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Free States and Automorphisms of the Clifford Algebra

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Abstract. We study automorphisms of the Clifford algebra which map the set of quasi-free states onto itself. We show that they are quasi-free if the one-particle space is infinite dimensional, and give counter examples in finite dimensions.

In a recent paper [1], Hugenholtz and Kadison have shown an automorphism of the *CAR* or Clifford algebra which maps the set of gauge invariant quasi-free states onto itself to be quasi-free. The same result is known for automorphisms which preserve the set of all quasi-free states [2]. We give simple, alterative proofs of these two results when the one particle space is infinite dimensional and counter-examples when it is not. Because of its economical description of non-gauge invariant free states and of non-unitary Bogoliubov transformations, we have worked in the real Hilbert space formalism of [3]. The connection between this and the complex Hilbert space formalism used in [1] is found in Section 2 of [3].

Let $(H, (,))$ be a real Hilbert space of even or infinite dimension. The C^*-Clifford algebra $\mathfrak{A}(H)$ over H is generated by the range of a linear map $f \to B(f)$ of H into self-adjoint part of $\mathfrak{A}(H)$, satisfying

$$B(f)B(g)+B(g)B(f)=2(f,g). \tag{1}$$

If H' is a subspace of even or infinite dimension we denote by $\mathfrak{A}(H')$ the C^*-subalgebra of $\mathfrak{A}(H)$ generated by $\{B(f) | f \in H'\}$. Every orthogonal transformation $\mathcal{O}$ on H defines a *-automorphism $\alpha_{\mathcal{O}}$ of $\mathfrak{A}(H)$ such that

$$\alpha_{\mathcal{O}} B(f) = B(\mathcal{O} f).$$

Such an automorphism is called quasi-free.

Every anti-hermitian operator A in the unit ball of $B(H)$ defines [3] a state ω_A such that

$$\omega_A(B(f_1) \dots B(f_N)) = \begin{cases} 0 \text{ if } N \text{ is odd} \\ \sum_{i=2}^{N} (-1)^i \omega_A(B(f_1)B(f_i))\omega_A(B(f_2) \dots \widehat{B(f_i)} \dots B(f_N)) & \text{otherwise} \end{cases} \tag{2}$$

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where $\widehat{\ }$ means the entry is omitted and $\omega_A(B(f)B(g))=(f,g)+i(Af,g)$. Such states are called quasi-free.

The quasi-free state $\omega_0(A\equiv 0)$ is the unique tracial state on $\mathfrak{A}(\mathscr{H})$ and so is invariant under any automorphism [4].

The first lemma answers the question, "When is the convex combination of quasi-free states quasi-free?" The proof is in the appendix.

Lemma 1. *Let A, B, C be anti-hermitian operators in the unit ball of $\mathscr{B}(\mathscr{H})$; let $\omega_A, \omega_B, \omega_C$ be the corresponding quasi-free states and let $0<\lambda<1$. Then,*

$$\omega_C=\lambda\omega_A+(1-\lambda)\omega_B \tag{3}$$

if and only if $C=\lambda A+(1-\lambda)B$ and there exists an orthonormal pair $\{f,g\}$ and $\mu>0$ such that

$$A-B=\mu(f\otimes g-g\otimes f).$$

If $\{f,g\}$ is an $o-n$ pair of vectors in H, we shall define the automorphism $\alpha_{\Theta}(f,g)$ as follows.

$$\alpha_{\Theta}(f,g)S=\exp((\Theta/2)B(g)B(f))S\exp((\Theta/2)B(f)B(g)) \quad \forall S\in\mathfrak{A}(H). \tag{4}$$

Let P be a 2-dimensional projector on H and let J be a complex structure on H commuting with P (i.e. $J^2=-1, J^+=-J$). Let $\tilde{P}_j$ denote the group of automorphisms generated by $\{\alpha_{\Theta}(f,Jf)|Pf=0\}$; and let $\mathfrak{A}(H,\tilde{P}_j)$ denote the algebra of $\tilde{P}_j$-fixed points of $\mathfrak{A}(H)$.

Lemma 2. *Let P and J be as above. If H is infinite dimensional, then $\mathfrak{A}(PH)=\mathfrak{A}(H,\tilde{P}_j)$.*

The proof is in the appendix.

Theorem 3a. *Let $\mathscr{H}$ be infinite dimensional; and, let α be a *-automorphism of $\mathfrak{A}(\mathscr{H})$ which maps the set of all quasi-free states onto itself. Then α is quasi-free.*

Proof. Let $\{f_1, f_2\}$ be an $o-n$ pair of vectors in $\mathscr{H}$. By α-invariance of ω_0, Lemma 1 and linearity of the transpose $\hat{\alpha}$ of α^{-1}, we have

$$(1-\lambda)\omega_0+\lambda\hat{\alpha}\omega_{f_1\otimes f_2-f_2\otimes f_1}=\hat{\alpha}\omega_{\lambda(f_1\otimes f_2-f_2\otimes f_1)} \quad 0<\lambda<1.$$

By hypothesis and Lemma 1, $\hat{\alpha}\omega_{f_1\otimes f_2-f_2\otimes f_1}=\omega_{\mu(g_1\otimes g_2-g_2\otimes g_1)}$ for some positive μ and $o-n$ pair $\{g_1,g_2\}$. Now, if $\{h,k\}$ is an arbitrary $o-n$ pair and η a real number ($|\eta|<1$), it follows easily that

$$\omega_{\eta(h\otimes k-k\otimes h)}=(1+i\eta B(h)B(k))\circ\omega_0.$$

Since ω_0 is faithful and invariant, it follows that $\alpha(B(f_1)B(f_2))=\mu B(g_1)B(g_2)$. Squaring both sides gives $\mu=1$. Now let $\{f_3, f_4\}$ be an $o-n$ pair, pairwise orthogonal to $\{f_1, f_2\}$, and let $\{g_3\, g_4\}$ be so that $\alpha(B(f_3)B(f_4))=B(g_3)B(g_4)$.

Now

$$\begin{aligned}0&=\omega_0(B(f_1)B(f_2)B(f_3)B(f_4))\\&=\omega_0(B(g_1)B(g_2)B(g_3)B(g_4))=(g_1,g_4)(g_2,g_3)-(g_1,g_3)(g_2,g_4).\end{aligned}$$

And,

$$\alpha[B(f_1)B(f_2), B(f_3)B(f_4)]=0=[B(g_1)B(g_2), B(g_3)B(g_4)].$$

Combining these two results, we have

$$0=\omega_0(B((g_2,g_3)g_4-(g_2,g_4)g_3)B(g_1)\,.$$
$$[B(g_1)B(g_2),\,B(g_3)B(g_4)])=2\{(g_2,g_3)^2+(g_2,g_4)^2\}\,.$$

Similary $(g_1,g_3)=(g_1,g_4)=0$. Thus, $\{g_1,g_2\}$ is pairwise orthogonal to $\{g_3,g_4\}$. Since α^{-1} has this property as well,

$$[\alpha(B(f_1)),\,B(h)B(h')]=0=[\alpha(B(f_2)),\,B(h)B(h')]$$

for all $o-n$ pairs $\{h,h'\}$ which are pairwise orthogonal to $\{g_1,g_2\}$. Let P be the projector upon the subspace spanned by $\{g_1,g_2\}$ and let J be a complex structure commuting with P. Clearly $\alpha B(f_1)$ and $\alpha B(f_2)$ are $\tilde{P}_j$-fixed points of $\mathfrak{A}(H)$. By Lemma 2, there exist numbers $\{X^j_\gamma\}^{j=1,\ldots,4}_{\gamma=1,2}$ such that

$$\alpha(B(f_\gamma))=X^1_\gamma B(g_1)+X^2_\gamma B(g_2)+X^3_\gamma B(g_1)B(g_2)+X^4_\gamma\,.$$

Clearly $\omega_0(\alpha B(f_\gamma))=\omega_0(\alpha B(f_\gamma)B(g_1)B(g_2))=0$.

Hence $X^3_\gamma=X^4_\gamma=0$. It follows from self-adjointness that X^1_γ and X^2_γ are real and from (1) that

$$\alpha B(f_1)=B(X_1g_1+X_2g_2)$$
$$\alpha B(f_2)=B(-X_2g_1+X_1g_2)$$

with $X_1^2+X_2^2=1$.

Now let $\{f_j\}_{j\in\mathbb{Z}}$ be an $o-n$ basis of H, and let $\{\mathcal{O}f_j\}$ be the orthonormal basis such that

$$\alpha B(f_j)=B(\mathcal{O}f_j)\,.$$

Let $\mathcal{O}$ be the unique linear, continuous extension of the basis transformation. $\mathcal{O}$ is orthogonal and, by linearity and by continuity,

$$\alpha B(f)=B(\mathcal{O}f)\qquad\forall f\in H\,.\qquad\text{Q.E.D.}$$

Theorem 3b. *Let H be finite dimensional* ($\dim H>2$). *There exists a *-automorphism of $\mathfrak{A}(H)$ which transforms the set of quasi-free states onto itself and which is not quasi-free.*

Proof. It suffices to exhibit a *-automorphism α which is non-free, which is trivial on even monomials and commutes with α_{-1}. Let $\{f_j\}_{j=1}^{2N}$ be an orthonormal basis of H, let

$$g_N=\begin{Bmatrix}1 & N\text{ odd}\\ i & N\text{ even}\end{Bmatrix}\quad\text{and let}$$

$U=2^{-\frac{1}{2}}(1-g_NB(f_1)\ldots B(f_{2N}))$. U is unitary and defines an automorphism with the above properties.

$$UB(f_k)U^*=g_NB(f_k)\{B(f_1)\ldots B(f_{2N})\}\qquad\forall k\,.\qquad\text{Q.E.D.}$$

We now indicate a simpler, alternative proof of Theorem 3.1 of [1]. Let J be a complex-structure on H, let H^J denote the J-complexification of H and let C denote a complex-conjugation on $H^J(C^2=1,\,CJ=-JC,\,C^+=C)$.

A quasi-free state ω_A is said to be gauge invariant if $[J, A]=0$. It is easily checked that $\mu(f\otimes g-g\otimes f)$ defines a gauge invariant quasi-free state if and only if $\mu(f\otimes g-g\otimes f)=\pm\mu(f\otimes Jf-Jf\otimes f)$. Thus, if α maps the set of gauge invariant quasi-free states onto itself, there exists, for every $f\in H$ a $g\in H$ such that $\alpha(B(f)B(Jf))=\pm B(g)B(Jg)$. Now

$$\|B(g)B(Jg)+B(h)B(Jh)\|$$

$\geqq|\omega_J(B(g)B(Jg)+B(h)B(Jh))|=\|g\|^2+\|h\|^2$. Further, since the map $f\to\alpha(B(f)B(Jf))$ is continuous, it follows that the two complementary subsets of $\{f|\,\|f\|\geqq\xi\}=H^\xi$

$$H^\xi_\pm=\{f|\,\|f\|\geqq\xi,\alpha(B(f)B(Jf))=\pm B(g)B(Jg)\quad\text{for some}\quad g\in H\}$$

are both closed and open. Since H^ξ is connected $H^\xi_+=\emptyset$ or H^ξ. Since ξ is arbitrary, either one of two cases is possible: For every $f\in H$ there exists $g\in H$ such that $\alpha B(f)B(Jf)=B(g)B(Jg)$; or, for every $f\in H$ there exists $g\in H$ such that $\alpha_C\alpha B(f)B(Jf)=B(g)B(Jg)$. Consider the first possibility. As above $\alpha B(f)$ is a fixed-point under the action of the group generated by $\{\alpha_\Theta(k, Jk)|\{g, Jg\}\perp\{k, Jk\}\}$. By Lemma 2, $\alpha B(f)$ and $\alpha B(Jf)$ are elements of the algebra generated by $\{B(g), B(Jg)\}$. It follows as above that there exists an orthogonal transformation $\mathcal{O}$ such that $\alpha(B(f))=B(\mathcal{O}f)$ for all $f\in H$. Further $[\mathcal{O}, J]=0$ implying that $\mathcal{O}$ is unitary on H^J. In the second case, $B(f)=B(\mathcal{O}f)$ with $\mathcal{O}=C\mathcal{O}'$ and $\mathcal{O}'$ unitary.

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Appendix

Proof of Lemma 1. Suppose (3) holds. Clearly $C=\lambda A+(1-\lambda)B$. Let $D=A-B$. Then, for arbitrary $\{f_j\}_{j=1}^4\subset H$, we have

$$\begin{aligned}0&=(\omega_{\lambda A+(1-\lambda)B}-(\lambda\omega_A+(1-\lambda)\omega_B))(B(f_1)\ldots B(f_4))\\&=-\lambda(\lambda-1)(f_1,\{(Df_3, f_4)D-Df_4\otimes Df_3+Df_3\otimes Df_4\}f_2)\,.\end{aligned}$$

Since f_1, f_2 are arbitrary, we have

$$(Df_3, f_4)D=Df_4\otimes Df_3-Df_3\otimes Df_4$$

Necessity follows by antisymmetry.

Conversely, suppose that $A-B=\mu(h\otimes g-g\otimes h)$ and $C=\lambda A+(1-\lambda)B$. It is clear that, for $N=2$ or $2S+1$ (S a positive integer),

$$(*)\qquad(\omega_{\lambda A+(1-\lambda)B}-(\lambda\omega_A+(1-\lambda)\omega_B))(B(f_1)\ldots B(f_N))=0\,.$$

To prove (*) in general it suffices, by the anticommutation relations (1) to choose $\{f_j\}$ from the elements of an orthonormal basis $\{k_i\}$ with $k_1=g, k_2=h$ and no entries repeated. We now proceed by induction: Suppose (*) holds for $N=2S$. By (2) and the induction hypothesis, we have

$$\begin{aligned}&(\omega_{\lambda A+(1-\lambda)B}-(\lambda\omega_A+(1-\lambda)\omega_B)(B(f_1)\ldots B(f_{2(S+1)}))\\&\qquad=i\mu\lambda(1-\lambda)\textstyle\sum_{j=2}^{2(S+1)}(-1)^j[(g, f_j)(h, f_1)-(f_1, g)(f_j, h)]\,.\\&\qquad(\omega_A-\omega_B)(B(f_2)\ldots\widehat{B(f_j)}\ldots B(f_{2(S+1)}))\,.\end{aligned}$$

We can assume $S \geqq 1$ and anti-commute on the left until $f_1 \neq g$ and $f_1 \neq h$, concluding that (*) holds for $N = 2(S+1)$. Q.E.D.

Proof of Lemma 2. Let C denote a complex-conjugation on H^J commuting with $P(C^2 = 1, C^+ = C$ and $CJ = -JC)$. Let $\{f_x\}_{x \in \mathbb{Z}}$ be an orthonormal basis of $(1-P)\left(\frac{1+C}{2}\right)H$ and let f be a normalized vector in $P\left(\frac{1+C}{2}\right)H$. $\mathfrak{A}((1-P)H)$ is the closed linear span of $\mathfrak{A}_0 \cup \{I\}$ where

$$\mathfrak{A}_0 \equiv \{B(g_1) \dots B(g_N) | \{g_n\}_{n=1}^N \subset \{f_x, Jf_x\}_{x \in \mathbb{Z}}\}$$

and $\mathfrak{A}(H)$ is the closed linear span of the set $\mathfrak{A}_1$ of elements of the type

$$S = B(f)S_1 + B(Jf)S_2 + B(f)B(Jf)S_3 + S_4$$

where $\{S_i\}_{i=1}^4 \subset \mathfrak{A}_0$.

Let

$$M = \frac{\alpha_{1+P(C-1)} + \alpha_1}{2} \cdot \frac{\alpha_{1-2P} + \alpha_1}{2}.$$

For $S \in \mathfrak{A}_1$, we clearly have

$$S = M(S) + B(f)M(B(f)S) + B(Jf)M(B(Jf)S) + B(f)B(Jf)M(B(Jf)B(f)S).$$

Since M is linear and continuous this identity holds for all $S \in \mathfrak{A}(H)$. Moreover $M\mathfrak{A}(H) = \mathfrak{A}((1-P)H)$.

It is clear that $S \in \mathfrak{A}(H, \tilde{P}_J)$ iff its coefficients in $\mathfrak{A}((1-P)H)$ are $\tilde{P}_J$-fixed points. Thus, it is sufficient to prove that

$$\mathfrak{A}((1-P)H) \cap \mathfrak{A}(H, \tilde{P}_J) = \{\mathbb{C}I\}.$$

To this end define the orthogonal shift T on H by $Tf_x = f_{x+1}$ and $TJf_x = Jf_{x+1}$, $TP = P$. It is clear that for all $A \in \mathfrak{A}_1, \alpha_T A = \lim_{L \to \infty} \alpha_{T_L} A$ where $[T_L, J] = 0$, $PT_L = P$ and

$$T_L f_x = \begin{cases} f_{x+1} & x \in [-L, L-1] \\ f_{-L} & x = L \\ f_x & x \notin [-L, L]. \end{cases}$$

By linearity and continuity $\alpha_T = \lim_{L \to \infty} \alpha_{T_L}$, pointwise. By standard Fourier methods one sees that

$$\alpha_{T_L} = \prod_{n=1}^{2L+1} \alpha_{\frac{2\pi n}{2L+1}}(k_n, Jk_n) \quad \text{where}$$

$$k_n = \frac{1}{\sqrt{2L+1}} \sum_{x=-L}^{L} e^{-2\pi J(x+L+1)n/2L+1} f_x.$$

Thus, if $S \in \mathfrak{A}((1-P)H) \cap \mathfrak{A}(H, \tilde{P}_J)$, then $\alpha_T S = S$.

Similarly $\alpha_{2P-1} = \lim_{L \to \infty} \prod_{x=-L}^{L} \alpha_\pi(f_x, Jf_x)$, so $\alpha_{2P-1} S = S$. Now the algebra of α_{2P-1}-fixed points of $\mathfrak{A}((1-P)H)$ is the range of the projector $M \cdot \frac{\alpha_1 + \alpha_{2P-1}}{2}$ and so it is the closed span $\mathfrak{A}_2$ of the even monomials in $\mathfrak{A}_0$.

Let S be an $\mathfrak{A}(H, \tilde{P}_J)$-fixed point of $\mathfrak{A}((1-P)H)$ and let S_n be a Cauchy sequence in $\mathfrak{A}_2$ converging to S. Now, for each n and $B \in \mathfrak{A}_1$, $\lim_{m \to \infty} \|[\alpha_{T^m} S_n, B]\| = 0$. By linearity, and a $3-\varepsilon$ argument,

$$\lim_{m \to \infty} [\alpha_{T^m} S, B] = [S, B] = 0 \qquad B \in \mathfrak{A}(H).$$

Thus $S = \lambda I$ for $\lambda \in \mathbb{C}$. Q.E.D.

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