Commun. math. Phys. 44, 1—7 (1975)

On Uniqueness of KMS States of One-dimensional Quantum Lattice Systems

Huzihiro Araki*

Institut für Theoretische Physik der Universität, Göttingen, Federal Republic of Germany

Received May 17, 1975

Abstract. Uniqueness of KMS states is proved for one-dimensional quantum lattice system. Sakai's theorem on uniqueness of KMS states is generalized to cases of non-commutative generators.

§ 1. Introduction

Uniqueness of equilibrium states for one-dimensional lattice system has been proved by Ruelle [7] for classical interactions and by Araki [1] for quantum interactions with a finite-range interaction. Simpler proofs have since been given for these cases (for example, see [8]. Also see Theorem 2 in [5]). It amounts to showing that any two states φ_1 and φ_2 satisfying the KMS condition are majorized by each other: $\varphi_1 \leqq \lambda\varphi_2 \leqq \lambda^2\varphi_1$ for some $\lambda > 0$.

We present here a proof of the uniqueness for one-dimensional quantum lattice system with an interaction Φ, which satisfies the same type of condition as known classical cases, namely surface energy has a bound independent of the volume. The key argument in the proof is Lemma 2 which states roughly that if the relative entropy of a state φ_1 with respect to a state φ_2 is finite, then the associated representation π_1 quasi-contains π_2.

To state the result more precisely, we use the following notation: The C^*-algebra $\mathfrak{A}$ under investigation will have the following structure as usual: For each integer ν, $\mathfrak{A}$ has a subalgebra $\mathfrak{A}_\nu$ mutually commuting for different ν. For any subset I of the set Z of all integers, $\mathfrak{A}(I)$ denotes the C^*-subalgebra of $\mathfrak{A}$ generated by $\mathfrak{A}_\nu$, $\nu \in I$. We assume that each $\mathfrak{A}_\nu$ is a type I finite factor and $\mathfrak{A}(Z) = \mathfrak{A}$. For each finite subset Λ of Z, an interaction potential $\Phi(\Lambda) \in \mathfrak{A}(\Lambda)$ is given such that

(0) $\Phi(\emptyset) = 0$,

(1) $\|\Phi\|_\alpha \equiv \sup_\nu \sum_\Lambda \{e^{\alpha N(\Lambda)} \|\Phi(\Lambda)\| ; \nu \in \Lambda\} < \infty$,

where $N(\Lambda)$ denotes the number of points in Λ and $\alpha > 0$,

(2) the following element $W(\Lambda_n)$ of $\mathfrak{A}$ for an increasing sequence of finite subsets Λ_n of Z is bounded in norm uniformly in n:

$$W(\Lambda) \equiv \sum_J \{\Phi(J); J \subset\subset Z, J \cap \Lambda \neq \emptyset, J \cap \Lambda^c \neq \emptyset\} . \tag{1.1}$$

Here Λ^c denotes the complement of Λ in Z and $\subset\subset$ denotes a finite subset.

* On leave from Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan.

The assumption (0) and (1) are sufficient condition for the existence of the limit

$$\alpha_t(Q) \equiv \lim_{\Lambda} e^{itU(\Lambda)} Q e^{-itU(\Lambda)}, \qquad Q \in \mathfrak{A}, \tag{1.2}$$

$$U(\Lambda) \equiv \sum_J \{\Phi(J); J \subset \Lambda\}, \tag{1.3}$$

which defines a one-parameter group α_t of automorphisms of $\mathfrak{A}$.

The assumption (2) is the key condition for the uniqueness of equilibrium states and is essentially the same as the classical cases [7].

Our main result:

Theorem 1. *For any β real, $\mathfrak{A}$ has one and only one α_t-KMS state at the inverse temperature β.*

The proof will be given under more abstract setting, which leads to a generalization of Sakai's result [8]: Let $\mathfrak{A}$ be a C^*-algebra generated by an increasing sequence of C^*-subalgebras $\mathfrak{A}_n$ of $\mathfrak{A}$, which are full matrix algebras. Let α_t be a one-parameter group of automorphisms of $\mathfrak{A}$ such that $\alpha_t(Q)$ is continuous in t for each $Q \in \mathfrak{A}$. Assume that there exists $h_n = h_n^* \in \mathfrak{A}$ for each n satisfying

$$(d/dt)\alpha_t(Q)|_{t=0} = i[h_n, Q] \tag{1.4}$$

for all $Q \in \mathfrak{A}_n$. Let τ be the unique tracial state on $\mathfrak{A}$ and $\bar{h}_n \in \mathfrak{A}_n$ be the conditional expectation of h_n: $\tau(h_n Q) = \tau(\bar{h}_n Q)$, $Q \in \mathfrak{A}_n$.

An abstract version of Theorem 1 is as follows:

Theorem 2. *Assume that*

$$\sup_n \|h_n - \bar{h}_n\| \equiv \lambda < \infty . \tag{1.5}$$

Then $\mathfrak{A}$ has at most one α_t-KMS state for each inverse temperature β.

Remark 1. If there exists $\hat{h}_n \in \mathfrak{A}_n$ satisfying

$$\sup_n \|h_n - \hat{h}_n\| < \infty , \tag{1.6}$$

then the condition (1.5) is satisfied: $\bar{h}_n - \hat{h}_n$ is the conditional expectation of $h_n - \hat{h}_n$, which implies

$$\|\bar{h}_n - \hat{h}_n\| \leqq \|h_n - \hat{h}_n\| .$$

Hence

$$\|h_n - \bar{h}_n\| \leqq 2\|h_n - \hat{h}_n\| .$$

Remark. 2. In the concrete case of Theorem 1, we may set $\mathfrak{A}_n = \mathfrak{A}(\Lambda_n)$, $h_n = U(\Lambda_n) + W(\Lambda_n)$, $\hat{h}_n = U(\Lambda_n)$. Then Theorem 2 and Remark 1 implies the uniqueness part of Theorem 1. The existence is well-known. Thus it is sufficient to prove Theorem 2.

§ 2. Quasi Containment

Two representations π_1 and π_2 of a C^*-algebra $\mathfrak{A}$ is said to be quasiequivalent if kernels of π_1 and π_2 coincide and the mapping $\pi_1(Q) \rightarrow \pi_2(Q)$, $Q \in \mathfrak{A}$, extends to a $*$-isomorphism of weak closures. In the present case, $\mathfrak{A}$ is simple and ker $\pi_1 =$

ker $\pi_2=0$. If a subrepresentation of π_1 is quasi-equivalent to π_2, then π_1 is said to quasi-contain π_2.

Let φ_1 and φ_2 be states of $\mathfrak{A}$. Let $\mathfrak{H}_j$, π_j and Ω_j be the space, representation and cyclic vector associated with φ_j, $j=1,2$.

Lemma 1. *If π_1 does not quasi-contain π_2, there exists a sequence of projections $e_m \in (\bigcup_n \mathfrak{A}_n)$ such that*

$$\lim_m \varphi_1(e_m)=0\,, \tag{2.1}$$

$$\lim_m \varphi_2(e_m)=a>0\,. \tag{2.2}$$

Proof. Consider the representation $\pi=\pi_1\oplus\pi_2$ on $\mathfrak{H}=\mathfrak{H}_1\oplus\mathfrak{H}_2$ with vectors $\Phi_1=\Omega_1\oplus 0$ and $\Phi_2=0\oplus\Omega_2$. Let $\mathfrak{M}=\pi(\mathfrak{A})''$, $\mathfrak{Z}$ be the center of $\mathfrak{M}$ and E_j be the $\mathfrak{Z}$-support of Φ_j, $j=1,2$. A condition that π_1 quasi-contains π_2 is equivalent to $E_1 \geqq E_2$. Since this condition is not satisfied, there exists a non-zero central projection E such that $EE_1=0$ and $E\leqq E_2$ Since $\pi(\bigcup_n \mathfrak{A}_n)$ is dense in $\mathfrak{M}$, there exists a sequence $a_m \in \mathfrak{A}_{n(m)}$ (for some $n(m)$) satisfying

$$\lim_m \pi(a_m)=E\,.$$

Let e_m be the spectral projection of a_m for an interval $[1-\delta, 1+\delta]$ where $\delta\in(0,1)$ is fixed. Then $e_m\in\mathfrak{A}_{n(m)}$ and

$$\lim_m \pi(e_m)=E$$

by a theorem of Kaplansky [6]. Since $EE_1=0$, $E\Phi_1=0$. Since $E\leqq E_2$ and $E\neq 0$, $E\Phi_2\neq 0$. Hence (2.1) and (2.2) are satisfied with $a=\|E\Phi_2\|^2>0$.

§ 3. Relative Entropy

For two states φ_1 and φ_2 of a matrix algebra, the relative entropy is defined by

$$S(\varphi_1/\varphi_2)=\varphi_2(\log\varrho_2)-\varphi_2(\log\varrho_1) \tag{3.1}$$

where ϱ_j is the density matrix for φ_j.

For two faithful states of a von Neumann algebra $\mathfrak{M}$ the definition has been extended with a help of relative modular operators [2], [3]. In particular, for a state φ^h obtained from a faithful state φ by a perturbation $h=h^*\in\mathfrak{M}$, we have

$$S(\varphi^h/\varphi)=-\varphi(h)\,, \tag{3.2}$$

$$S(\varphi/\varphi^h)=\varphi^h(h)\,. \tag{3.3}$$

If N is a von Neumann subalgebra of $\mathfrak{M}$ and φ_j^N denotes the restriction of φ_j to $\mathfrak{N}$, the monotonicity

$$0\leqq S(\varphi_1^N/\varphi_2^N)\leqq S(\varphi_1/\varphi_2) \tag{3.4}$$

has been proved for hyperfinite $\mathfrak{M}$ and $\mathfrak{N}$ [2]. (For finite matrices, non-faithful φ_j are allowed.)

If $e \in \mathfrak{M}$ is a projection operator, the inequality (3.4) for $\mathfrak{N}$ generated by e and $(1-e)$ yield

$$S(\varphi_1/\varphi_2) \geqq \varphi_2(e) \log \{\varphi_2(e)/\varphi_1(e)\} + \varphi_2(1-e) \log \{\varphi_2(1-e)/\varphi_1(1-e)\}\,. \tag{3.5}$$

Lemma 2. *Let φ_1 and φ_2 be states of $\mathfrak{A}$ and φ_j^n denote the restriction of φ_j to $\mathfrak{A}_n$. If*

$$\sup_n S(\varphi_1^n/\varphi_2^n) \equiv \lambda_1 < \infty\,, \tag{3.6}$$

then π_1 quasi-contains π_2 where π_j is the cyclic representation of $\mathfrak{A}$ associated with φ_j.

Proof. Assume that π_1 does not quasi-contain π_2. By Lemma 1, there exists a sequence of projections $e_m \in \mathfrak{A}_{n(m)}$ such that $\varphi_1(e_m) \to 0$ and $\varphi_2(e_m) \to a > 0$. Then

$$-\varphi_2(e_m) \log \varphi_1(e_m) \to +\infty\,,$$

while

$$\varphi_2(e_m) \log \varphi_2(e_m) + \varphi_2(1-e_m) \log \varphi_2(1-e_m) \geqq -\log 2\,,$$

$$-\varphi_2(1-e_m) \log \varphi_1(1-e_m) \geqq 0\,.$$

These estimates contradicts with the bound (3.6) when $\varphi_j^{n(m)}$ and e_m are substituted into φ_j and e of the inequality (3.5).

§ 4. Gibbs Condition

Let $\mathfrak{A}'_N$ denote the commutant of $\mathfrak{A}_N$ in $\mathfrak{A}$. Then $\mathfrak{A} = \mathfrak{A}_N \otimes \mathfrak{A}'_N$. Let τ_N and τ'_N denote the restriction of the tracial state τ of $\mathfrak{A}$ to $\mathfrak{A}_N$ and $\mathfrak{A}'_N$. Let

$$\varphi_N^G(Q) = \tau_N(e^{-\beta \bar{h}_N} Q)/\tau_N(e^{-\beta \bar{h}_N})\,. \tag{4.1}$$

Let $W(N) \equiv h_N - \bar{h}_N$. A state φ of $\mathfrak{A}$ is said to satisfy the Gibbs condition at β if

(i) The normal extension $\hat{\varphi}$ of φ to the weak closure $\mathfrak{M} = \pi_\varphi(\mathfrak{A})''$ of the associated representation is faithful on $\mathfrak{M}$ and

(ii) for every N, $\varphi^{\beta W(N)} = \varphi_N^G \otimes \varphi'_N$ for some linear positive functional φ'_N on $\mathfrak{A}'_N$.

Theorem 3. *If φ satisfies the KMS condition at β, it satisfies the Gibbs condition at β.*

Proof. The condition (i) is known to follow from the KMS condition. Let

$$\psi = \{\varphi^{\beta W(N)}(1)\}^{-1} \varphi^{\beta W(N)} \tag{4.2}$$

be a state on $\mathfrak{A}$ obtained from φ by a perturbation $\beta W(N) - \{\log \varphi^{\beta W(N)}(1)\}\, 1$.

Let σ_t^φ and σ_t^ψ be modular automorphisms of $\mathfrak{M}$ for states $\hat{\varphi}$ and $\hat{\psi}$ (the normal extensions of φ and ψ to $\mathfrak{M}$). Then

$$(d/dt)\{\sigma_t^\psi(x) - \sigma_t^\varphi(x)\}_{t=0} = i\beta[\pi_\varphi(W(N)), x] \tag{4.3}$$

for $x \in \mathfrak{M}$. The KMS condition implies

$$\sigma_t^\varphi(\pi_\varphi(Q)) = \pi_\varphi(\alpha_{-\beta t}(Q)), \qquad Q \in \mathfrak{A}\,. \tag{4.4}$$

By (1.4), (4.4) and (4.3), we obtain

$$(d/dt)\sigma_t^{\psi}(\pi_{\varphi}(Q))|_{t=0} = -i\beta\pi_{\varphi}([\bar{h}_N, Q]) \tag{4.5}$$

for $Q \in \mathfrak{A}_N$. By the group property,

$$(d/dt)\sigma_t^{\psi}(x) = \sigma_t^{\psi}\{(d/ds)\sigma_s^{\psi}(x)|_{s=0}\}\,. \tag{4.6}$$

Let

$$\alpha_t^N(Q) = e^{it\bar{h}_N} Q e^{-it\bar{h}_N}\,. \tag{4.7}$$

Then

$$(d/dt)\alpha_t^N(Q) = i[\bar{h}_N, \alpha_t^N(Q)]\,. \tag{4.8}$$

From (4.5), (4.6) and (4.8), we obtain

$$(d/dt)\sigma_t^{\psi}(\pi_{\varphi}\{\alpha_{\beta t}^N(Q)\}) = 0 \tag{4.9}$$

for $Q \in \mathfrak{A}_N$. This implies

$$\sigma_t^{\psi}\{\pi_{\varphi}(Q)\} = \pi_{\varphi}\{\alpha_{-\beta t}^N(Q)\}, \qquad Q \in \mathfrak{A}_N\,. \tag{4.10}$$

In particular

$$\pi_{\varphi}(\bar{h}_N) \in \mathfrak{M}^{\psi}\,. \tag{4.11}$$

where the centralizer $\mathfrak{M}^{\psi}$ is the set of $x \in \mathfrak{M}$ invariant under σ_t^{ψ}. If we set $\psi_1 = \psi^{\beta\bar{h}_N}$, then (4.11) implies

$$\psi_1(Q) = \psi(e^{\beta\bar{h}_N} Q), \qquad Q \in \mathfrak{A}\,, \tag{4.12}$$

and

$$\sigma_t^{\psi_1}(Q) = e^{i\beta\pi_{\varphi}(\bar{h}_N)} \sigma_t^{\psi}(x) e^{-i\beta\pi_{\varphi}(\bar{h}_N)} \tag{4.13}$$

for $x \in \mathfrak{M}$. The last equation together with (4.10) imply

$$\pi_{\varphi}(\mathfrak{A}_N) \in \mathfrak{M}^{\psi_1}\,. \tag{4.14}$$

If $Q_1, Q_2 \in \mathfrak{A}_N$ and $Q' \in \mathfrak{A}'_N$, then

$$\begin{aligned}\psi_1(Q_1(Q_2Q')) &= \psi_1((Q_2Q')Q_1) \quad \text{(by (4.14))}\\ &= \psi_1(Q_2Q_1Q')\end{aligned}$$

which implies

$$\psi_1([Q_1, Q_2]Q') = 0\,. \tag{4.15}$$

Since $\mathfrak{A}_N$ is a full matrix algebra, any element $Q \in \mathfrak{A}_N$ can be written as

$$Q = \tau_N(Q)\mathbf{1} + \textstyle\sum_j [Q_{j1}, Q_{j2}] \tag{4.16}$$

for some $Q_{j1}, Q_{j2} \in \mathfrak{A}_N$. Hence (4.15) implies

$$\psi_1(QQ') = \tau_N(Q)\psi_1(Q') \tag{4.17}$$

for $Q \in \mathfrak{A}_N$, $Q' \in \mathfrak{A}'_N$. Namely $\psi_1 = \tau_N \otimes \psi'_1$ where ψ'_1 is the restriction of ψ_1 to $\mathfrak{A}'_N$. Because of (4.12), we obtain (ii) of the Gibbs condition.

Remark. What we need in the subsequent application is a part of the Gibbs condition, which says that the restriction of $\varphi^{\beta W(N)}$ to $\mathfrak{A}_N$ is the Gibbs state φ_N^G up to a normalization constant $\varphi^{\beta W(N)}(1)$. This much is deduced immediately from (4.10) by the uniqueness of KMS states for full matrix algebra.

§ 5. Proof of Theorem 2

Let φ_∞ be any one of the accumulation points of the sequence of states $\varphi_n^G \otimes \tau'_n$ at $n=\infty$. Let φ be an arbitrary extremal α_t-KMS state at β. By a known general result, φ is primary.

Let p be a fixed positive integer. Since $\mathfrak{A}_p$ is of finite dimension and φ_∞ is a weak accumulation point of $\varphi_n^G \otimes \tau'_n$, there exists an integer $N(\varepsilon)$ for any given $\varepsilon > 0$ such that $N(\varepsilon) \geqq p$ and

$$\|(\varphi_{N(\varepsilon)}^G)_p - (\varphi_\infty)_p\| < \varepsilon \tag{5.1}$$

where $(\varphi)_p$ denotes the restriction of φ to $\mathfrak{A}_p$. Note that $(\varphi_N^G)_p = (\varphi_N^G \otimes \tau'_N)_p$ for $N \geqq p$.

By (3.4), we have

$$0 \leqq S((\varphi)_p/(\psi_N)_p) \leqq S(\varphi/\psi_N) \tag{5.2}$$

where ψ_N denotes the state ψ given by (4.2). By (3.3), we have the following estimate:

$$\begin{aligned} S(\varphi/\psi_N) &= \psi_N(\beta W(N)) - \log \varphi^{\beta W(N)}(1) \\ &\leqq \psi_N(\beta W(N)) - \varphi(\beta W(N)) \\ &\leqq 2|\beta|\lambda \end{aligned} \tag{5.3}$$

where we have used (1.5) and the following Peierls-Bogolubov inequality [4]

$$\log \varphi^{\beta W(N)}(1) \geqq \varphi(\beta W(N))$$

which follows from $S(\psi_N/\varphi) \geqq 0$ for example.

By the Gibbs condition, $(\psi_N)_p = (\varphi_N^G)_p$ for $N \geqq p$. Hence (5.2) and (5.3) imply

$$0 \leqq S((\varphi)_p/(\varphi_N^G)_p) \leqq 2|\beta|\lambda\,. \tag{5.4}$$

The function $\mathrm{tr}(\varrho \log \varrho)$ of the density matrices ϱ for a finite dimensional case is bounded and continuous. If σ is strictly positive, $\mathrm{tr}(\varrho \log \sigma)$ is also bounded and continuous as a function of ϱ. Hence

$$S((\varphi)_p/(\varphi_\infty)_p) = \lim_{\varepsilon \to 0} S((\varphi)_p/(\varphi_{N(\varepsilon)}^G)_p)$$

due to (5.1). By (5.4), we obtain

$$0 \leqq S((\varphi)_p/(\varphi_\infty)_p) \leqq 2|\beta|\lambda\,. \tag{5.5}$$

Since p is any positive integer, Lemma 2 implies that the cyclic representation π associated with φ quasi-contains the cyclic representation π_∞ associated with φ_∞. Since π is primary, this implies that π and π_∞ are quasiequivalent. Since φ_∞ is fixed, any primary KMS states are mutually quasiequivalent. The proof of Theorem 2 is then completed by the following Lemma.

Lemma 3. *If two extremal KMS-states φ and φ' of a C^*-algebra $\mathfrak{A}$ at the same β have quasi-equivalent associated cyclic representations, then $\varphi = \varphi'$.*

Proof. Let $\mathfrak{H}$, π and Ω be canonically associated with φ and $\mathfrak{M} = \pi(\mathfrak{A})''$. Since φ is a KMS-state, Ω is separating (and cyclic by definition). By quasi-equivalence, there exists $\Omega' \in V_\Omega^{1/4}$ such that the associated vector states is φ', where $V_\Omega^{1/4}$ denotes the natural positive cone (see [3], for example). Since φ' is a KMS-state, Ω' is separating for $\mathfrak{M}$ and hence is also cyclic (see [3], for example). Let the unitary cocycle (the intertwining operator for modular automorphisms) be denoted by

$$u_t^{\varphi\varphi'} = \Delta_{\Omega',\Omega}^{it} \Delta_\Omega^{-it} .$$

Since the KMS condition characterizes the modular automorphisms, we have $\sigma_t^{\varphi'} = \sigma_t^{\varphi} (= \pi_\varphi \alpha_{-\beta t} \pi_\varphi^{-1}$ on $\pi_\varphi(\mathfrak{A}))$ and hence

$$u_t^{\varphi\varphi'} \in \mathfrak{M} \cap \mathfrak{M}' .$$

Since φ is an extremal KMS state, the center $\mathfrak{M} \cap \mathfrak{M}'$ is trivial and hence $u_t^{\varphi\varphi'} = e^{ict}$ for some real c. By analytic continuation, we have

$$\Omega' = u_{-i/2}^{\varphi'\varphi} \Omega = e^{c/2} \Omega .$$

Hence $\varphi = \varphi'$.

Acknowledgement. The present work is completed while the author is a Gauß Professor of the Göttingen Academy of Sciences. The author would like to thank the Akademie der Wissenschaften zu Göttingen for financial support. Hospitality of Professor H. J. Borchers and members of the Institut für Theoretische Physik, Universität Göttingen, is gratefully acknowledged.

References

1. Araki, H.: Commun. math. Phys. **14**, 120—157 (1968)
2. Araki, H.: Commun. math. Phys. **38**, 1—10 (1974)
3. Araki, H.: Recent developments in the theory of operator algebras and their significance in theoretical physics. To appear in Proceedings of convegno sulle algebre C^*e loro applicazioni in Fisica Teorica, Rome, 1975
4. Araki, H.: Commun. math. Phys. **34**, 167—178 (1973)
5. Dobrushin, R. L.: Funktsional'nyi Analiz i Ego Prilozheniya **2** (4), 44—57 (1968)
6. Kaplansky, I.: Pacific J. Math. **1**, 227—232 (1951)
7. Ruelle, D.: Commun. math. Phys. **9**, 267—278 (1968)
8. Sakai, S.: On commutative normal *-derivations II (preprint)

Communicated by G. Gallavotti

H. Araki
Research Institute
for Mathematical Sciences
Kyoto University
Kyoto 606, Japan