c} + 0.9(1 - 1/\sqrt{c})) \\ &= d_{n+1}(0.9 + 0.1/\sqrt{c}) = d_{n_i} d_{n+1}/d_{n_i}(0.9 + 0.1/\sqrt{c}) < d_{n_i} \end{aligned} \tag{6.9}$$

since

$$\begin{aligned} d_{n+1}/d_{n_i} = d_{n_{i+1}}/d_{n_i} &= ((\ln \varepsilon^{-1} + n_{i+1} \ln(1-\varepsilon)^{-1})/(\ln \varepsilon^{-1} + n_i \ln(1-\varepsilon)^{-1}))^{\frac{1}{2}} \\ &< ((\ln \varepsilon^{-1} + n_i(\omega - 1)\ln(1-\varepsilon)^{-1} + n_i \ln(1-\varepsilon)^{-1})/(\ln \varepsilon^{-1} + n_i \ln(1-\varepsilon)^{-1}))^{\frac{1}{2}} \\ &< \sqrt{1 + (\omega - 1)} = \sqrt{\omega} \underset{\varepsilon \to 0}{\longrightarrow} 1. \end{aligned}$$

From (6.9) it follows that in estimating the value I_1 we may employ the properties of functions $f_n(z;a) - \bar{f}_i(z)$ and $g_n(z;a)$, $z \in D_{n_i}$. Using v_{13}) and (6.5), we obtain

$$\begin{aligned} I_1 &= (2/\sqrt{\pi}) |\textstyle\int_{|u| < u_n} e^{-u^2}(f_n(z/\sqrt{c} - u; a) - \bar{f}_i(z/\sqrt{c} - u)) g_n(z/\sqrt{c} + u; a) du \\ &\leqq \varepsilon^{7/3}(1-\varepsilon/3)^{n_i}\varepsilon^{7/3}(1-\varepsilon/2)^n 3^{n-n_i+1} \leqq (\varepsilon^4/4)(1-2\varepsilon/3)^n 3^{n-n_i+1}. \end{aligned}$$

Performing summation of the estimates for I_1 and I_2 we obtain (6.2). The relation (6.3) can be proved analogously.

Let us now prove (6.4). We have

$$\nabla_a t_n'(z;a) = L\nabla_{af_n} g_n + (L_{f_n} - L_{\bar f_i})\nabla_a g_n .$$

We shall use the inequalities, resulting from (V_n):

$$|\nabla_a f_n(z;a)| \leqq K_0(1+|z|^{2k+1}),$$
$$|\nabla_a g_n(z;a)| \leqq K_0(1+|z|^{2k+1}),$$
$$|\nabla_a g_n(z;a)| \leqq \varepsilon^{5/2}(1-3\varepsilon/5)^{n_i} 3^{n-n_i} \quad \text{for} \quad |z| \leqq d_n .$$

Let us divide the integral, which determines $L_{\Delta_a f_n} g_n$, into the sum of two integrals:

$$L_{\Delta_a f_n} g_n(z;a) = (2/\sqrt{\pi})\left[\int_{-u_n}^{u_n} + \int_{|u| \geqq u_n}\right] e^{-u^2} \nabla_a f_n(z/\sqrt{c} - u; a) g_n(z/\sqrt{c} + u; a) du .$$

For the external integral we have the estimate:

$$K_1 \int_{|u|>u_n} e^{-u^2}(1+|z/\sqrt{c} - u|)^{\alpha+1}(1+|z/\sqrt{c}+u|)^{\alpha+1} du$$
$$\leqq K_2 \int_{|u|>u_n} e^{-u^2}|u|^{2\alpha+2} du < \varepsilon^{7/2}(1-2\varepsilon/3)^n ,$$

so far as for $z \in D_n$ and $|u| > u_n$, we shall evidently have $|z/\sqrt{c} \pm u| < K_3 u$. The last inequality follows from the fact that $\exp(-u_n^2) < \varepsilon^4(1-\varepsilon)^n$ due to the definition of u_n. For the internal integral on the base of the condition v_{14}):

$$(2/\sqrt{\pi}) \int_{|u|<u_n} e^{-u^2} \nabla_a f_n(z/\sqrt{c} + u; a) g_n(z/\sqrt{c} - u; a) du$$
$$< (2/\sqrt{\pi}) \int_{|u|<u_n} e^{-u^2} du \cdot \varepsilon^{8/3}(1-2\varepsilon/3)^n \cdot u_n^{\alpha+1} \leqq \varepsilon^{5/2}(1-5\varepsilon/8)^n .$$

Summing the estimates of the external and internal integrals, we obtain the estimate for the value $|L_{\Delta_a f_n} g_n|$. Analogously $(L_{f_n} - L_{f_i})\nabla_a g_n$ can be estimated. So, the first inequality in (6.4) is proved. In a similar way we prove the second inequality for $|\nabla_a t_n''|$ in (6.4).

Now we make use of the representation for $g_n(z;a)$ involved in v_1). In the expression

$$L_{\bar f_i} g_n(z;a) = \sum_{j=0}^{k-1} a_j L_{\bar f_i} e_j^{(i)} + \sum_{j=k}^{N} \delta_n^{(j)}(a) L_{\bar f_i} e_j^{(i)} + L_{\bar f_i} h_n + L_{\bar f_i} t_n$$

we consider each term separately, beginning from the right one. Let us introduce the operator

$$\bar L_{\bar f_i} g = (2/\sqrt{\pi}) \int_{-u_n}^{u_n} e^{-u^2} \bar f_i(z/\sqrt{c} - u) g(z/\sqrt{c} + u) du .$$

We shall show that for $z \in D_{n+1}$

$$|(L_{\bar f_i} - \bar L_{\bar f_i}) t_n(z;a)| < \varepsilon^4(1-\varepsilon)^{n+1} . \tag{6.10}$$

From the conditions V_1)

$$\|\bar f_i\|_{C(R^1)} \leqq 2, \qquad \|t_n(z;a)(1+|z|)^{-2N-2}\|_{C(R^1)} < 1 .$$

Hence

$$|(L_{\bar f_i} - \bar L_{\bar f_i}) t_n(z;a)| \leqq (4/\sqrt{\pi}) \int_{|u|>u_n} e^{-u^2}(1+|z/\sqrt{c} - u|)^{2N+2} du$$
$$\leqq K \int_{|u|>u_n} e^{-u^2}|u|^{2N+2} du < K_1 \exp(-u_n^2) u_n^{2N+2}$$
$$< \varepsilon^4(1-\varepsilon)^{n+1} K_1 \exp(-0.1 u_n^2) u_n^{2N+2} ,$$

that proves (6.10), since $K_1 \exp(-0.1u_n^2)u_n^{2N+2} \to 0$ at $u_n \to \infty$, whereas due to smallness of ε we may consider all u_n to be sufficiently large.

For $z \in D_{n+1}$, $|u| < u_n$ it is obvious that $z/\sqrt{c} \pm u \in D_n$. Thus,

$$\|\bar{L}_{\bar{f}_i} t_n(z;a)\|_{C(D_{n+1})} \leqq 2,1 \|t_n(z;a)\|_{C(D_n)} . \tag{6.11}$$

Let us put $t_{n+1}(z;a) = t_n'(z;a) + t_n''(z;a) + L_{\bar{f}_i} t_n(z;a)$. Having summed the estimates (6.2), (6.3), (6.10), (6.11), we obtain the resulting estimate for $t_{n+1}(z;a)$:

$$\|t_{n+1}(z;a)\|_{C(D_{n+1})} \leqq \varepsilon^3 (1 - 2\varepsilon/3)^{n_{ni}+1} 3^{n-n_i+1}$$

The same consideration allows to obtain an estimate for the vector-function $\nabla_a t_{n+1}(z;a)$:

$$\|\nabla_a t_{n+1}(z;a)\|_{C(D_{n+1})} \leqq \varepsilon^{5/2} (1 - 3\varepsilon/5)^{n_i+1} 3^{n-n_i+1} .$$

Now we turn to the function $\bar{h}_{n+1}(z;a) = \chi_{D_{n+1}}(z) L_{\bar{f}_i} h_n(z;a)$.

According to the assumption of v_{12}) $h_n \in H_{\bar{f}_i,N}$. Therefore from the Theorem 3.1 it follows:

$$\begin{aligned}\|\bar{h}_{n+1}(z;a)\|_{L^2(R^1;\exp(-\gamma z^2))} &\leqq \|L_{\bar{f}_i} h_n(z;a)\|_{L^2(R^1;\exp(-\gamma z^2))} \\ &\leqq \nu_{N+1} \|h_n(z;a)\|_{L^2(R^1;\exp(-\gamma z^2))} .\end{aligned} \tag{6.12}$$

So far as $h_n(z;a) \in H_{\bar{f}_i,N}$ for any $a \in \mathfrak{A}_n$, then

$$\partial h_n(z;a)/\partial a_j \in H_{\bar{f}_i,N}, \qquad j = 0, \ldots, k-1 .$$

So, analogously to (6.12), we shall have

$$\|\partial \bar{h}_{n+1}(z;a)/\partial a_j\|_{L^2(R^1;\exp(-\gamma z^2))} \leqq \nu_{N+1} \|\partial h_n(z;a)/\partial a_j\|_{L^2(R^1;\exp(-\gamma z^2))} .$$

From the inequalities

$$\|L_{\bar{f}_i} h_n(z;a)\|_{C(D_{n+1})} \leqq 2,1 \|h_n(z;a)\|_{C(D_n)}$$
$$\|L_{\bar{f}_i} h_n(z;a)\|_{C^1(D_{n+1})} \leqq K \|h_n(z;a)\|_{C(D_n)}$$

we get directly the following

$$\|\bar{h}_{n+1}(z;a)\|_{C(D_{n+1})} \leqq 2,1 \|h_n(z;a)\|_{C(D_n)} , \tag{6.13}$$

$$\|\bar{h}_{n+1}(z;a)\|_{C^1(D_{n+1})} \leqq K \|h_n(z;a)\|_{C(D_n)} , \tag{6.14}$$

$$\|\nabla_a \bar{h}_{n+1}(z;a)\|_{C(D_{n+1})} \leqq 2,1 \|\nabla_a h_n(z;a)\|_{C(D_n)} , \tag{6.15}$$

$$\|\nabla_a \bar{h}_{n+1}(z;a)\|_{C^1(D_{n+1})} \leqq K \|\nabla_a h_n(z;a)\|_{C(D_n)} . \tag{6.16}$$

Now we have

$$g_{n+1}(z;a) = \sum_{j=0}^{k-1} \lambda_j a_j e_j(z) + \sum_{j=k}^{N} \lambda_j \delta_n^{(j)}(a) e_j(z) + \bar{h}_{n+1}(z;a) + t_{n+1}(z;a) \tag{6.17}$$

where $\lambda_0, \ldots, \lambda_N$ are the eigenvalues of the operator $L_{\bar{f}_i}$. From the invariance of $H_{\bar{f}_i,N}$ it follows that $L_{\bar{f}_i} h_n \in H_{\bar{f}_i,N}$ but, generally speaking, $h_{n+1} \notin H_{\bar{f}_i,N}$.

Thus, we consider the expansion

$$\bar{h}_{n+1}(z;a) = \sum_{j=0}^{N} \bar{\delta}_{n+1}^{(j)}(a) \chi_{D_{n+1}}(z) e_j(z) + h_{n+1}(z;a) , \tag{6.18}$$

where $h_{n+1} \in H_{\bar{f}_i, N}$. From the Theorem 3.1

$$\|h_{n+1}(z;a)\|_{L^2(R^1;\exp(-\gamma z^2))} \leqq 1,1 \|\bar{h}_{n+1}(z;a)\|_{L^2(R^1;\exp(-\gamma z^2))}$$

$$\|\nabla_a h_{n+1}(z;a)\|_{L^2(R^1;\exp(-\gamma z^2))} \leqq 1,1 \|\nabla_a \bar{h}_{n+1}(z;a)\|_{L^2(R^1;\exp(-\gamma z^2))}$$

$$|\delta_{n+1}^{(j)}(a)| \leqq 1,1 \|\bar{h}_{n+1}(z;a)\|_{L^2(R^1;\exp(-\gamma z^2))}$$

$$|\nabla_a \bar{\delta}_{n+1}^{(j)}(a)| \leqq 1,1 \|\nabla_a \bar{h}_{n+1}(z;a)\|_{L^2(R^1;\exp(-\gamma z^2))}\,.$$

Let us introduce the mapping $\psi_n : \mathfrak{A}_n \to R^k$, putting

$$\psi_n(a_0, \ldots, a_{k-1}) = (\lambda_0 a_0 + \bar{\delta}_{n+1}^{(0)}(a), \ldots, \lambda_{k-1} a_{k-1} + \delta_{n+1}^{(k-1)}(a))\,.$$

From the last estimates, from (6.13), (6.15) and from the condition v_{12})

$$\|\psi_n - \psi^{(0)}\|_C \leqq \varepsilon^{7/3}(1-2\varepsilon/3)^n \tag{6.19}$$

$$\|\psi_n - \psi^{(0)}\|_{C^1} \leqq \varepsilon^{7/3}(1-2\varepsilon/3)^n \tag{6.20}$$

where $\psi^{(0)}(a_0, \ldots, a_{k-1}) = (\lambda_0 a_0, \ldots, \lambda_{k-1} a_{k-1})$. Let us insert expansion (6.18) into (6.17) and denote the vector $\psi(a)$ by a:

$$g_{n+1}(z;a) = \sum_{j=0}^{k-1} a_j e_j(z) + \sum_{j=k}^{N} (\lambda_j \delta_n^{(j)}(\psi^{-1}(a)) + \bar{\delta}_{n+1}^{(j)}(\psi^{-1}(a)) e_j(z)$$
$$+ h_{n+1}(z;\psi^{-1}(a)) + t_{n+1}(z;\psi^{-1}(a))\,.$$

From (6.19) it follows that $\psi(\mathfrak{A}_n) \supset \mathfrak{A}_{n+1}$. Let us put

$$\delta_{n+1}^{(j)}(a) = \lambda_j \delta_n^{(j)}(\psi^{-1}(a)) + \bar{\delta}_{n+1}^{(j)}(\psi^{-1}(a)),\ h_{n+1}(z;a)$$
$$= h_{n+1}(z;\psi^{-1}(a)),\ t_{n+1}(z;a) = t_{n+1}(z;\psi^{-1}(a))\,.$$

Lemma 2 is proved.

§ 7. Proof of Lemma 3

In this section we assume $n = n_{i+1}$

$$\bar{R}_n(z;a) = \sum_{j=k+1}^{N} \delta_n^{(j)}(a) e_j(z) + h_n(z;a) + t_n(z;a)\,.$$

We shall estimate firstly $|\bar{R}_n(z;a)|$ for $z \in D_n = [-d_n, d_n]$. From the condition v_{11}) and the Theorem 3.1 we obtain the estimate

$$|\delta_n^{(j)}(a) e_j(z)| < v_j^{n-n_i} \varepsilon^{8/3} (1-2\varepsilon/3)^{n_i} d_n^{2j+1}\,.$$

Hence

$$\sum_{j=k+1}^{N} |\delta_n^{(j)}(a) e_j(z)| \leqq \varepsilon^{8/3}(1-2\varepsilon/3)^{n_i}(1-3\varepsilon/4)^{n-n_i} \sum_{j=k+1}^{N} d_n^{2j+1}(v_j/(1-3\varepsilon/4))^{n-n_i} \tag{7.1}$$

Let us show that for $j > k$

$$(v_j/(1-3\varepsilon/4)^{n-n_i} d_n^{2j+1} < c^{-\frac{1}{2}(j-k)n_0}\,. \tag{7.2}$$

Indeed:

$$v_j/(1-3\varepsilon/4) \leqq c^{-2(j-k)/3}/(1-3\varepsilon/4) \leqq c^{-5(j-k)/8},\ n - n_i = [(\omega-1)n_i] + n_0\,.$$

The left-hand side of Eq. (7.2) does not thus exceed

$$c^{-(5/8)(j-k)([(\omega-1)n_i]+n_0)}d_n^{2j+1} \leqq c^{-\frac{1}{2}(j-k)n_0}c^{-(1/8)(j-k)([(\omega-1)n_i]+n_0)}d_n^{2j+1}\,.$$

In order to prove the inequality (7.2) it is sufficient to show, that

$$c^{-(1/8)(j-k)([(\omega-1)n_i]+n_0)}d_n^{2N+1}<1\,.$$

From the definition of the numbers $n_0, n_1, \ldots, \omega, d_n$ we have

$$[(\omega-1)n_i] \geqq (\omega-1)n_i-1 \geqq (\omega-1)n/\omega-1-(\omega-1)n_0/\omega\,,$$

$$c^{-(1/8)(\omega-1)/\omega}=(5^{\omega-1})^{(\log_5 c)/(8\omega)} \leqq (1-\varepsilon)^\lambda, \qquad \lambda=(\log_5 c)/320\,,$$

$$c^{-n_0/8} \leqq \varepsilon^\mu, \qquad \mu=(\log_5 c)/50>\lambda\,,$$

$$d_n=4/(1-c^{-\frac{1}{2}})\sqrt{\ln\varepsilon^{-1}+n\ln(1-\varepsilon)^{-1}}=\text{const}\sqrt{\ln\varepsilon^{-\lambda}+n\ln(1-\varepsilon)^{-\lambda}}\,.$$

Consequently, if we denote $x=\varepsilon^\lambda(1-\varepsilon)^{n\lambda}$, then

$$c^{-(1/8)(j-k)([(\omega-1)n_i]+n_0)}d_n^{2N+1}<\text{const}\,x\ln^{(2N+1)/2}x\underset{\varepsilon\to 0}{\longrightarrow}0$$

the fact that should have been shown. Turning to the inequality (7.1) we can see that

$$\begin{aligned}\textstyle\sum_{j=k+1}^N|\delta_n^{(j)}(a)e_j(z)| &< \varepsilon^{8/3}(1-2\varepsilon/3)^{n_i}(1-3\varepsilon/4)^{n-n_i}\textstyle\sum_{j=k+1}^N c^{-\frac{1}{2}(j-k)n_0}\\ &<\varepsilon^{8/3}(1-2\varepsilon/3)^{n_i}(1-3\varepsilon/4)^{n-n_i}c^{-\frac{1}{2}n_0}/(1-c^{-\frac{1}{2}n_0})\\ &<\varepsilon^{8/3+\log_5 c}(1-2\varepsilon/3)^{n_i}(1-3\varepsilon/4)^{n-n_i}\,,\end{aligned} \tag{7.3}$$

since $c^{-\frac{1}{2}n_0}=5^{-\frac{n_0}{2}\log_5 c}<\varepsilon^{\log_5 c}$ due to the choice of n_0.

Let us estimate now the other terms entering into $\bar{R}_n(z;a)$. From the condition v_{15}) we have $|t_n(z;a)|<\varepsilon^{7/2}(1-2\varepsilon/3)^n$. From v_{12})

$$\begin{aligned}\|h_n(z;a)\|_{L^2(R^1)} &\leqq \exp(\gamma d_n^2)\|h_n(z;a)\|_{L^2(R^1;\exp(-\gamma z^2))}\\ &\leqq e^{\gamma d_n^2}c^{-(2/3)(N-k)(n-n_i)}\varepsilon^{8/3}(1-2\varepsilon/3)^{n_i}=S\,.\end{aligned}$$

It may be shown now that $S<\varepsilon^3(1-2\varepsilon/3)^n$. The idea of proving consists in the fact that by choosing the number N sufficiently large the increase of $e^{+\gamma d_n^2}$ will be compensated by the decrease of the value $c^{-(2/3)(N-k)(n-n_i)}$. We have

$$\begin{aligned}n-n_i=[(\omega-1)n_i]+n_0 &\geqq (\omega-1)n_i+n_0-1 \geqq (\omega-1/\omega)(n-n_0+1)\\ &+n_0-1 \geqq ((\omega-1)/\omega)n+n_0/\omega-2\,,\end{aligned}$$

$$c^{-(2/3)(\omega-1)/\omega}=(5^{\omega-1})^{2(\log_5 c)/(3\omega)} \leqq (1-\varepsilon)^\lambda, \quad \lambda=(\log_5 c)/(60\omega), \quad c^{-2n_0/3\omega}=\varepsilon^\mu\,,$$

$$\mu=(\log_5 c)/(72\omega), \qquad e^{\gamma d_n^2}=[(1-\varepsilon)^{-n}\varepsilon^{-1}]^\nu, \qquad \nu=16(\sqrt{c}+1)/(\sqrt{c}-1)\,.$$

Hence

$$S \leqq \varepsilon^{-\nu+(N-k)\mu+8/3}(1-\varepsilon)^{n(-\nu+(N-k)\lambda)}(1-2\varepsilon/3)^{n_i}c^2\,.$$

The required inequality for S is obtained, provided $(N-k)\mu$, $(N-k)\lambda \geqq \nu+1$.

Let us estimate now $|h_n(z;a)|$ for $z\in D_n$. The discussion presented below has been already employed in [2]. Suppose $0\leqq z\leqq d_n$ and

$$\Pi(u)=\begin{cases}0, & u\notin[0,-2]\\ 1+\frac{1}{2}u, & u\in[0,-2].\end{cases}$$

From v_{13})

$$|h_n(u;a)|>|h_n(z;a)|\Pi(u-z)$$

and, therefore:

$$|h_n(z;a)|<\|h_n(u;a)\|_{L^2(R^1)}/\|\Pi(u)\|_{L^2(R^1)}<\|h_n(u;a)\|_{L^2(R^1)}<\varepsilon^3(1-2\varepsilon/3)^n. \tag{7.4}$$

Performing the summation of the estimates (7.3), (7.4), we obtain

$$|\bar{R}_n(z;a)|<\varepsilon^{8/3+\frac{1}{2}\log_5 c}(1-2\varepsilon/3)^n.$$

Analogously the following inequality may be proved

$$|\nabla_a\bar{R}_n(z;a)|<\varepsilon^{5/2+\frac{1}{2}\log_5 c}(1-5\varepsilon/8)^n.$$

Let us turn now directly to proving the conditions (U_n). We have

$$g_n(z;a)=\sum_{j=0}^{k-1}a_je_j(z)+\delta_n^{(k)}(a)e_k(z)+\bar{R}_n(z;a). \tag{7.5}$$

It should be recalled that here e_j are the eigenfunctions of the operator $L_{\bar{f}_i}$. We verify first that the function $f_{n_i+1}(z;0)=f_n(z;0)$ satisfies all the conditions of the Theorem 3.1. Suppose $b^{(j)}=\varphi_j^{-1}(\varphi_{j+1}^{-1}(\dots\varphi_{n-1}^{-1}(0)\dots))$, $j=n_i,\dots,n-1$ where $\varphi_{\cdot}:\mathfrak{A}_l\to\mathfrak{A}_{l+1}$ are the mappings constructed in Lemma 2. Then from v_{14}) for $z\in D_n$ we have:

$$|f_{j+1}(z;b^{(j+1)})-f_j(z;b^{(j)})|=|g_j(z;b^{(j)})|<\varepsilon^{7/3}(1-2\varepsilon/3)^n3^{j-n}.$$

Whence

$$|f_n(z;0)-f_{n_i}(z;b^{(n_i)})|<2\varepsilon^{7/3}(1-2\varepsilon/3)^n. \tag{7.6}$$

Then, so far as $\|f_{n_i}(z;a^{(0)})-(1-\varepsilon G_{2k}(z;\gamma))\|_{C^1(D_0)}=O(\varepsilon^{4/3})$ the analogous equality is valid for $f_n(z;0)$ also. Thus, in the segment D_0 function f_n satisfies the condition of the theorem. From the conditions (V_n) it also follows that it satisfies the conditions of the Theorem 3.1 outside D_0 too. Consequently, Theorem 3.1 is applicable, and we may introduce the eigenfunctions $e_j(z;\bar{f}_{i+1})$, $\bar{f}_{i+1}=f_{n_{i+1}}(z;0)$, $j=0,1,\dots,N$. From (7.5), v_2) and u_{23})

$$\|f_n(z;0)-f_{n_i}(z;b^{(n_i)})\|_{L^2(R^1;\exp(-\gamma z^2))}<(4/\sqrt{\gamma})\varepsilon^{7/3}(1-2\varepsilon/3)^n$$

$$\|f_{n_i}(z;b^{(n_i)})-f_{n_i}(z;a^{(0)})\|_{L^2(R^1;\exp(-\gamma z^2))}\leqq\sup_a|\nabla_af_{n_i}(z;a)|$$
$$\cdot|b^{(n_i)}-a^{(0)}|<\varepsilon^3(1-2\varepsilon/3)^n$$

where $a^{(0)}$ is introduced in u_1). Thus,

$$\|f_n(z;0)-f_{n_i}(z;a^{(0)})\|_{L^2(R^1;\exp(-\gamma z^2))}<(5/\sqrt{\gamma})\varepsilon^{7/3}(1-2\varepsilon/3)^n.$$

From this inequality, using the consideration of the Theorem 3.1, it is easy to derive the estimates

$$\|e_j(z;\bar{f}_{i+1}) - e_j(z;\bar{f}_i)\|_{L^2(R^1;\exp(-\gamma z^2))} < \varepsilon^{9/4}(1-2\varepsilon/3)^n$$

for $j=0, 1, \ldots, N$. From here

$$e_j(z;\bar{f}_i) = \sum_{m=0}^{k}(\delta_m^j + c_{jm})e_m(z;\bar{f}_{i+1}) + \tilde{R}_n(z;a),$$

where δ_m^j is the Kronecker symbol,

$$|c_{jm}|, \quad \|\tilde{R}_n(z;a)\|_{C(D_n)} < \varepsilon^{11/5}(1-2\varepsilon/3)^n, \quad \tilde{R}_n(z;a) \in H_{\bar{f}_{i+1},k}.$$

Inserting this expansion into the equality (7.3), and performing the substitution of the variables similar to the identical one in the space of the parameters $a=(a_0, \ldots, a_{k-1})$ we obtain the condition (U_n) at $n=n_{i+1}$. Lemma 3 is proved.

§ 8. Derivation of Formulas for Indices

In papers [2, 8] there have been obtained results concerning the indices of the asymptotic hierarchical models under the condition $\sqrt{r} < c < r$. As it will be seen in what follows the cases $\sqrt{r} < c < r$ and $c = \sqrt{r} - \varepsilon$ differ essentially. For the sake of simplicity we consider the case $r=2$.

The values of the critical indices we derive by studying the asymptotic behaviour of the recursive relations (1.2) when $n \to \infty$. Function $f_n(z;\beta)$ in (1.2) is defined on the discrete finite lattice of points $M_n = \{c^{n/2}(-1+i/2^{n-1})\}_{i=0}^{2^n}$ with the step $\Delta_n = 2(\sqrt{c}/2)^n$, since $\sum_{x \in V_n} u(x)$ is an even number, which does not exceed 2^n in modulus. The summation is carried out in (1.2) so, that $z/\sqrt{c} \pm u \in M_n$.

As in the papers [2] and [8] we obtain the critical indices for a.h.m., their initial distribution $f_{n_0}(z;\beta)$ satisfying some relations of the inequality type for a sufficiently large value of n_0. These inequalities determine the open set Ω which is deliberately non-empty in the space of all a.h.m. In this way we show that the branch g_{c_1} is thermodynamically stable.

Let us suppose c be fixed, and $\sqrt{2} - c > 0$ is small. Let $f^{(0)}(z;\beta) \not\equiv \text{const}$ be the solution of Eq. (1.3) constructed in the Theorem 1, and $e_i(z;\beta)$ are the eigenfunctions of the operator $L_{g^{(0)}}$ (see the Theorem 3.1) with the eigen numbers λ_i. The eigenfunctions are considered to be normalized by the condition

$$\|e_i(z;\beta)\|_{L^2(R^1;\exp(-\gamma z^2)}^{-1} = (\gamma^{\frac{1}{2}}/\pi^{\frac{1}{4}}) \cdot ((2j)!^{\frac{1}{2}}/2^j) \cdot (2\gamma^{\frac{1}{2}})^{2j}/(2j)!,$$

so that $e_i(z;\beta)$ for $z \sim \sqrt{\ln \varepsilon^{-1}}$ has the asymptotics z^{2j}. The set Ω consists of the families of the probability distributions $f_{n_0}(z;\beta) = \exp(-a_0(\beta)z^2)p_{n_0}(z;\beta)$ depending on β, which satisfy the following conditions (the number n_0 is large, it is enough for it to exceed $10^7\varepsilon^{-2}$):

The condition (U). There exists a segment of inverse temperatures $[\beta^-, \beta^+]$ and C^1-function $b(\beta)$ defined on $[\beta^-, \beta^+]$, such that

$$p_{n_0}(z;\beta) = p^{(0)}(z;\beta) + \lambda_1^{n_0} b(\beta) e_1(z;p^{(0)}) + R(z;\beta) \tag{8.1}$$

and in this case

u$_1$) $\lambda_1^{n_0} b(\beta^{\pm}) = \mp \varepsilon^{3/2}$; $\quad |b'(\beta)| > \sqrt{\varepsilon}$ for $\beta \in [\beta^-, \beta^+]$;

u$_2$) $|R(z;\beta)| + |\partial R(z;\beta)/\partial\beta| + |\partial R(z;\beta)/\partial z| < \varepsilon^2$

for $|z| < 4\beta\sqrt{\ln \varepsilon^{-1}}$, $\beta \in [\beta^-, \beta^+]$;

u$_3$) $0 < p_{n_0}(z;\beta) < \exp(-(\beta\varepsilon'/2)|z|^{\alpha})$;

$$|\partial p_{n_0}(z;\beta)/\partial z| < |z|^4 \exp(-(\beta\varepsilon'/2)|z|^{\alpha});$$

$$|\partial p_{n_0}(z;\beta)/\partial \beta| < |z|^5 \exp(-(\beta\varepsilon'/2)|z|^{\alpha});$$

$$|\partial^2 p_{n_0}(z;\beta)/\partial z\partial\beta| < |z|^9 \exp(-(\beta\varepsilon'/2)|z|^{\alpha})$$

for $|z| > 4\beta\sqrt{\ln(1/\varepsilon)}$, $\quad \beta \in [\beta^-, \beta^+]$.

Theorem 8.1. *Suppose the value $\sqrt{2} - c$ is sufficiently small, and the condition (U) is fulfilled. Then in the segment $[\beta^-, \beta^+]$ there is one and only one critical point β_{cr}, for which $f_{n_0}(z;\beta_{cr}) \Rightarrow f_c^{(0)}(z;\beta_{cr})$.*

Note. It follows from Theorem 8.1 that the value of the critical index $\eta = 0$, provided the dimension of the model is $d_a = \dfrac{2}{\log_2 2/c}$. ⌟

Theorem 8.2. *Suppose the value $\sqrt{2} - c$ is sufficiently small, the condition (U) is fulfilled, and $\beta \in [\beta_+, \beta_{cr})$. Then*

$$(c/2)^n f_n(z(c/2)^{n/2};\beta) \Rightarrow (2\pi\sigma_1(\beta))^{-\frac{1}{2}} \exp(-(\sigma_1(\beta)/2)z^2).$$

For $\beta \to \beta_{cr}$ asymptotically $\sigma_1(\beta) \sim |\beta_{cr} - \beta|^{-\gamma}$, $\gamma = 1 - \log_{\lambda_1}(c\lambda_1/2)$. ⌟

Theorem 8.3. *Suppose the value $\sqrt{2} - c$ is sufficiently small, the condition (U) is fulfilled, and $\beta \in [\beta_-, \beta_{cr})$. Then there exists a sequence of the numbers $0 < M_1(\beta) < M_2(\beta) < \ldots$, $\lim_{n\to\infty} M_n(\beta) = M(\beta)$ such, that $(c/2)^{n/2} f_n(z(c/2)^{n/2};\beta) - G_n(z;\beta) \Rightarrow 0$, where*

$$G_n(z;\beta) = \tfrac{1}{2}(2\pi\sigma_2(\beta))^{-\frac{1}{2}}(\exp(-(z - 2^{n/2}M_n(\beta))^2/2\sigma_2(\beta))$$
$$+ \exp(-(z + 2^{n/2}M_n(\beta))^2/2\sigma_2(\beta))).$$

For $\beta \to \beta_{cr}$ the asymptotical formulas

$$M(\beta) \sim |\beta - \beta_{cr}|^{\omega}, \quad \omega = \tfrac{1}{2}\log_{\lambda_1} c; \quad \sigma_2(\beta) \sim |\beta - \beta_{cr}|^{\gamma}, \quad \gamma = 1 - \log_{\lambda_1}(c\lambda_1/2)$$

are valid.

Refinement of Theorem 8.3 (calculation of the correlation radius). In the assumptions of the Theorem 8.3 there is a number $N = N(\beta)$ such that for $n < N$ the condition $|(c/2)^n f_n(z(c/2)^n;\beta) - f_0(z;\beta)| < \varepsilon^{2/3}$ is fulfilled, and for $n > N$ the condition $|(c/2)^n f_n(z(c/2)^n;\beta) - G_n(z;\beta)| < \varepsilon^{2/3}$ is fulfilled; the value $\xi = \xi(\beta) = 2^{N(\beta)/d_a}$ is the correlation radius, and for $\beta \to \beta_{cr}$ $\xi(\beta) \sim |\beta - \beta_{cr}|^{-\nu}$, $\nu = \frac{1}{2}\log_{\lambda_1}(2/c)$.

Let us consider the Gibbs distribution in the volume V_n at the external field value H and at the inverse temperature β and put

$$f_n(z;\beta,H) = ((2/\sqrt{c})^n/\Xi_n) \sum_{(\sqrt{c}/2)^n \sum_{x\in V_n}\sigma(x) = z} \exp(-\beta H_n(\sigma) + H\sum_{x\in V_n}\sigma(x)). \quad (8.2)$$

Theorem 8.4. *Suppose the value* $\sqrt{2}-c$ *is sufficiently small, the condition* (*U*) *is fulfilled,* $\beta\in[\beta^-,\beta^+]$, *and* $|H|<\varepsilon|\beta^+-\beta^-|$, $H\neq 0$. *Then there is a sequence of numbers* $M_n(\beta,H)\xrightarrow[n\to\infty]{} M(\beta,H)$ *such that*

$$(c/2)^n f_n(z(c/2)^n;\beta)-(2\pi\sigma(\beta,H))^{-\frac{1}{2}}\exp(-(z+2^{n/2}M_n(\beta,H))^2/2\sigma(\beta,H))\Rightarrow 0\,.$$

Theorem 8.5. *In the assumptions of Theorem 8.4 we have:*

a) $M(\beta,H)$ *is the monotonously increasing function of* H;

b) $\lim\limits_{H\to\pm 0} M(\beta,H)=\pm M(\beta)$ *(see Theorem 8.3)*;

c) *function* $H=H(\beta,M)$, *which is an inverse one to the function* $M(\beta,H)$, *permits for* $\beta, M\to 0$ *the expansions;*

c_1) *in the region* $|M|/|\tau|^{\omega}>\ln\varepsilon^{-1}$, $\tau=(\beta-\beta_{cr})/\beta_{cr}$, $\omega=\frac{1}{2}\log_{\lambda_1}c$

$$H=(L_1(\beta_{cr}-\beta)|M|^{\delta_1}+L_2|M|^{\delta_2}+\ldots)\operatorname{sgn}M\,, \tag{8.3}$$

where $\delta_1=1-2\log_c(c\lambda_1/2)$, $\delta_2=3+2\log_c(2/c^2)$, $L_1>0$, $L_2>0$ *are constants,* … *are the terms of higher order in the expansions; for* $c\to\sqrt{2}$ $L_1\sim 1$, $L_2\sim(\sqrt{2}-c)$;

c_2) *in the region* $|M|/|\tau|^{\omega}<(\ln\varepsilon^{-1})^{-1}$, $\beta<\beta_{cr}$

$$H=L_3(\beta_{cr}-\beta)^{-\gamma}M+\ldots \tag{8.4}$$

where $\gamma=1-\log_{\lambda_1}(c\lambda_1/2)$, L_3 *is a constant; for* $c\to\sqrt{2}$ $L_3\sim 1$.

Note. The presence of two asymptotical expansions in different regions of the equation of state $H(\beta,M)$ in the neighbourhood of the critical point is a very important phenomenon. It shows the type of the expansion $H(\beta,M)$, when the Landau theory cannot be applied.

The Theorem 8.1 is derived in the same way as the proof of the basic theorem (see also [2]), and we shall omit it. The proofs of the remaining theorems also involve essentially the technique of paper [8]. We present two lemmas without proof which elucidate the derivation of Theorem 8.3. These lemmas are proved analogously to the corresponding lemmas in paper [8], and we shall omit it too. Let us denote $N=N(\beta)=\min\{n:\lambda_1^n b'(\beta_{cr})|\cdot|\beta-\beta_{cr}|>(4/5)\beta_{cr}/(2/c-1)\}$, $\varepsilon_0(u)=\varepsilon^{1/3}$ for $\sqrt{\ln(1/\varepsilon)}<|u|\leqq\varepsilon^{-2/3}$, $\varepsilon_0(u)=\varepsilon^{1/3}|\varepsilon^{2/3}u|^{-1.5}$ for $\varepsilon^{-2/3}<|u|\leqq c^{(n-n_0)/2}\sqrt{\ln\varepsilon^{-1}}$.

Lemma 8.1. *Suppose* $n_0<n<N(\beta)$, $\sqrt{\ln\varepsilon^{-1}}\leqq|z^{(0)}|\leqq c^{(n-n_0)/2}\sqrt{\ln\varepsilon^{-1}}$. *Then there exist the numbers* $L_n=L_n(\beta,z^{(0)})$, $\mu_n=\mu_n(\beta,z^{(0)})$, $s_n=s_n(\beta,z^{(0)})$ *independent on* z, *such that*

$$f_n(z;\beta)=L_n\exp(-\mu_n(z-s_n)^2)(1+R_n(z))\,, \tag{8.5}$$

where $|R_n(z)|=|R_n(z;z^{(0)},\beta)|<\varepsilon_0(z_0)$ *for* $|z-z^{(0)}|<\sqrt{(1/\mu_n)\ln(z^{(0)}/\varepsilon)}$ *and the recursive relations are fulfilled*

$$\mu_{n+1}(\beta,\sqrt{c}z^{(0)})=(a_0(\beta)+(2/c)(\mu_n(\beta,z^{(0)})-a_0(\beta)))(1+O(\varepsilon_0(z^{(0)})))\,, \tag{8.6}$$

$$s_{n+1}(\beta,\sqrt{c}z^{(0)})=(2/\sqrt{c})(\mu_n(\beta,z^{(0)})/\mu_{n+1}(\beta,\sqrt{c}z^{(0)}))s_n(\beta,z^{(0)})(1+O(\varepsilon_0(z^{(0)})))\,. \tag{8.7}$$

Moreover for $\sqrt{\ln\varepsilon^{-1}}<|z^{(0)}|<2\sqrt{\ln\varepsilon^{-1}}$

$$\mu_n(\beta,z^{(0)})=(a_0(\beta)-L_1\lambda_1^n(\beta-\beta_{cr})/\beta_{cr}+L_2|z^{(0)}|^2)(1+O(|z^{(0)}|^{-1}))\,, \tag{8.8}$$

$$s_n(\beta,z^{(0)})=(L_3|z^{(0)}|^3/\mu_n(\beta,z^{(0)}))(1+O(|z^{(0)}|^{-1}))\,, \tag{8.9}$$

where $L_1, L_2, L_3 > 0$ *are independent on* β *and* $z^{(0)}$, *and for* $\varepsilon \to 0$ $L_1 \sim 1$, $L_2, L_3 \sim \varepsilon$, $L_2 L_3^{-1} \underset{\varepsilon\to 0}{\longrightarrow} \frac{3}{2}$. *And finally, for* $z^{(0)} = c^{(n-n_0)/2}\sqrt{\ln \varepsilon^{-1}}$ *and* $z > z^{(0)} + \sqrt{(1/\mu_n)\ln(z^{(0)}/\varepsilon)}$ $(1 + R_n(z)) < 1$.

Lemma 8.2. *For* $\beta \in [\beta^-, \beta_{cr}]$ *there is a sequence of the numbers* $z_{N(\beta)+1}(\beta) < z_{N(\beta)+2}(\beta) < \ldots$, $\varepsilon^{-0.49} < z_{N(\beta)+1} < \varepsilon^{-0.51}$ *such that all the statements of the Lemma 8.1 are valid for* $n > N(\beta)$ *for the points* $z^{(0)}$, $|z^{(0)}| > u_n(\beta) = z_n(\beta) - \sqrt{\mu_n^{-1}(\beta, z_n(\beta)) \ln z_n(\beta)}$ *and* $t = z_n(\beta)$ *is the solution of the equation* $s_n(\beta, t) = t$. *Besides, for* $|z| < u_n(\beta)$ $f_n(z;\beta) < 2 f_n(u_n(\beta);\beta)$. *For* $n \to \infty$ *there exists the limit* $c^{-n/2} z_n(\beta) \to M(\beta)$. ⌟

Let us elucidate the derivation of the critical index, connected with magnetization, and the equation of state in the vicinity of the critical point.

Suppose $n > N(\beta)$. Using Lemma 8.2 we may show [8], that

$$\| f_n(z;\beta) - G_n(z; z_n, \mu_n) \|_{C(R^1)} < \varepsilon_0(z_n),$$

where

$$G_n(z; z_n, \mu_n) = L_n[\exp(-\mu_n(z - z_n)^2) + \exp(-\mu_n(z + z_n)^2)],$$

$z_n = z_n(\beta)$ is the solution of the equation $s_n(\beta, t) = t$, $\mu_n = \mu_n(z_n(\beta))$. Therefore, the spontaneous magnetization is determined by the formula $M(\beta) = \lim_{n\to\infty} c^{-n/2} z_n(\beta)$. Denote $M_n(\beta) = z_n(\beta)/c^{n/2}$ and consider such m, that $\sqrt{\ln \varepsilon^{-1}} < c^{m/2} M_n(\beta) < 2\sqrt{\ln \varepsilon^{-1}}$. It may be shown that $m < N(\beta)$ [8]. From the asymptotical formulas (8.6), (8.7) it follows that

$$s_n(\beta, z_n(\beta)) = (2/\sqrt{c})^{n-m} (\mu_m(\beta, z^{(0)})/\mu_n(\beta, z_n(\beta))) s_m(\beta, z^{(0)})(1 + O(\varepsilon^{1/3})), \tag{8.10}$$

$$\mu_n(\beta, z_n(\beta)) = a_0(\beta) + (2/c)^{n-m}(\mu_m(\beta, z^{(0)}) - a_0(\beta))(1 + O(\varepsilon^{1/3})). \tag{8.11}$$

$$z^{(0)} = c^{m/2} M_n(\beta).$$

So far as $\sqrt{\ln \varepsilon^{-1}} < z^{(0)} < 2\sqrt{\ln \varepsilon^{-1}}$, we may use the formulas (8.8) and (8.9). The errors in the relations (8.8)–(8.11) can be neglected. Then for $M_n(\beta)$ we derive the equation:

$$s_n = z_n,$$

$$\mu_n s_n = \mu_n z_n,$$

$$(2/\sqrt{c})^{n-m} \mu_m s_m = (a_0 + (2/c)^{n-m}(\mu_m - a_0)) c^{(n-m)/2} z^{(0)},$$

$$\mu_m s_m = (\mu_m - a_0) z^{(0)},$$

[the term $(\sqrt{c}/4)^{n-m} a_0 \underset{n\to\infty}{\longrightarrow} 0$, which is inessential in deriving the asymptotics, may be omitted]

$$L_3 (z^{(0)})^3 = (-L_1 \lambda_1^m \tau + L_2 (z^{(0)})^2) z^{(0)},$$

$$L_3 M_n^2 = -L_1 (\lambda_1/c)^m \tau + L_2 M_n^2, \tag{8.12}$$

$$M_n^2 = L_4 (\lambda_1/c)^m \tau,$$

where $\tau=(\beta-\beta_{cr})/\beta$, $L_4=(L_2-L_3)/L_1>0$, since, as has been formulated in Lemma 8.1, $L_2L_3^{-1}\to\frac{3}{2}$ for $\varepsilon\to 0$. Further $\sqrt{\ln\varepsilon^{-1}}<c^{m/2}M_n<2\sqrt{\ln\varepsilon^{-1}}$, whence it follows that

$$(\lambda_1/c)^m=L_5M_n^{-2\log_c(\lambda_1/c)}, \qquad (\sqrt{\ln\varepsilon^{-1}})^{2\log_c(\lambda_1/c)}<L_5<(2\sqrt{\ln\varepsilon^{-1}})^{2\log_c(\lambda_1/c)}.$$

Thus,

$$\begin{aligned} &M_n^2=L_6M_n^{-2\log_c(\lambda_1/c)}\tau, \qquad L_6=L_4\cdot L_5, \\ &M_n=\tau^{[2(1+\log_c(\lambda_1/c))]^{-1}}. \end{aligned} \tag{8.13}$$

As a result we have found the critical index $\beta=[2(1+\log_c(\lambda_1/c)]^{-1}$. The neglecting of errors in the formulas (8.8)–(8.11) is substantiated as it has been done in the paper [8].

Let us derive now the equation of state. Suppose $f_n(z;\beta,H)$ is the density of the distribution of the random value $(\sqrt{c/2})^n\sum_{x\in V_n}u(x)$ in the Gibbs ensemble at the inverse temperature β, and at the external field H. It may be easily seen that

$$f_n(z;\beta,H)=L_n\exp(\beta H(2/\sqrt{c})^n)f_n(z;\beta,0).$$

For large values of n the function, as may be derived from the Lemmas 8.1, 8.2 [8] is close to the Gaussian density with the average $z_n=z_n(\beta,H)$ satisfying the equation

$$s_n(z_n)+(2c^{-\frac{1}{2}})^n\beta H/(2\mu_n(z_n))=z_n.$$

Let us denote $M_n=M_n(\beta,H)=c^{-n/2}z_n(\beta,H)$. It is clear that $M_n(\beta,H)$ is an odd function of M, thus, we may consider $H>0$.

Let us consider such m that $\sqrt{\ln\varepsilon^{-1}}<z^{(0)}<2\sqrt{\ln\varepsilon^{-1}}$, $z^{(0)}=M_nc^{m/2}$. It is easy to show that $m<N(\beta)$ for $\beta>\beta_{cr}$ [8]. Then analogously to (8.13) we obtain the equation

$$H=(L_7\tau M^{1-2\log_c(c\lambda_1/2)}+L_8M^{3+2\log_c(2/c^2)})\operatorname{sgn}M. \tag{8.14}$$

$\tau=(\beta-\beta_{cr})/\beta_{cr}$, $L_7, L_8>0$ are the constants, which gives the asymptotics of the equation of state in the neighbourhood of the critical point in the region $\beta\geqq\beta_{cr}$, $|M|\geqq M(\beta)$.

At $\beta<\beta_{cr}$ the asymptotics (8.14) holds true, provided $|M|c^{N(\beta)/2}>\sqrt{\ln\varepsilon^{-1}}$. Since

$$N(\beta)=\min\{n\cdot|\lambda_1^nb'(\beta)\tau|\}>(4/5)(2/c-1)^{-1},$$

this condition is equivalent to the following one

$$|M|\,|\tau|^{-\frac{1}{2}\log_{\lambda_1}c}<(\ln\varepsilon^{-1})^{-1}.$$

In fulfilling this condition the number m, determined from the condition $\sqrt{\ln\varepsilon^{-1}}<Mc^{m/2}<2\sqrt{\ln\varepsilon^{-1}}$, is less than $N(\beta)$, and therefore the asymptotics (8.14) takes place.

If the following condition is fulfilled

$$|M|/(|\tau|^{\frac{1}{2}}\log_{\lambda_1}c)<(\ln\varepsilon^{-1})^{-1}$$

then $m \gg N(\beta)$, and

$$f_m(z;\beta)=L_m\exp(-\mu_m z^2)(1+O(\varepsilon/2^{(m-N(\beta))/2})).$$

Thus, in this case the derivation of the equation state is reduced to the case of the Gaussian fixed point studied in [8]. The asymptotics of the equation of state in this case can be given by:

$$H=\text{const.}|\tau|^{-\gamma}M,$$

where $\gamma=1-\log_{\lambda_1}(c\lambda_1/2)$ is the critical index calculated in Theorem 2.

Appendix 1
Calculation of the Number

$$a=\int_{-\infty}^{\infty}e^{-\gamma z^2}G_{2k}(\sqrt{\gamma}z)\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty}e^{-u^2}G_{2k}(\sqrt{\gamma/c}z-\sqrt{\gamma}u)G_{2k}(\sqrt{\gamma/c}z+\sqrt{\gamma}u)du dz,$$
$$\gamma=1-c^{-1}.$$

Let us make a substitution of $z=t\gamma^{-\frac{1}{2}}$:

$$a=\int_{-\infty}^{\infty}e^{-t^2}G_{2k}(t)\Phi(t)dt,$$

where

$$\Phi(t)=(\pi\gamma)^{-\frac{1}{2}}\int_{-\infty}^{\infty}e^{-u^2}G_{2k}(t/\sqrt{c}-\sqrt{\gamma}u)G_{2k}(t/\sqrt{c}+\sqrt{\gamma}u)du.$$

Using the equality $e^{-t^2}G_{2k}(t)=(\pi^{\frac{1}{4}}2^k(2k)!^{\frac{1}{2}})^{-1}(d^{2k}e^{-t^2}/dt^{2k})$ and integrating by parts, we obtain

$$a=(\pi^{\frac{1}{4}}2^k(2k)!^{\frac{1}{2}})^{-1}\int_{-\infty}^{\infty}e^{-t^2}(d^{2k}\Phi(t)/dt^{2k})dt=(\pi^{\frac{1}{4}}2^k(2k)!^{\frac{1}{2}}(\pi\gamma)^{\frac{1}{2}})^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-t^2-u^2}$$
$$\cdot(d^{2k}/dt^{2k})[G_{2k}(t/\sqrt{c}-\sqrt{\gamma}u)G_{2k}(t/\sqrt{c}+\sqrt{\gamma}u)]du dt.$$

Lemma. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-t^2-u^2}G_j(t/\sqrt{c}+\sqrt{\gamma}u)G_i(t/\sqrt{c}-\sqrt{\gamma}u)dt du=(2/c-1)^i\delta_j^i\sqrt{\pi}$.

Proof. Suppose $i\leqq j$. We have $(1/\sqrt{c})^2+(\sqrt{\gamma})^2=c^{-1}+1-c^{-1}=1$. Therefore, the matrix

$$U=\begin{pmatrix}1/\sqrt{c} & -\sqrt{\gamma}\\ \sqrt{\gamma} & 1/\sqrt{c}\end{pmatrix}$$

is orthogonal. Let us make a substitution of the variables in the integral

$$\begin{pmatrix}w\\ v\end{pmatrix}=U\begin{pmatrix}t\\ u\end{pmatrix}.$$

We obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-w^2-v^2} G_j(w) G_i((-1+2/c)w+2\sqrt{\gamma/c}\,v)\,dw\,dv$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-w^2-v^2} G_j(w)\,[2/c-1)^i G_i(w)+Q(w,v)]\,dw\,dv$$

where the degree of the polynomial $Q(w, v)$ with respect to w does not exceed $i-1<j$, and, therefore,

$$\int_{-\infty}^{\infty} e^{-t^2} G_j(t) Q(t,v)\,dt=0\,.$$

Since $\int_{-\infty}^{\infty} G_j(t) G_i(t) e^{-t^2} dt=\delta_i^j$, the lemma is proved.

Let us use the known property of the Hermite polynomials:

$$d^i G_j(z)/dz^i=\sqrt{2^i j(j-1)\ldots(j-i+1)}\,G_{j-i}(z)\,.$$

In a combination with lemma this gives the following:

$$a=(\pi^{\frac{1}{4}} 2^k (2k)!^{\frac{1}{2}}(\gamma\pi)^{\frac{1}{2}})^{-1} C_{2k}^k 2^k\cdot((2k)!/k!)c^{-k}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-u^2-z^2} G_k(z/\sqrt{c}-u\sqrt{\gamma})$$
$$\cdot\, G_k(z/\sqrt{c}+u\sqrt{\gamma})\,dz\,du=([(2k)!]^{\frac{3}{2}}/(k!)^3)\,(2/c^2-1/c)^k(\pi^{\frac{1}{4}}(\pi\gamma)^{\frac{1}{2}})^{-1}\,.$$

The number a is calculated.

Appendix 2

One of the authors (Bleher) has investigated the renormalization group transformation for the hierarchical model in the case $d=1$, $r=2$, with the help of the computer. As a result all the critical indices for all the values of the parameter of the hierarchical model were found.

In the case under consideration the renormalization group transformation can be considered as the nonlinear integral mapping:

$$Q: f(z)\to \text{const}\int_{-\infty}^{\infty} e^{-u^2} f(z/\sqrt{c}+u)\,f(z/\sqrt{c}-u)\,du$$

where $1<c<2$ is a parameter of the hierarchical model. The first aim of the computations was to find all the thermodinamically-stable fixed points (TSFP) of the mapping Q. From the mathematical point of view it means that we seek fixed points for which the linearized mapping $L_f Q$ has explicitly one eigenvalue the modulo of which is bigger than one.

It is one of the results of the computations that for all the values of the parameter c, $1<c<2$, there exists one and only one TSFP of the transformation Q. For $\sqrt{2}<c<2$ this is the evident fixed point $f(x)\equiv$ const. The graphs of the TSFP for various values of the parameter c, $1<c<\sqrt{2}$, are shown on the Figs. 1–5. Probably for $c\to 1$ TSFP degenerates in a discrete measure. The branch of nonconstant TSFP have been considered rigorously before for sufficiently small $\varepsilon=\sqrt{2}-c>0$, where $\varepsilon=0$ is the point of the bifurcation of TSFP. The numerical computations show that there are not bifurcations of this branch of TSFP for all ε, $0<\varepsilon<\sqrt{2}-1$.

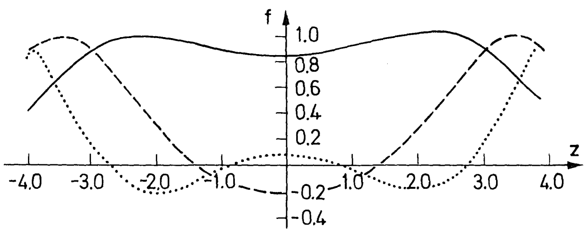

Fig. 1. The graphs of the TSFP (the continuous line), of the first eigenfunction (the interrupted line) and of the second eigenfunction (the dotted line) are plotted for $c=2^{0.45}$

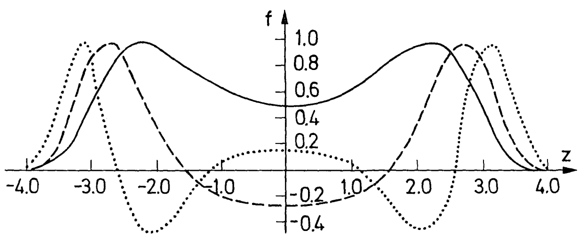

Fig. 2. $c=2^{1/3}$

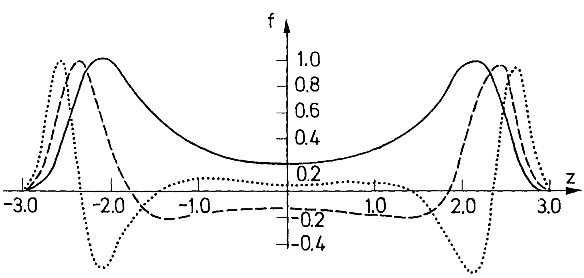

Fig. 3. $c=2^{0.2}$

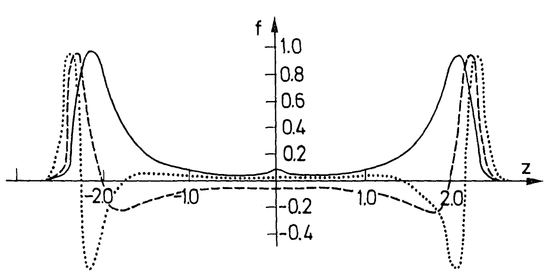

Fig. 4. $c=2^{0.1}$

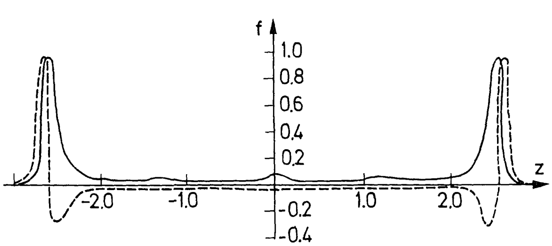

Fig. 5. $c=2^{0.03}$

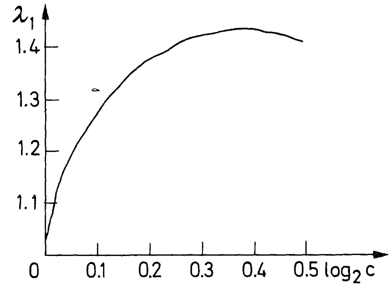

Fig. 6. The dependence of the first eigenvalue on the parameter $\log_2 c$

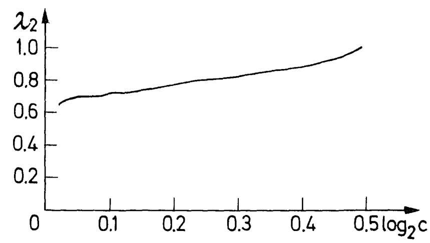

Fig. 7. The dependence of the second eigenvalue on the parameter $\log_2 c$

Our second aim was to compute the spectrum of the linearized mapping $L_f Q$ for TSFP. It is a very interesting problem because of as it was pointed out before all the critical indices of the asymptotically hierarchical models can be expressed via the first eigenvalue $\lambda_1 > 1$ of the operator $L_f Q$. On the Fig. 6 the dependence of λ_1 on the parameter c is plotted. One can see that there is a good agreement of this curve with the theoretical ε-expansions $\lambda_1 = (1 + \varepsilon/3 + O(\varepsilon^2))\sqrt{2}$ for $c = \sqrt{2} - \varepsilon$ and $\lambda_1 = 1 + \sqrt{\varepsilon}$ for $c = 1 + \varepsilon$. The last expansion is taken from the paper by Kosterlitz [13].

Finally on the Fig. 7 it is plotted the dependence of the second eigenvalue of the linearized operator $L_f Q$ on the parameter c. It is evident that $0 < \lambda_2 < 1$ for all the values c, $1 < c < \sqrt{2}$. This points out that the considering branch of TSFP has not any other bifurcation for $1 < c < \sqrt{2}$ indeed.

On the Fig. 1–5 the two first eigenfunctions of the operator $L_f Q$ are also plotted for some values of the parameter c.

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