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Analyticity Properties and Many-Particle Structure in General Quantum Field Theory

II. One-Particle Irreducible n-Point Functions

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Abstract. The extraction of one-particle singularities from the n-point functions is performed in the framework of L.S.Z. field theory in the case of a single massive scalar field. It is proved that the "one-particle irreducible" functions thus obtained enjoy the analytic and algebraic primitive structure of general n-point functions (up to a finite number of generalized C.D.D. singularities). Finally under an additional technical assumption, it is shown that the Glaser-Lehmann-Zimmermann relations stating the completeness of asymptotic states yield similar relations satisfied in any given channel by the corresponding one-particle irreducible functions.

I. Introduction

In the first paper of this series [1] and in a previous work [2], we described a general method for investigating the analyticity properties of the n-point Green's functions implied by the non-linear structure of general quantum field theory.

This non-linear program makes use conjointly of the rigorous results of the linear program expressed in complex momentum space, and of the many-particle structure analysis (M.P.S.A.) due to Symanzik [3] where an essential role is played by the notion of p-particle irreducible (p.i.) part of a Green's function.

The advantage for a synthesis of these two approaches is twofold. On one hand the M.P.S.A. program can be carried out there on a rather rigorous level since it can be developed in the framework of the analyticity properties of Green's functions which is well-established in the L.S.Z. field theory [5–7]. Then it is expected that (apart from certain technical postulates which have to be added to the axioms of the theory) the introduction of irreducible n-point functions can be made rigorous in complex momentum space and that their analytic structure can be clearly exhibited (except for the possible occurence of generalized C.D.D. poles as explained in the following).

On the other hand the use of the non-linear information of field theory, expressed through the *completeness relations* of Glaser, Lehmann and Zimmermann [8], should lead in the M.P.S.A. context to important improvements in the knowledge of the global analytic structure of the n-point functions. Actually the rigorous introduction of the irreducible functions in axiomatic field theory seems to enrich the latter by a powerful algorithm which is borrowed from the perturbative approach [3].

This paper is devoted to the study of the first and simplest case $p = 1$, namely to the extraction of one-particle singularities from the n-point functions of a local field.

This had been already performed by Zimmermann in his work on time-ordered Green's functions [4]. Our construction of the one-p.i. functions (with respect to *one* channel and to *all* channels) can be considered as the incorporation of the ideas of [3, 4] in the Steinmann-Ruelle-Araki framework of the analyticity properties of the n-point functions (which was not at that time completely analysed). Such a point of view had been already adopted by Völkel and results obtained in the case $n = 4$ [9].

For the sake of simplicity, we consider the case of a single Bose self-interacting field describing massive scalar particles. On the basis of the Wightman axioms [10, 11] supplemented by the convenient postulate that "sharp" retarded operators should exist [7], a set of one p.i. n-point functions can be constructed. They are proved to enjoy the same analytic and linear algebraic structure as the complete n-point functions, with additional analyticity properties on appropriate polar manifolds of the latter. The proof makes use of the conservation of the primitive structure of n-point functions by G-convolution in the simplest case of tree-products [1].

It is clear that this procedure cannot be unique since there is an infinite number of analytic functions which have the same polar singularities. However this ambiguity can be removed if one is concerned with the exact analogues of the perturbative formal sums of one-p.i. ("proper") Feynmann graphs. Then it can be proved that these "*physical*" one-p.i. n-point functions satisfy a system of *non-linear* relations similar to those of G.L.Z. [8] linking the complete n-point functions (these relations being always considered here in the formulation given to them by Steinmann [7], which involves generalized retarded functions).

The derivation of this system is given on the basis of asymptotic completeness but it can only be achieved if an additional technical postulate is assumed. This postulate requires that appropriate matrix elements of the spectral measure $dE(p)$ of the translation group in the Hilbert space of the theory should have a certain smoothness property as a function of the four-momentum p.

We must now point out the occurrence of a well-known disease in the construction of the physical one-p.i. n-point functions. It is the possibility of producing new singularities induced by the zeros of the two-point function. By using Symanzik's analysis [3] it is recognized that these singularities can be divided into two classes:

i) a finite number of poles which cross the primitive domain of the n-point functions and whose possible occurrence is connected with the existence of ultraviolet divergences for the propagator.

ii) singularities lying in the real space inside the continuous part of the spectral region: these are singularities of the C.D.D. type [12].

Whether both types of singularities are spurious or not is not known at present, and we shall not discuss this point here.

In Section II we begin by a recall of the axioms of the theory; in particular the additional technical postulate which we need is described there, under the name of "smooth spectral condition". Section III is devoted to a review of the linear and non-linear properties of the n-point functions, in particular to the completeness relations for generalized retarded distributions [7].

In Section IV we extract the one-particle singularities from the n-point functions in *one* arbitrary channel: the general property of "factorization of residues" is proved and the one-p.i. functions are constructed and shown to satisfy the primitive structure of the n-point functions. In Section V, the "smooth spectral condition" is used conjointly with asymptotic completeness to investigate their non-linear structure. Finally, Section VI is devoted to the construction of the one-p.i. n-point functions with respect to *all* channels.

The proof of completeness relations for the one-p.i. functions raises questions of mathematical rigor concerning the restriction of distributions to the mass-shell and the integration of these restrictions over the mass-shell. These questions are ruled out by using the local analytic structure of general n-point functions in the way of [26]; the desired results are established in Appendix B. Some mathematical facts concerning the local analytic structure of distributions are listed in Appendix A.

II. The Basic Postulates

We first recall the postulates of the rigorous L.S.Z. field theory. They are in brief

A. *Wightman Axioms* [10, 11]

i) Relativistic Quantum Mechanics: There exists a Hilbert space of states $\mathscr{H}$ in which a unitary representation $U(\Lambda, a)$ of the Poincaré group operates.

ii) Spectral Assumption: In the spectral decomposition of the translation group $U(1, a) = \int e^{ip\cdot a}\, dE(p)$ the support of the spectral measure $dE(p)$ is the following

$$\mathrm{Sp} = \{0\} \cup H_m^+ \cup \bar{V}_{2m}^+$$

with

$$H_m^+ = \{p \in \mathbb{R}^4 : p^0 = \omega(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2 + m^2}\}$$

$$\bar{V}_{2m}^+ = \{p \in \mathbb{R}^4 : p^0 \geqq \sqrt{\boldsymbol{p}^2 + 4m^2}\}.$$

In Sections IV and VI, the case of several masses m_μ is also considered, namely $\mathrm{Sp} = \{0\} \cup \left(\bigcup_\mu H_{m_\mu}^+\right) \cup \bar{V}_{2m}^+$, with $m = \inf_\mu \{m_\mu\}$, $\sup_\mu \{m_\mu\} \langle 2m$.

To this structure of Sp corresponds the following decomposition of $\mathscr{H}$:

$$\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1 \oplus \mathscr{H}'.$$

where $\mathscr{H}_0$ is one-dimensional and generated by a unique vector $|\Omega\rangle$ called the vacuum, $\mathscr{H}_1$ is a (non-degenerate) subspace of one-particle states with mass(es) m, (resp. $\{m_\mu\}$) and $\mathscr{H}'$ corresponds to the continuous part $\bar{V}_{2m}^+$ of the spectrum.

iii) Field: there exists an operator-valued distribution $A(x)$ which satisfies

$$[A(x), A(y)] = 0 \quad \text{for} \quad (x-y)^2 < 0 \quad \text{(locality)}$$

$$U(\Lambda, a)\, A(x)\, U(\Lambda, a)^{-1} = A(\Lambda x + a) \quad \text{(relativistic invariance)}.$$

The vectors $A^n(\varphi)|\Omega\rangle = [\int A(x_1) \dots A(x_n)\, \tilde{\varphi}(x_1, \dots, x_n)\, dx_1 \dots dx_n]\, |\Omega\rangle$ span a dense subspace D of $\mathscr{H}$, and $A(x)$ is assumed to have non-vanishing matrix elements between $|\Omega\rangle$ and the subspace $\mathscr{H}_1$. More precisely for any test-function f in $\mathscr{S}(\mathbb{R}^4)$ with its support inside a neighbourhood of $H^+_{m_\mu}$ which does not intersect the rest of the spectrum, and with restriction $\hat{f}_\mu$ to $H^+_{m_\mu}$, we write

$$\|A(f)\,|\Omega\rangle\|^2 = 2\pi Z_\mu \int |\hat{f}_\mu(p)|^2 \frac{dp}{2\omega_\mu} \tag{1}$$

with the wave-function normalization constant $Z_\mu \neq 0$ and $\omega_\mu = \sqrt{p^2 + m_\mu^2}$.

Under these axioms, the Haag-Ruelle-Hepp asymptotic theory can be performed [6, 13, 14] and the sets of asymptotic states there constructed span two subspaces $\mathscr{H}_{\text{in}}$ and $\mathscr{H}_{\text{out}}$ of $\mathscr{H}$. Then the two following postulates are necessary to carry out further steps.

B. Asymptotic Completeness: $\mathscr{H}_{\text{in}} = \mathscr{H}_{\text{out}} = \mathscr{H}$

C. Existence of Sharp Retarded Operators

There exists an infinite set of operator-valued retarded distributions $\tilde{R}(x_1; x_2, \dots, x_n)$ which satisfy all the algebraic relations and support properties suggested by the corresponding formal definitions (in terms of products of multiple commutators by step functions). The reader is referred to [7] for the complete list of these properties.

From this postulate it can be shown [7] that all generalized retarded products (g.r.p.) $\tilde{R}_{\mathscr{S}}(x_1, \dots, x_n)$ can be defined (here $\mathscr{S}$ is a certain index described in Section III.1). The operators

$$\begin{aligned} R_{\mathscr{S}}(\varphi) &= \int \tilde{R}_{\mathscr{S}}(x_1, \dots, x_n)\, \tilde{\varphi}(x_1, \dots, x_n)\, dx_1 \dots dx_n \\ &= \int R_{\mathscr{S}}(-p_1, \dots, -p_n)\, \varphi(p_1, \dots, p_n)\, dp_1 \dots dp_n \end{aligned}$$

act on the dense domain D and take their values in D. Moreover they satisfy the relativistic invariance condition

$$\tilde{R}_{\mathscr{S}}(\Lambda x_1 + a, \dots, \Lambda x_n + a) = \tilde{R}_{\mathscr{S}}(x_1, \dots, x_n)\,.$$

Postulate C is needed to make simpler the exploitation of the primitive structure although it is irrelevant as far as one is not concerned with slow increase behaviour in the tubes at infinity.

However it is known that the axioms A–C are not sufficient to avoid local pathologies in momentum space. In particular the development of the M.P.S.A. non-linear program seems to necessitate[1] an additional "smoothness postulate" which can be conveniently formulated as follows:

D. Smooth Spectral Condition

Let us call Π the projection operator on $\mathscr{H}'$. We shall assume the following property of the spectral measure $dE(p)$:

[1] This was recognized and investigated by Martin [15] in the similar context of the mass-shell non-linear program (use of unitarity). In this connection, see also the notion of "permanence of smoothness" introduced by Williams [29].

For any couple vectors Φ_1, Φ_2 in D, the measure $\langle \Phi_1, \Pi dE(p)\Pi \Phi_2\rangle$ is a Hölder-continuous [25] *function of p.*

Remark. It is known [11] that all matrix elements of the form

$$\int \langle \Phi_1, \Pi dE(\boldsymbol{p}, p_0)\Pi \Phi_2\rangle \chi(p_0)\, dp_0 \quad \text{with} \quad \chi \in \mathscr{S}(\mathbb{R}) \quad \text{and} \quad \Phi_j (j=1,2) \text{ in } D$$

are C^∞ functions of $\boldsymbol{p}$ as a consequence of relativistic invariance. Postulate D is therefore a statement about the behaviour of $\langle \Phi_1, \Pi dE(p)\Pi \Phi_2\rangle$ with respect to the variable p_0 (or $p^2 = p_0^2 - \boldsymbol{p}^2$) when $\Phi_{1,2}$ belongs to D.

III. Properties of the *n*-Point Functions

III.1. Cells and Generalized Retarded Operators

In the following N will always denote the set $\{1, 2, \ldots, n\}$ of indices numbering the different four-vectors $\{k_1, k_2, \ldots, k_n\}$. $\mathscr{P}^*(N)$ will denote the set of proper subsets I of N and $(I, N\backslash I)$ any partition of N in two (disjoint) subsets. We shall consider the space $\mathbb{C}_{(k)}^{4(n-1)}$ of the n complex four-vectors $\{k_i = p_i + iq_i,\ 1 \leqq i \leqq n\}$ linked by the relation $\sum_{i=1}^{n} k_i = 0$. Moreover k_I will always denote $\sum_{i\in I} k_i$.

We consider the space $\mathbb{R}_{(s)}^{n-1}$ of the n scalar variables $\{s_1, s_2, \ldots, s_n\}$ linked by $\sum_{i\in N} s_i = 0$ and the "triangulation" of this space by the family of planes $\{s_I = 0, I \in \mathscr{P}^*(N)\}$. The various open convex cones thus obtained [21] are called *geometrical cells* and denoted $\{\gamma_{\mathscr{S}}, \mathscr{S} \in S(N)\}$. Here the index $\mathscr{S}$ stands for the set of *all* proper subsets of N with s_I positive in $\gamma_{\mathscr{S}}$. $\mathscr{S}$ is called a *cell* of N and satisfy compatibility conditions which we shall not recall here [16, 17]. Two cells $\mathscr{S}_+$ and $\mathscr{S}_-$ are called adjacent and separated by a partition $(I, N\backslash I)$ if $\gamma_{\mathscr{S}_+}$ and $\gamma_{\mathscr{S}_-}$ have a common $(n-2)$-dimensional face in the hyperplane $s_I = s_{N\backslash I} = 0$.

With any cell $\mathscr{S}$ it is possible to associate an element $\hat{\mathscr{S}}$ in an abstract graded Lie algebra $\mathscr{D}$, the so-called Steinmann algebra [18–21]. The set of all products of Green's operators generates a certain representation $\mathscr{D}_A$ of this algebra, each element $\hat{\mathscr{S}}$ in $\mathscr{D}$ being represented in $\mathscr{D}_A$ by a generalized retarded product (g.r.p.) $\tilde{R}_{\mathscr{S}}(x_1, x_2, \ldots, x_n)$ which we assume here to exist in the sense of sharp operators (Postulate C). We denote by $r_{\mathscr{S}}(p)$ the generalized retarded distribution (g.r.d.) which is the Fourier transform of $\langle \Omega, \tilde{R}_{\mathscr{S}}(x)\Omega\rangle$.

Moreover in $\mathscr{D}$ we have the Ruelle "discontinuity formula" [21]:

$$\hat{\mathscr{S}}_+ - \hat{\mathscr{S}}_- = [\hat{\mathscr{S}}_1, \hat{\mathscr{S}}_2], \tag{2a}$$

where $\mathscr{S}_+$ and $\mathscr{S}_-$ are two adjacent cells separated by $(I, N\backslash I)$; $\mathscr{S}_1 = \{J \in \mathscr{S}_+, J \subset I, J \neq I\}$ is a cell of I; $\mathscr{S}_2 = \{J \in \mathscr{S}_+, J \subset N\backslash I\}$ a cell of $N\backslash I$. When I (resp. $N\backslash I$) is reduced to one element $\{i\}$, $\hat{\mathscr{S}}_1$ (resp. $\hat{\mathscr{S}}_2$) stands for an element $\hat{\imath}$ of $\mathscr{D}$ whose image in $\mathscr{D}_A$ is the field operator $A(x_i)$. We write in $\mathscr{D}_A$:

$$R_{\mathscr{S}_+} - R_{\mathscr{S}_-} = [R_{\mathscr{S}_1}, R_{\mathscr{S}_2}]. \tag{2b}$$

III.2. Linear Properties

From the Wightman axioms A the following primitive structure can be proved for the n-point function $H^{(n)}(k)$:

i) analyticity and slow increase near the real inside a domain which is composed of the union of a certain family of tubes $\{\mathcal{T}_{\mathcal{S}}, \mathcal{S} \in S(N)\}$ in the linear manifold $\left\{k \in \mathbb{C}^{4n} : \sum_{i \in N} k_i = 0\right\}$ with appropriate complex neighbourhoods of real regions which connect the various tubes together. Here $\mathcal{T}_{\mathcal{S}}$ is defined by:

$$\mathcal{T}_{\mathcal{S}} = \{k \in \mathbb{C}^{4(n-1)} : \operatorname{Im} k_I \in V^+, \forall I \in \mathcal{S}\}$$

and the real boundary value of $H^{(n)}$ inside $\mathcal{T}_{\mathcal{S}}$

$$H^{(n)}_{\mathcal{S}}(p) = \lim_{\substack{q \to 0 \\ p+iq \in \mathcal{T}_{\mathcal{S}}}} H^{(n)}(p+iq)$$

coincides with the g.r.d. $r_{\mathcal{S}}(p)$ up to an overall δ factor:

$$r_{\mathcal{S}}(p) = H^{(n)}_{\mathcal{S}}(p)\, \delta\left(\sum_{i \in N} p_i\right).$$

ii) Steinmann relations hold between the various boundary values $\{H^{(n)}_{\mathcal{S}}(p), \mathcal{S} \in S(N)\}$.

iii) Coincidence of two adjacent boundary values $H^{(n)}_{\mathcal{S}_+}(p)$ and $H^{(n)}_{\mathcal{S}_-}(p)$ separated by a partition $(I, N \backslash I)$ on a real region $\mathcal{R}_I$ thus defined:

$$\mathcal{R}_I = \{p \in \mathbb{R}^{4(n-1)} : p_I^2 \neq m_\mu^2,\, p_I^2 < 4m^2\}.$$

Properties ii) and iii) are consequences of relation (2b) and of the spectral assumption. We note that in view of iii) the support of the distribution $[H^{(n)}_{\mathcal{S}_+} - H^{(n)}_{\mathcal{S}_-}](p)$ is the union of the two following disconnected sets

$$\Sigma_I^{\pm} = \left\{p \in \mathbb{R}^{4(n-1)} : p_I \in \left(\bigcup_\mu H^{\pm}_{m_\mu}\right) \cup \bar{V}^{\pm}_{2m}\right\},$$

where $H^+_{m_\mu} = -H^-_{m_\mu}$ and $\bar{V}^+_{2m} = -\bar{V}^-_{2m}$ have been defined in Section II. Taking into account (2b) we then choose to write

$$H^{(n)}_{\mathcal{S}_+} - H^{(n)}_{\mathcal{S}_-} = \Delta_{\mathcal{S}_1 \mathcal{S}_2} H^{(n)} - \Delta_{\mathcal{S}_2 \mathcal{S}_1} H^{(n)}, \tag{2c}$$

where $\Delta_{\mathcal{S}_1 \mathcal{S}_2} H^{(n)}$ (resp. $\Delta_{\mathcal{S}_2 \mathcal{S}_1} H^{(n)}$) is defined as the value of $[H^{(n)}_{\mathcal{S}_+} - H^{(n)}_{\mathcal{S}_-}]$ on Σ_I^+ (resp. Σ_I^-) and zero outside.

$\Delta_{\mathcal{S}_1 \mathcal{S}_2} H^{(n)}$ is called an absorptive part of $H^{(n)}$. This absorptive part is one of the boundary values of a function $\Delta^I H^{(n)}$ which is the common analytic continuation inside the manifold $q_I = q_{N \backslash I} = 0$ of all the *discontinuities* $[H^{(n)}_{\mathcal{S}_+} - H^{(n)}_{\mathcal{S}_-}](k)$, with the couple $(\mathcal{S}_+, \mathcal{S}_-)$ separated by the partition $(I, N \backslash I)$. Actually $\Delta^I H^{(n)}$ is analytic[2] inside the union of all the "flat" tubes

$$\mathcal{T}_{\mathcal{S}_1 \mathcal{S}_2} = \{k \in \mathbb{C}^{4(n-1)} : q_I = q_{N \backslash I} = 0; \quad \forall J \in \mathcal{S}_{1,2}, q_J \in V^+\}$$

with appropriate real regions connecting these flat tubes together. Its real boundary

[2] More precisely, $\Delta^I H^{(n)}$ is a distribution in p_I which depends analytically on the $4(n-2)$ remaining independent four-vectors k_j.

value taken in Σ_I^+ from the tube $\mathscr{T}_{\mathscr{S}_1\mathscr{S}_2}$ is $\Delta_{\mathscr{S}_1\mathscr{S}_2}H^{(n)}$. The absorptive parts $\Delta_{\mathscr{S}_1\mathscr{S}_2}H^{(n)}$ and the discontinuity $\Delta^I H^{(n)}$ in a given channel $(I, N\backslash I)$ will play a basic role further in the formulation of the non-linear properties of the n-point functions.

Finally, following [1], we call *general* n-point function any function F enjoying properties i)–iii) stated above but having no longer any physical connection with products of n fields. We write $F_{\mathscr{S}}$ for its real boundary value inside the tube $\mathscr{T}_{\mathscr{S}}$, $\Delta^I F$ for its discontinuity on the manifold $q_I = q_{N\backslash I} = 0$ and similarly $\Delta_{\mathscr{S}_1\mathscr{S}_2}F$ for its absorptive parts with support in Σ_I^+.

III.3. Non-linear Properties of the n-Point Functions

We shall now write the "*completeness relations*" which link together the various n-point generalized retarded distribution (g.r.d.), and are consequences of the postulate of asymptotic completeness. Obtained through an extensive use of reduction formulae, they are the field theoretic off-shell extrapolations of the unitarity relations. First derived by Glaser, Lehmann, and Zimmermann [8], they have been presented in a completely rigorous and general version by Steinmann [7]. However, for a reason which will appear in the following we shall not fully adopt Steinmann's point of view. For the sake of simplicity, we shall only consider the case of a single discrete mass m in the spectrum.

We shall recall the following algebraic notion:

Proposition 1 [16]. *Let $\mathscr{S}$ be a cell of N and K be a non-empty finite set of integers, with $K \cap N = \emptyset$. We define*

$$K \uparrow \mathscr{S} = \{J \subset N \cup K : \emptyset \neq J \subset K \quad or \quad J \cap N \in \mathscr{S}\}.$$

Then $K \uparrow \mathscr{S}$ is a cell of $N \cup K$.

Similarly, the following set $K \downarrow \mathscr{S}$ is a cell of $N \cup K$:

$$K \downarrow \mathscr{S} = \{J \subset N \cup K : N \subset J \neq N \cup K \quad or \quad J \cap N \in \mathscr{S}\}.$$

Then with any n-point g.r.p. $R_{\mathscr{S}}(p'_1, \ldots, p'_n)$, we can associate, for all values of $k = |K|$, two $(n+k)$-point g.r.p.'s, which we denote $R_{K\uparrow\mathscr{S}}(p_1, \ldots, p_k; p'_1, \ldots, p'_n)$ and $R_{K\downarrow\mathscr{S}}(p_1, \ldots, p_k; p'_1, \ldots, p'_n)$. We similarly define the corresponding general retarded distributions (g.r.d.) $r_{K\uparrow\mathscr{S}}$ and $r_{K\downarrow\mathscr{S}}$. Note that if N reduces to a single element, then $R_{K\uparrow\mathscr{S}}$ (resp. $R_{K\downarrow\mathscr{S}}$) is an ordinary retarded (resp. advanced) $(k+1)$-point operator.

Now some care is needed for giving the correct definition of the mass-shell restrictions $\hat{r}_{K\updownarrow\mathscr{S}}$ of the amputated g.r.d. $r_{K\updownarrow\mathscr{S}}$, since the existence of these restrictions as distributions on the mass-shell was the key-point in the rigorous proof of reduction formulae (see Hepp [6], Steinmann [7], and also in this connection Bros, Epstein, and Glaser [26]).

On the mass-shell manifold $\mathscr{M}_K^+ = (H_m^+)^k$, let us define the subset Ω of all configurations $\{p_j \in H_m^+, j = 1, 2, \ldots, k\}$ such that at least two vectors p_i, p_j coincide. Ω is a closed set of measure zero, and we denote by Ω' its complement in $\mathscr{M}_K^+$. Conventionally, every wave-packet $\prod_{j=1}^{k} \hat{f}_j(p_j)$ with support in Ω' is called "non-overlapping"; we shall thus call Ω the "overlap set".

Then we have the following:

Lemma 1. i) *It is meaningful to define the following distribution in the open set* $\Omega' \times \mathbb{R}^{4n}_{(p')}$:

$$\hat{r}_{K\downarrow\mathscr{S}}(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k; p') = \left[\prod_{j=1}^{k} (p_j^2 - m^2)\, r_{K\downarrow\mathscr{S}}(p_1, \ldots, p_k; p')\right]\Bigg|_{\substack{p_j^0 = \omega_j \\ 1 \le j \le k}} \tag{3}$$

with $\omega_j = \sqrt{\boldsymbol{p}_j^2 + m^2}$.

ii) *For every* $\varphi \in \mathscr{S}(\mathbb{R}^{4n}_{(p')})$, *the distribution*

$$\hat{r}_{K\downarrow\mathscr{S}}(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k; \varphi) = \int \hat{r}_{K\downarrow\mathscr{S}}(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k; p')\, \varphi(-p')\, dp' \tag{4}$$

defined in Ω', *can be identified with a square-integrable function on the* whole *manifold* $\mathscr{M}_K^+$, *and the following reduction formulae hold:*

$$\langle \Phi^{\text{out}}_{\hat{f}_1 \ldots \hat{f}_k}, R_{\mathscr{S}}(\varphi)\Omega\rangle = \int \hat{r}_{K\downarrow\mathscr{S}}(\boldsymbol{p}_1 \ldots \boldsymbol{p}_k; \varphi) \cdot \prod_{j=1}^{k} \hat{f}_j^*(\boldsymbol{p}_j) \frac{d\boldsymbol{p}_j}{2\omega_j} \tag{5}$$

for arbitrary wave packets $\prod_{j=1}^{k} \hat{f}_j(\boldsymbol{p}_j)$ *in* $L^2(\mathscr{M}_K^+)$ (*the meaning of* * *being "complex conjugation"*).

The following similar relations hold:

$$\text{in } \Omega' \times \mathbb{R}^{4n}_{(p')}: \hat{r}_{K\uparrow\mathscr{S}}(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k; p') = \prod_{j=1}^{k} (p_j^2 - m^2)\, r_{K\uparrow\mathscr{S}}(p_1, \ldots, p_k; p')\Bigg|_{\substack{p_j^0 = \omega_j \\ 1 \le j \le k}} \tag{6}$$

$$\text{in } \mathscr{M}_K^+: \langle \Phi^{\text{in}}_{\hat{f}_1 \ldots \hat{f}_k}, R_{\mathscr{S}}(\varphi)\Omega\rangle = \int \hat{r}_{K\uparrow\mathscr{S}}(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k; \varphi) \prod_{j=1}^{k} \hat{f}_j^*(\boldsymbol{p}_j) \frac{d\boldsymbol{p}_j}{2\omega_j}. \tag{7}$$

Proof and Comments. The first part of the lemma results from Hepp's analysis [6], which can be carried through for the g.r.p.'s [7] as well as for the chronological products; this property can also be obtained in the local analytic structure approach which we shall recall below and in Appendices A and B.

The second part is based on the following argument: from Haag-Ruelle asymptotic theory, $\langle(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k)^{\text{out}}, R_{\mathscr{S}}(\varphi)\Omega\rangle$ is defined as a square-integrable function on $\mathscr{M}_K^+$ $\left(L^2 \text{ with respect to the measure } \prod_{j=1}^{k} \frac{d\boldsymbol{p}_j}{2\omega_j}\right)$; moreover it is proved in the rigorous theory of reduction formulae [6, 7, 22], that this L^2 function must coincide *in the sense of distributions in* Ω' with $\hat{r}_{K\downarrow\mathscr{S}}(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k; \varphi)$, the latter being defined in Ω' in view of the first part. Now since Ω is a set of measure zero, it is legitimate to say that $\hat{r}_{K\downarrow\mathscr{S}}(\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k; \varphi)$ represents a *unique* square-integrable function on the *whole* set $\mathscr{M}_K^+$ (this defines a "canonical extension" of $\hat{r}_{K\downarrow\mathscr{S}}$ as a distribution on $\mathscr{M}_K^+ \times \mathbb{R}^{4n}_{(p')}$ since any other extension would contain polynomials of δ-functions with support in Ω). With this extended definition of $\hat{r}_{K\downarrow\mathscr{S}}$ reduction formula (4) holds for arbitrary L^2-wave packet in $\mathscr{M}_K^+$. Therefore, the latter definition is also suitable for the rigorous derivation of completeness equations of the G.L.Z. type for the g.r.d.'s [7]; in doing so, we avoid using the definition of $\hat{r}_{K\updownarrow\mathscr{S}}$ as a "refined average" in the neighbourhood of Ω, introduced in [7]; such a refinement would be difficult to reproduce in the further case of one-p.i. functions.

Now, in view of later use, it is important to give an interpretation of $\hat{r}_{K\updownarrow\mathscr{S}}$ in Ω' from the point of view of the local analytic structure of $H^{(n+k)}$ [26]. A rather general way of studying the existence of the restriction of a distribution to a submanifold is based on the following fact (see [26], Theorem 2 and [28]): assume that a distribution T can be expressed on a certain open set U of R^n as a finite sum of boundary values T_β of analytic functions F_β, respectively defined in local tubes $U+i\mathscr{C}_\beta$ (in $\mathbb{C}^n$), each $\mathscr{C}_\beta$ being an open convex cone; then T admits a restriction to every submanifold $\mathscr{M}$ of U, which is "*transverse*" to all cones $\mathscr{C}_\beta$ (i.e. such that the tangent plane to $\mathscr{M}$ at every point $p\in\mathscr{M}$ has non-empty intersections with *all* cones $\mathscr{C}_\beta$): in fact, it is then possible to restrict each F_β to the complexified manifold $\mathscr{M}^c$ of $\mathscr{M}$, and then take the real boundary value of this restriction $\hat{F}_\beta$ which is a distribution in $\mathscr{M}$. In [26], such an analysis was made for the fully amputated chronological products, and the restriction of the latter to the mass-shell $\mathscr{M}_n$ could be defined in this way in the complement of the overlap set of $\mathscr{M}_n$. A similar analysis is done here in Appendix B for the $\hat{H}_{K\updownarrow\mathscr{S}}$'s, and yields:

Lemma 2. *Let* $\mathscr{M}'=\left\{(p,p');p\in\Omega';\sum_{j\in K}p_j+\sum_{i\in N}p'_i=0\right\}$.

For every point (p,p') in the manifold $\mathscr{M}'$, there exists a neighbourhood U of (p,p') in $\mathbb{R}^{4(n+k-1)}$ in which any distribution $\hat{H}_{K\updownarrow\mathscr{S}}$ can be expressed as a finite sum of boundary values of analytic functions h_β, each h_β being analytic in a local tube which is transverse *to the complex mass-shell manifold $\mathscr{M}'_c$ (i.e. $\mathscr{M}'$ complexified).*

According to the above comments, this gives an alternative proof of the first part of Lemma 1 $\left(\text{since } \hat{r}_{K\updownarrow\mathscr{S}}=\delta\left(\sum_{j\in K}p_j+\sum_{i\in N}p'_i\right)\cdot\hat{H}^{(n+k)}_{K\updownarrow\mathscr{S}}\right)$.

The advantage of this procedure is to belong to the linear program (use of causality and spectrum) and therefore to also apply to *general* n-point functions, such as the one-p.i. functions (see Appendix B and Section V). We now turn to the algebraic description of the completeness relations which link the various g.r.d.'s [7].

Being given a channel $(I,N\backslash I)$ let us introduce the following family of sets $\{\Sigma^l_{I,N\backslash I}, l\geqq 1\}$ in $\mathbb{R}^{4(n-1)}$:

$$\Sigma^1_{I,N\backslash I}=\{p\in\mathbb{R}^{4(n-1)}:p_I^2=m^2\} \tag{8}$$

$$\Sigma^l_{I,N\backslash I}=\{p\in\mathbb{R}^{4(n-1)}:l^2m^2\leqq p_I^2<(l+1)^2m^2\}\qquad l\geqq 2 \tag{9}$$

with p_I^2 the squared total energy in the considered channel.

Being given an absorptive part $\Delta_{\mathscr{S}_1,\mathscr{S}_2}H^{(n)}$ of the n-point function $H^{(n)}$ in the channel $(I,N\backslash I)$, the completeness of one-particle states yields on $\Sigma^1_{I,N\backslash I}$:

$$\delta\left(\sum_{i\in N}p_i\right)\Delta_{\mathscr{S}_1\mathscr{S}_2}H^{(n)}=\frac{2i\pi}{Z}\int dp_\lambda\delta^-_\lambda\,\hat{r}_{\lambda\uparrow\mathscr{S}_1}\hat{r}_{\underline{\lambda}\uparrow\mathscr{S}_2}=\frac{2i\pi}{Z}\int dp_\lambda\delta^-_\lambda\,\hat{r}_{\lambda\downarrow\mathscr{S}_1}\hat{r}_{\underline{\lambda}\downarrow\mathscr{S}_2}, \tag{10a}$$

where the notation $\underline{\lambda}$ means that $p_{\underline{\lambda}}=-p_\lambda$, $\delta^-_\lambda=\theta(-p^0_\lambda)\,\delta(p^2_\lambda-m^2)$ with $\theta(t)=\frac{1}{2}(t+|t|)$. Z is the normalization constant defined in (1) and the various $\hat{r}_{\lambda\updownarrow\mathscr{S}}$ are defined by (3) and (6). This can be rewritten:

$$\Delta_{\mathscr{S}_1\mathscr{S}_2} H^{(n)} = \frac{2i\pi}{Z}\,\delta_\lambda^-\,\hat{H}^{(n_1+1)}_{\lambda\uparrow\mathscr{S}_1}\hat{H}^{(n_2+1)}_{\underline{\lambda}\uparrow\mathscr{S}_2} = \frac{2i\pi}{Z}\,\delta_\lambda^-\,\hat{H}^{(n_1+1)}_{\lambda\downarrow\mathscr{S}_1}\hat{H}^{(n_2+1)}_{\underline{\lambda}\downarrow\mathscr{S}_2} \tag{10b}$$

with $n_1 = |I|$ and $n_2 = |N\backslash I|$. Here $-p_\lambda = p_{\underline{\lambda}} = p_I$ and the various $\hat{H}^{(n_j+1)}_{\lambda\updownarrow\mathscr{S}_j}$ are defined by:

$$\hat{H}^{(n_j+1)}_{\lambda\updownarrow\mathscr{S}_j} = (p_I^2 - m^2)\,H^{(n_j+1)}_{\lambda\updownarrow\mathscr{S}_j}$$

Similarly on any $\Sigma^l_{I,N\backslash I}$ with $l \geqq 2$, the completeness of incoming (resp. outgoing) states with $(2, 3, \ldots, l)$ particles yields

$$\begin{aligned}\delta\Big(\sum_{i\in N} p_i\Big)\,\Delta_{\mathscr{S}_1\mathscr{S}_2} H^{(n)} &= \sum_{k=2}^{l}\left(\frac{2i\pi}{Z}\right)^k \frac{1}{k!}\,\hat{r}_{K\uparrow\mathscr{S}_1} * \hat{r}_{\underline{K}\uparrow\mathscr{S}_2} \\ &= \sum_{k=2}^{l}\left(\frac{2i\pi}{Z}\right)^k \frac{1}{k!}\,\hat{r}_{K\downarrow\mathscr{S}_1} * \hat{r}_{\underline{K}\downarrow\mathscr{S}_2}\,.\end{aligned} \tag{11a}$$

Here some explanation of the notation is needed: with any value of $k \geqq 2$ we associate a set of k indices $K = \{\lambda_1, \ldots, \lambda_k\}$ and two g.r.d.'s $\hat{r}_{K\uparrow\mathscr{S}_j}$ and $\hat{r}_{\underline{K}\uparrow\mathscr{S}_j}$ (resp. $\hat{r}_{K\downarrow\mathscr{S}_j}$ and $\hat{r}_{\underline{K}\downarrow\mathscr{S}_j}$) defined as in Lemma 1. The notation $\underline{K} = \{\underline{\lambda}_1, \ldots, \underline{\lambda}_k\}$ means that $p_{\underline{\lambda}_j} = -p_{\lambda_j}$ and the notation $*$ is defined as follows

$$\hat{r}_{K\uparrow\mathscr{S}_1} * \hat{r}_{\underline{K}\uparrow\mathscr{S}_2} = \int_{\mathbb{R}^{4k}} \prod_{j=1}^{k} [\delta^-_{\lambda_j}\, dp_{\lambda_j}]\,\hat{r}_{K\uparrow\mathscr{S}_1}\hat{r}_{\underline{K}\uparrow\mathscr{S}_2}\,. \tag{12}$$

In view of Lemma 2, the product of the distributions $\hat{r}_{K\uparrow\mathscr{S}_1}$ and $\hat{r}_{\underline{K}\uparrow\mathscr{S}_2}$ which occurs in (10a) and (12) is well defined (after testing in the external variables) in the sense of functions square-integrable with respect to the measure $\prod_{j=1}^{k} \frac{d\boldsymbol{p}_{\lambda_j}}{2\omega_j}$: their product is integrable and (12) is meaningful.

Finally we can rewrite (11a) under the equivalent form (without factors δ)

$$\begin{aligned}\Delta_{\mathscr{S}_1\mathscr{S}_2} H^{(n)} &= \sum_{k=2}^{l}\left(\frac{2i\pi}{Z}\right)^k \frac{1}{k!}\,\hat{H}^{(n_1+k)}_{\underline{K}\uparrow\mathscr{S}_1} * \hat{H}^{(n_2+k)}_{K\uparrow\mathscr{S}_2} \\ &= \sum_{k=2}^{l}\left(\frac{2i\pi}{Z}\right)^k \frac{1}{k!}\,\hat{H}^{(n_1+k)}_{\underline{K}\downarrow\mathscr{S}_1} * \hat{H}^{(n_2+k)}_{K\downarrow\mathscr{S}_2}\,.\end{aligned} \tag{11b}$$

Here we have put: $\hat{H}^{(n_j+k)}_{K\updownarrow\mathscr{S}_j} = \prod_{j=1}^{k} (p_j^2 - m^2) H^{(n_j+k)}_{K\updownarrow\mathscr{S}_j}$, and the notation $*$ is defined by

$$\hat{H}^{(n_1+k)}_{K\uparrow\mathscr{S}_1} * \hat{H}^{(n_2+k)}_{\underline{K}\uparrow\mathscr{S}_2} = \int_{\mathbb{R}^{4k}} \prod_{j=1}^{k} [\delta^-_{\lambda_j}\, dp_{\lambda_j}]\,\delta(p_{I\cup K})\,\hat{H}^{(n_1+k)}_{K\uparrow\mathscr{S}_1}\hat{H}^{(n_2+k)}_{\underline{K}\uparrow\mathscr{S}_2}\,.$$

Relations (10) and (11) are the asymptotic completeness relations: written for any absorptive part $\Delta_{\mathscr{S}_1\mathscr{S}_2} H^{(n)}$ in any channel $(I, N\backslash I)$ they express the *whole non-linear information* of general quantum field theory.

III.4. Consequences of the Smooth Spectral Condition

In this section we shall now be concerned with the smoothness properties which can be deduced from Postulate D for the absorptive parts and the boundary values of the n-point functions.

Consider the set $\mathscr{S}_n(\overline{V_{2m}^+})$ of all test-functions $\varphi(p_1, \ldots, p_n)$ in $\mathscr{S}(\mathbb{R}_p^{4n})$ such that the projection of the support of φ into the space $\mathbb{R}_P^4$ of the total four momentum $P = \sum_{i=1}^{n} p_i$ only intersects the continuous part $\overline{V}_{2m}^+$ of the spectrum. We also introduce the space $\mathbb{R}_{\underline{p}}^{4(n-1)}$ of "barycentric variables" $\underline{p} = \left\{\underline{p}_i = p_i - \frac{P}{n}; \sum_{i=1}^{n} \underline{p}_i = 0\right\}$.

Proposition 2. *Let $\mathscr{S}_1$ (resp. $\mathscr{S}_2$) be some cell of I (resp. $N\backslash I$) and $\Delta_{\mathscr{S}_1\mathscr{S}_2}H^{(n)}$ the corresponding absorptive part. The distribution*

$$\Delta_{\mathscr{S}_1\mathscr{S}_2}H^{(n)}(p_I; \underline{\varphi}_1, \underline{\varphi}_2) = \int \Delta_{\mathscr{S}_1\mathscr{S}_2}H^{(n)}(p_I, \underline{p}^{(n_1)}, \underline{p}^{(n_2)})\, \underline{\varphi}_1(\underline{p}^{(n_1)})\, \underline{\varphi}_2(\underline{p}^{(n_2)})\, d\underline{p}^{(n_1)}\, d\underline{p}^{(n_2)} \quad (13)$$

with $\underline{\varphi}_j \in \mathscr{S}(\mathbb{R}_{\underline{p}^{(n_j)}}^{4(n_j-1)})$ and $\underline{p}^{(n_1)}$ (resp. $\underline{p}^{(n_2)}$) the barycentric variables of the set $\{p_i, i \in I\}$ (resp. $\{p_j, j \in N\backslash I\}$), can be identified with a Hölder-continuous function of the variable p_I.

Proof. Let ψ_j $(j = 1, 2)$ be the following vector in D:

$$\psi_j = R_{\mathscr{S}_j}(\varphi_j)\,|\Omega\rangle = \int R_{\mathscr{S}_j}(-p_1, \ldots, -p_{n_j})\, \varphi_j(p_1, \ldots, p_{n_j})\, dp_1 \ldots dp_{n_j}\,|\Omega\rangle\,.$$

Using a classical argument ([11], p. 70) based on the translation invariance of the operator $R_{\mathscr{S}_j}$, one easily shows that if $\varphi_j \in \mathscr{S}_{n_j}(\overline{V}_{2m}^+)$, then:

$$\Pi\, R_{\mathscr{S}_j}(\varphi_j)|\Omega\rangle = R_{\mathscr{S}_j}(\varphi_j)|\Omega\rangle\,,$$

so that:

$$\langle \psi_1, \Pi\, dE(p)\, \Pi\, \psi_2\rangle = \langle \psi_1, dE(p)\psi_2\rangle\,.$$

But using again translation invariance, we have:

$$\begin{aligned}\langle \psi_1, dE(p)\psi_2\rangle &= \frac{1}{(2\pi)^4} \int e^{-ipa} \cdot \langle \psi_1, U(1, a)\psi_2\rangle\, da \\ &= \int \varphi_1^*(p_1, \ldots, p_{n_1})\, \varphi_2(-p_1' \ldots -p_{n_2}')\, \langle \Omega, R_{\mathscr{S}_1}(p_1, \ldots, p_{n_1}) \cdot R_{\mathscr{S}_2}(p_1', \ldots, p_{n_2}')\Omega\rangle \\ &\qquad \delta_4(p - p_I)\, dp_1 \ldots dp_{n_1}\, dp_1' \ldots dp_{n_2}'\,.\end{aligned} \quad (14)$$

Now putting

$$p_i = \frac{p_I}{n_1} + \underline{p}_i,\ i \in I;\ p_j' = \frac{p_{N\backslash I}}{n_2} + \underline{p}_j',\ j \in N\backslash I$$

and

$$\underline{p}^{(n_1)} = \left\{\underline{p}_i, i \in I; \sum_{i \in I} \underline{p}_i = 0\right\}, \quad \underline{p}^{(n_2)} = \left\{\underline{p}_j', j \in N\backslash I;\ \sum_{j \in N\backslash I} \underline{p}_j' = 0\right\},$$

we can integrate over $p_I, p_{N\backslash I}$. Taking into account that

$$\langle R_{\mathscr{S}_1} \cdot R_{\mathscr{S}_2}\rangle = \delta(p_I + p_{N\backslash I})\Delta_{\mathscr{S}_1\mathscr{S}_2}H^{(n)}(p_I; \underline{p}^{(n_1)}, \underline{p}^{(n_2)})\,,$$

relation (14) yields:

$$\langle \psi_1, dE(p)\psi_2\rangle = \Delta_{\mathscr{S}_1\mathscr{S}_2}H^{(n)}(p_I = p; \underline{\varphi}_1, \underline{\varphi}_2)$$

[i.e. (13)] provided that one chooses:

$$\varphi_j = \underline{\varphi}_j(\underline{p}^{(n_j)})\chi(p_I);\ j = 1, 2$$

with $\chi(p_I)$ locally equal to 1 at $p_I = p$.

Then postulate D immediately gives the announced result.

Proposition 3. *Let $\mathscr{S}_+$ and $\mathscr{S}_-$ be two adjacent cells separated by the partition $(I, N\backslash I)$. Then for every $\underline{\varphi}_1, \underline{\varphi}_2$ as in Proposition 2, the expression*

$$H^{(n)}_{\mathscr{S}_\pm}(p_I; \underline{\varphi}_1, \underline{\varphi}_2) = \int H^{(n)}_{\mathscr{S}_\pm}(p_I, \underline{p}^{(n_1)}, \underline{p}^{(n_2)})\, \underline{\varphi}_1(\underline{p}^{(n_1)})\, \underline{\varphi}_2(\underline{p}^{(n_2)})\, d\underline{p}^{(n_1)}\, d\underline{p}^{(n_2)} \tag{15}$$

is a Hölder-continuous function of p_I.

Proof. By taking a sufficiently large integer N, we can introduce the auxiliary functions:

$$I^\pm(\boldsymbol{p}_I, k_I^0; \underline{k}) = (k_I^0)^N \int \frac{(\Delta_{\mathscr{S}_1\mathscr{S}_2} H^{(n)} - \Delta_{\mathscr{S}_2\mathscr{S}_1} H^{(n)})(\boldsymbol{p}_I, u; \underline{k})}{u^N(u-k_I^0)}\, du \tag{16}$$

which are respectively analytic in the tubes

$$\{\operatorname{Im} k_i^0 \gtrless 0; \operatorname{Im} \boldsymbol{k}_I = 0; \underline{k} \in \mathscr{T}_{\mathscr{S}_1\mathscr{S}_2}\},$$

and such that (in the sense of distributions on their common face $\{\operatorname{Im} k_I = 0\}$):

$$I^+ - I^- = H^{(n)}_{\mathscr{S}^+} - H^{(n)}_{\mathscr{S}^-}.$$

As a result of the edge of the wedge theorem, there exists an analytic function $G(\boldsymbol{p}_I, k_I^0; \underline{k})$ in the convex envelope $\mathscr{T}_c$ of $\mathscr{T}_{\mathscr{S}^+} \cup \mathscr{T}_{\mathscr{S}^-}|_{\operatorname{Im} \boldsymbol{k}_I = 0}$ such that in $\mathscr{T}_{\mathscr{S}^\pm}$:

$$H^{(n)}_{\mathscr{S}^\pm} = G + I^\pm.$$

After testing both sides of this relation (on the real) with $\underline{\varphi}_1, \underline{\varphi}_2$, we get:

$$H^{(n)}_{\mathscr{S}^\pm}(p_I; \underline{\varphi}_1, \underline{\varphi}_2) = G(p_I; \underline{\varphi}_1, \underline{\varphi}_2) + I^\pm(p_I; \underline{\varphi}_1, \underline{\varphi}_2).$$

The Hölder continuity of $H^{(n)}_{\mathscr{S}^\pm}(p_I; \underline{\varphi}_1, \underline{\varphi}_2)$ is then entailed by the two following facts:

i) Hölder-continuity of $I^\pm(p_I; \underline{\varphi}_1, \underline{\varphi}_2)$: it is a consequence of Proposition 2, since Hölder-continuity is preserved by convolution with the Cauchy kernel (i.e. by Hilbert transform; see [24]); formula (16) therefore yields the desired result, after testing with $\underline{\varphi}_1, \underline{\varphi}_2$.

ii) $G(p_I; \underline{\varphi}_1, \underline{\varphi}_2)$ is a C^∞ function of p_I, since it is obtained by testing the boundary value of G on a submanifold which is transverse to the tube $\mathscr{T}_c$ in which G is analytic (see Proposition A.3).

IV. Construction of One-p.i. n-Point Functions

IV.1. Poles and Residues of the n-Point Functions

Let $\hat{H}^{(n)}$ denote the "fully-amputated" n-point function, namely

$$\hat{H}^{(n)}(k_1, \ldots, k_n) = \prod_{\mu} \prod_{(I, N\backslash I)} (k_I^2 - m_\mu^2)\, H^{(n)}(k_1, \ldots, k_n), \tag{17}$$

where the product extends to all partitions of N into two subsets $(I, N\backslash I)$.

$\hat{H}^{(n)}$ is a general n-point function whose boundary values $\{\hat{H}^{(n)}_{\mathscr{S}}, \mathscr{S} \in S(N)\}$ satisfy the following coincidence relations. For any couple $(\mathscr{S}_+, \mathscr{S}_-)$ of adjacent cells separated by a partition $(I, N\backslash I)$,

$$\hat{H}^{(n)}_{\mathscr{S}_+}(p) = \hat{H}^{(n)}_{\mathscr{S}_-}(p) \quad \text{in} \quad \hat{\mathscr{R}}_I = \{p \in \mathbb{R}^{4(n-1)} : p_I^2 \langle 4m^2\}.$$

To show this it is enough to concentrate on a fixed partition $(I, N\backslash I)$ and to prove that

$$\int \Delta_{\mathscr{S}_1\mathscr{S}_2}\left[\prod_{\mu}(p_I^2-m_\mu^2)H^{(n)}\right](p_I,\underline{p}^{(n_1)},\underline{p}^{(n_2)})\,\underline{\varphi}_1^*(p^{(n_1)})\,\underline{\varphi}_2(-p^{(n_2)})\,d\underline{p}^{(n_1)}\,d\underline{p}^{(n_2)}\,dp_I=0$$

for all test-functions $\varphi_j(j=1,2)$ whose support does not intersect the region $\{p_I\in \bar{V}_{2m}^+\}$. Here $\mathscr{S}_1$ (resp. $\mathscr{S}_2$) is a cell of I (resp. $N\backslash I$) with $\hat{\mathscr{S}}_+-\hat{\mathscr{S}}_-=[\hat{\mathscr{S}}_1,\hat{\mathscr{S}}_2]$. Applying the argument of Subsection III.4 (using the translation invariance of the operators $R_{\mathscr{S}_j}$), the left-hand side can be rewritten:

$$\int \prod_{\mu}(p^2-m_\mu^2)\,\langle\psi_1, dE(p)\psi_2\rangle$$

with $\psi_j=R_{\mathscr{S}_j}(\varphi_j)|\Omega\rangle$ $(j=1,2)$. This matrix element of $dE(p)$ is a measure with support $\left\{p\in\bigcup_{\mu} H_{m_\mu}^+\right\}$; it therefore factorizes $\sum_{\mu} c_\mu\delta_\mu^+(p)$ [3] and yields zero for the integral, q.e.d.

Let us now denote $\hat{\mathscr{D}}^{(n)}$ the primitive domain of $\hat{H}^{(n)}$ associated with this set of coincidence relations and let $\mathscr{E}(\hat{\mathscr{D}}^{(n)})$ be the holomorphy envelope of $\hat{\mathscr{D}}^{(n)}$ in $\mathbb{C}^{4(n-1)}$: it is crossed by $\mathscr{M}_{I,\mu}=\{k\in\mathbb{C}^{4(n-1)}:k_I^2=m_\mu^2\}$. Thus taking (17) into account, we have

Proposition 4. *The n-point function $H^{(n)}$ is meromorphic in the holomorphy envelope $\mathscr{E}(\hat{\mathscr{D}}^{(n)})$ and it is given there by*

$$H^{(n)}(k)=\frac{\hat{H}^{(n)}(k)}{\prod\limits_{\mu}\prod\limits_{(I,N\backslash I)}(k_I^2-m_\mu^2)}$$

with $\hat{H}^{(n)}$ holomorphic in $\mathscr{E}(\hat{\mathscr{D}}^{(n)})$.

Remark. This result contains as a special case the treatment of poles in the s-plane dispersion relations [4], since all processes of analytic continuation for $\hat{H}^{(n)}$ yield corresponding meromorphic continuation for $H^{(n)}$.

We shall now compute the residues of all the functions $H^{(n)}$ on their various polar manifolds $\mathscr{M}_{I,\mu}$. We define

$$G_\mu^{I,N\backslash I}(k_I;k')=(k_I^2-m_\mu^2)\,H^{(n)}(k_I;k')|_{\mathscr{M}_{I,\mu}} \tag{18}$$

where k' denotes a set of $(n-2)$ independent four-vectors. $G_\mu^{I,N\backslash I}$ is the *residue function* of $H^{(n)}$ on $\mathscr{M}_{I,\mu}$. In view of Proposition 4, it is meromorphic in $\mathscr{E}(\hat{\mathscr{D}}^{(n)})\cap\mathscr{M}_{I,\mu}$.

Now the restriction of $G_\mu^{I,N\backslash I}$ to the "real" submanifold

$$\mathscr{M}_{I,\mu}^+=\{k\in\mathbb{C}^{4(n-1)}:q_I=0,\,p_I\in H_{m_\mu}^+\}$$

satisfies the following relation, which is valid for p_I in a neighbourhood of $H_{m_\mu}^+$:

$$G_\mu^{I,N\backslash I}(p_I;k')\,\delta_\mu^+(p_I)=-\frac{1}{2i\pi}\,\Delta^I H^{(n)}(p_I;k')\,. \tag{19}$$

Here

$$\Delta^I H^{(n)}(p;k')=\lim_{\substack{\varepsilon\in V^+\\ \varepsilon\to 0}}[H^{(n)}(p+i\varepsilon;k')-H^{(n)}(p-i\varepsilon;k')]$$

[3] With $\delta_\mu^+(p)=\theta(p^0)\,\delta(p^2-m_\mu^2)$.

is the analytic absorptive part (with various boundary values $\Delta_{\mathscr{S}_1\mathscr{S}_2}H^{(n)}$) introduced in Section III.2.

The above relation (19) is the exact analogue in several variables of the formula

$$\operatorname{Res} f|_{z=0} \cdot \delta(x) = \frac{1}{2i\pi}\,[f(x-i\varepsilon) - f(x+i\varepsilon)]$$

for functions having a simple pole at the origin; it holds as an identity between distributions in the real variable p_I, analytically valued in the variables k'.

Let Z_μ be the "wave-function renormalization constants" of the field defined in (1) by:

$$\|A(f)\,|\Omega\rangle\|^2 = 2\pi Z_\mu \int |\hat{f}_\mu(\boldsymbol{p})|^2 \,\frac{d\boldsymbol{p}}{2\omega_\mu}\,.$$

Then for $n=2$, relation (19) yields (in a neighbourhood of $H^+_{m_\mu}$):

$$\begin{aligned} G^{\{1\},\{2\}}_\mu \delta^+_\mu(p_1)\,\delta(p_1+p_2) &= -\frac{1}{2\pi}\langle \Omega, A(p_1)\,A(p_2)\Omega\rangle = -Z_\mu \delta^+_\mu(p_1)\,\delta(p_1+p_2) \\ G^{\{1\},\{2\}}_\mu &= -Z_\mu\,. \end{aligned} \tag{20}$$

For a general value of n, it follows from the completeness relation (10b) that

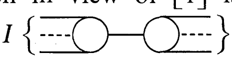
$$\Delta_{\mathscr{S}_1\mathscr{S}_2}H^{(n)}(p_I;p') = \frac{2i\pi}{Z_\mu}\,\delta^+_\mu(p_I)\,[(p_I^2-m_\mu^2)\,H^{(n_1+1)}_{\lambda\uparrow\mathscr{S}_1}]\cdot[(p_{N\setminus I}^2-m_\mu^2)\,H^{(n_2+1)}_{\underline{\lambda}\uparrow\mathscr{S}_2}] \tag{21}$$

which is valid for p_I in a neighbourhood of $H^+_{m_\mu}$.

Here $n_1=|I|$, $n_2=|N\setminus I|$ and λ (resp. $\underline{\lambda}$) stands for the internal four-vector in the argument of $H^{(n_1+1)}$ (resp. $H^{(n_2+1)}$), that is $p_\lambda = p_{N\setminus I}$.

Let us now consider the analytic function

$$\begin{aligned} H^\Gamma_\mu(k_1,\ldots,k_n) = {}&[(k_I^2-m_\mu^2)\,H^{(n_1+1)}(\{k_i, i\in I\};k_\lambda)] \\ &\cdot[(k_{N\setminus I}^2-m_\mu^2)\,H^{(n_2+1)}(\{k_j, j\in N\setminus I\};k_{\underline{\lambda}})] \end{aligned}$$

which in view of [1] is a general n-point function associated with the tree $\Gamma = I$  $N\setminus I$. H^Γ_μ is analytic in the same domain as $H^{(n)}$ and amputation arguments similar to those used in the proof of Proposition 4 allow to conclude that it is analytic in $\mathscr{E}(\hat{\mathscr{D}}^{(n)})$ minus the following set of polar manifolds

$$\{k_J^2 = m_{\mu'}^2;\, J\in\mathscr{P}^*(I), \quad J\in\mathscr{P}^*(N\setminus I), \quad \forall\mu'; \quad k_I^2 = m_{\mu'}^2, \quad \mu'\neq\mu\}\,.$$

We shall then show that the two meromorphic functions $-Z_\mu G^{I,N\setminus I}_\mu$ and H^Γ_μ coincide in the complex submanifold $\mathscr{E}(\hat{\mathscr{D}}^{(n)})\cap\mathscr{M}_{I,\mu}$.

First we notice that the left-hand (resp. right-hand) side of (21) is the boundary value inside the given tube $\mathscr{T}_{\mathscr{S}_1,\mathscr{S}_2}$ of the analytic function $\Delta^I H^{(n)}(p_I;k')_{|\mathscr{M}^+_{I,\mu}}$ (resp. $H^\Gamma_\mu(p_I;k')|_{\mathscr{M}^+_{I,\mu}}$). As a consequence of the edge of the wedge theorem, (21) then implies the coincidence of the corresponding analytic functions throughout their common domain $\mathscr{E}(\hat{\mathscr{D}}^{(n)})\cap\mathscr{M}^+_{I,\mu}$, namely:

$$\frac{Z_\mu}{2i\pi}\,\Delta^I H^{(n)}(p_I;k') = \delta^+_\mu(p_I)\,H^\Gamma_\mu(p_I;k')$$

which yields, in view of (19):

$$-Z_\mu G_\mu^{I,N\backslash I}(p_I;k') = H_\mu^\Gamma(p_I;k')$$

inside $\mathscr{E}(\hat{\mathscr{D}}^{(n)}) \cap \mathscr{M}_{I,\mu}^+$. Through analytic continuation in k_I, this holds in the whole domain[4] $\mathscr{E}(\hat{\mathscr{D}}^{(n)}) \cap \mathscr{M}_{I,\mu}$ and we have the following set of relations linking the various residue functions [due to the expression of H_μ^Γ and to (18)]:

$$G_\mu^{I,N\backslash I} = -\frac{G_\mu^{I,\{\lambda\}} G_\mu^{\{\underline{\lambda}\},N\backslash I}}{Z_\mu}. \tag{22}$$

Remark. If either I or $N\backslash I$ contains a single element, (22) is reduced to (20).

We can then state:

Theorem 1. (Factorization of residues). *The various residue functions $G_\mu^{I,N\backslash I}$ of the n-point functions $H^{(n)}$ on the polar manifolds $\mathscr{M}_{I,\mu}$ satisfy the following set of relations:*

$$G_\mu^{\{1\},\{2\}} = -Z_\mu$$
$$G_\mu^{I,N\backslash I} = -\frac{G_\mu^{I,\{\lambda\}} \cdot G_\mu^{\{\underline{\lambda}\},N\backslash I}}{Z_\mu}. \tag{23}$$

IV.2. The One-p.i. Functions

In this section our purpose is to extract the one-particle singularities from the n-point function $H^{(n)}$ in one given channel $(I, N\backslash I)$. In other words we want to define a function $F^{I,N\backslash I}$ enjoying the following properties:

i) $F^{I,N\backslash I}$ is a general n-point function.

ii) Its coincidence region in the channel $(I, N\backslash I)$ has the form

$$\hat{\mathscr{R}}_I = \{p \in \mathbb{R}^{4(n-1)} : p_I^2 < 4m^2\},$$

$2m$ being the mass threshold of the continuum. Actually such a function is analytic inside the manifolds $\mathscr{M}_{I,\mu}$ for all values of μ. We shall say that it is *one-particle irreducible with respect to the channel* $(I, N\backslash I)$.

As shown in Proposition 4, the amputation procedure yields such one-p.i. functions. However in the cases when $|I|$ and $|N\backslash I|$ are larger than one, the factorization of residues allows a more interesting construction of one-p.i. functions.

In fact let us consider any general two-point function $\Pi(k)$ having simple *zeros* at all values $p^2 = m_\mu^2$ and such that $\Pi(p) \sim -\frac{1}{Z_\mu}(p^2 - m_\mu^2)$ at each of these points. We can prove:

Theorem 2. *For any channel $(I, N\backslash I)$ with n arbitrary and $|I| > 1$, $|N\backslash I| > 1$ the function*

$$F^{I,N\backslash I}(k_1, \ldots, k_n)$$
$$= H^{(n)}(k_1, \ldots, k_n) - H^{(n_1+1)}(\{k_j, j \in I\}; k_\lambda)\Pi(k_\lambda)H^{(n_2+1)}(\{k_j, j \in N\backslash I\}; k_{\underline{\lambda}}) \tag{24}$$

is a general n-point function, one-particle irreducible with respect to this channel.

[4] If $\mathscr{E}(\hat{\mathscr{D}}^{(n)}) \cap \mathscr{M}_{I,\mu}$ contains several connected components, (22) will still hold in *all* these components, as a result of Theorem 2, since the one-p.i. function $F^{I,N\backslash I}$ can be analytically continued in $\mathscr{E}(\hat{\mathscr{D}}^n)$ [note that the proof of Theorem 2 only makes use of (22) in the neighborhood of the real].

Proof. The function H^{Γ} subtracted from $H^{(n)}$ in (24) has the same properties as the H_{μ}^{Γ}'s introduced in the proof of Theorem 1: as a result of the study of tree-products performed in [1], it is a general n-point function. Moreover from our assumption on the zeros of Π, it admits all the sets $\mathscr{M}_{I,\mu}$ in the channel $(I, N\backslash I)$ as simple polar manifolds with the corresponding residue functions

$$\frac{-G_{\mu}^{I,\{\lambda\}} \cdot G_{\mu}^{\{\underline{\lambda}\},N\backslash I}}{Z_{\mu}}.$$

Therefore, in view of Theorem 1, these residues cancel those of $H^{(n)}$ in (24) and the theorem is proved.[5]

A natural choice of a function Π satisfying the above conditions is $\Pi = [H^{(2)}]^{-1}$. Actually a formal definition of one-p.i. functions can be found in perturbation theory as the sum of all "proper" diagrams (i.e. with no internal line carrying the four-momentum k_I) occurring in the expansion of any n-point Green's function. It turns out that the various objects thus introduced are linked with the original Green's functions by the same linear relations as (24), satisfied as formal series of Feynmann graphs, with the special choice $\Pi = [H^{(2)}]^{-1}$. Following the perturbative theory as an heuristic guide we shall choose these relations as *definitions* and call "physical" one-p.i. n-point functions the special set of functions $F^{I,N\backslash I}$ thus exhibited.

However this has the disadvantage of producing new singularities induced by the zeros of the two-point function. From Lorentz invariance, it is known that such polar manifolds have the form $p^2 = \alpha_\nu$, with α_ν lying in the cut-plane $\{z \neq 4m^2 + \varrho,\ \varrho \geqq 0\}$. They were called *non-C.D.D. singularities* by Symanzik: their number is finite and equal to the number of substractions in the Källen-Lehmann representation of $H^{(2)}$. Moreover α_ν can always be chosen real between 0 and m^2: see [3] and below in Section IV.3.

As for the occurrence of *proper C.D.D. singularities* which correspond to zeros of $H^{(2)}$ on the cut $\{p^2 = 4m^2 + \varrho,\ \varrho \geqq 0\}$, it does not prevent $F^{I,N\backslash I}$ to have boundary values in the sense of distributions and to satisfy the algebraic structure (Steinmann relations) of a general n-point function.

Finally let us list the relations extracted from the perturbative framework which we shall choose as definitions of the complete set of functions $F^{I,N\backslash I}$:

i) $|I| = |N\backslash I| = 1$

$$F^{\{1\},\{2\}} = -[H^{(2)}]^{-1} \tag{25a}$$

(in connection with the perturbative definition of the self-energy).

ii) $|I| = 1,\ |N\backslash I| \neq 1$

$$F^{\{i\},N\backslash\{i\}}(k_1, \ldots, k_n) = [H^{(2)}(k_i)]^{-1}\, H^{(n)}(k_1, \ldots, k_n). \tag{25b}$$

iii) $|I| \neq 1,\ |N\backslash I| \neq 1$

$$\begin{aligned} F^{I,N\backslash I}(k_1, \ldots, k_n) &= H^{(n)}(k_1, \ldots, k_n) \\ &\quad - H^{(n_1+1)}(\{k_i, i\in I\}; k_\lambda)\, F^{(\underline{\lambda}),N\backslash I}(k_{\underline{\lambda}}; \{k_j, j\in N\backslash I\}). \end{aligned} \tag{25c}$$

[5] We are indebted to Prof. G. F. Chew for pointing out to us the idea of such a simultaneous extraction of singularities in the case with several masses.

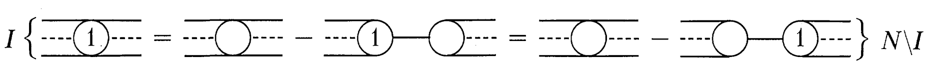
Note that (25c) is a variant of (24) in the special case $\Pi = [H^{(2)}]^{-1}$ and can also be written:

$$F^{I,N\setminus I}(k_1, \dots, k_n) = H^{(n)}(k_1, \dots, k_n) - F^{I,\{\lambda\}}(\{k_i, i \in I\}; k_\lambda) H^{(n_2+1)}(k_{\underline{\lambda}}; \{k_j, j \in N\setminus I\}).$$

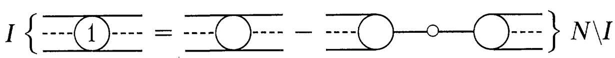
Finally in view of Proposition 4 and Theorem 2, we can state:

Proposition 5. *The functions $F^{I,N\setminus I}$ thus introduced satisfy all the properties of general n-point functions except for a finite set of poles $\{k_I^2 = \alpha_\nu\}$ due to the zeros of $H^{(2)}$; they are one-particle irreducible in the channel $(I, N\setminus I)$.*

Note that the definitions (25c) can be illustrated as follows:



and also



IV.3. Complements on the Zeros of $H^{(2)}$

Proposition 6. *The set of retarded operators $R(x_1; x_2, \dots, x_n)$ introduced in Postulate C can always be chosen so that the two-point function $H^{(2)}$ has a finite number of real zeros $\{k^2 = \alpha_\nu\}$ in its analyticity domain with $0 < \alpha_\nu < \inf_\mu m_\mu^2$. Then $[H^{(2)}]^{-1}$ is a general two-point function which is analytic in $\{k \in \mathbb{C}^4 : k^2 \neq \alpha_\nu, \forall \nu; k^2 \neq 4m^2 + \varrho, \varrho \geqq 0\}$ and has temperate boundary values on the real.*

Proof. Postulates A determine the Källen-Lehmann measure $\Delta H^{(2)} = \varrho$ with $\varrho(p^2) = C\langle \Omega, A(p) A(-p) \Omega \rangle$. From ϱ we can construct a function $\hat{H}^{(2)}(k^2)$ in the following way:

$$\hat{H}^{(2)}(k^2) = g(k^2)\, P(k^2) \quad \text{with} \quad g(z) = \int \frac{\varrho(t)\, dt}{P(t)\,(t - z)}. \tag{26}$$

Here $P(t)$ is a polynomial which is positive for $t \geqq 4m^2$ and whose degree is sufficient for the convergence of (26). If it is chosen with a finite number of real zeros inside $\left]0, \inf_\mu m_\mu^2\right[$, it is clear that $\hat{H}^{(2)}$ satisfies the same property.

Let $H^{(2)}$ denote the two-point function associated with the retarded and advanced operators $R(x, y)$ and $A(x, y)$ of Postulate C. Then $[H^{(2)} - \hat{H}^{(2)}]$ is a polynomial whose Fourier transform in x-space is a finite sum of derivatives of $\delta(x - y)$ which we denote $Q(x - y)$. Now if we make the substitutions

$$R(x, y) \to R'(x, y) = R(x, y) + Q(x - y)$$

$$A(x, y) \to A'(x, y) = A(x, y) + Q(x - y)$$

all the $R(x_1; x_2, \dots, x_n)$ with $n > 2$ being unchanged, we obtain a new set of retarded operators satisfying postulate C (indeed the non-linear relations between them only involve the discontinuity of the two-point function). Now it is clear that with such a choice of the two-point function, the factor $[H^{(2)}]^{-1}$ does not spoil the primitive analytic structure of the one-p.i. functions $F^{I,N\setminus I}$, apart from the possible occurrence of the finite number of real poles $k_I^2 = \alpha_\nu$.

We shall now go back to the case of a single mass m and use the additional technical postulate D (smooth spectral condition) to derive regularity properties of the boundary values of $F^{I,N\backslash I}$ in the region $\Sigma_{I,N\backslash I}=\{p\in\mathbb{R}^{4(n-1)}: p_I^2\geqq 4m^2\}$.

In view of Proposition 3, the real boundary values of $H^{(2)}$ are Hölder continuous functions of p and therefore of p^2, for $p^2\geqq 4m^2$. Then the set of points p^2 at which $H^{(2)}$ vanishes (the so-called C.D.D. zeros) is *a closed set without interior points* (if it had any it would contain at least one interval but the vanishing of $H^{(2)}$ there would imply $H^{(2)}\equiv 0$ by analytic continuation). We denote Λ the dense open subset of $\{p\in\mathbb{R}^4: p^2\geqq 4m^2\}$ in which $H^{(2)}(p)\neq 0$. The boundary values of $[H^{(2)}]^{-1}$ are Hölder-continuous in Λ and we have:

Proposition 7. *Let $\mathscr{S}_+$ and $\mathscr{S}_-$ two adjacent cells separated by the partition $(I, N\backslash I)$. The distribution*

$$F^{I,N\backslash I}_{\mathscr{S}_\pm}(p_I;\underline{\varphi}_1,\underline{\varphi}_2)=\int F^{I,N\backslash I}_{\mathscr{S}_\pm}(p_I,\underline{p}^{(n_1)},\underline{p}^{(n_2)})\,\underline{\varphi}_1(\underline{p}^{(n_1)})\,\underline{\varphi}_2(\underline{p}^{(n_2)})\,d\underline{p}^{(n_1)}\,d\underline{p}^{(n_2)}$$

with $\underline{\varphi}_j\in\mathscr{S}(\mathbb{R}^{4(n_j-1)}_{p^{(n_j)}})$ and $\underline{p}^{(n_1)}$ (resp. $\underline{p}^{(n_2)}$) the barycentric variables of the set $\{p_i, i\in I\}$ (resp. $\{p_j, j\in N\backslash I\}$), can be identified with a Hölder-continuous function of p_I in the region $\hat{\Sigma}_{I,N\backslash I}=\{p\in\mathbb{R}^{4(n-1)}: p_I\in\Lambda\}$.

Proof. i) If $I=\{\lambda\}$, in view of (25b) we write

$$F^{\{\lambda\},N\backslash\{\lambda\}}_{\mathscr{S}_\pm}(p_\lambda;\underline{\varphi}_2)=[H^{(2)}(p_\lambda)]^{-1}\,H^{(n)}_{\mathscr{S}_\pm}(p_\lambda;\underline{\varphi}_2)\,.$$

From Proposition 3, $H^{(n)}_{\mathscr{S}_\pm}(p_\lambda;\underline{\varphi}_2)$ is Hölder-continuous on $\{p_\lambda: p_\lambda\geqq 4m^2\}$. Since $[H^{(2)}]^{-1}$ admits Hölder continuous boundary values on Λ, the property is proved.

ii) In the general case $|I|>1$, $|N\backslash I|>1$, it is the consequence of the definition (25c) and of the previous argument.

V. Non-Linear Properties of the One-p.i. Functions

In this section we investigate the non-linear properties of the set of "physical" one-p.i. functions $\{F^{I,N\backslash I}\}$ which have been constructed above as the exact analogues of the perturbative formal sums of "proper" Feynmann graphs.

As a consequence of the completeness of asymptotic states, we shall prove that they satisfy a set of *non-linear* relations similar to the completeness relations described in Section III.3.

More precisely let us recall the definition of the l-particle region $\Sigma^l_{I,N\backslash I}$ associated with a channel $(I, N\backslash I)$:

$$\Sigma^l_{I,N\backslash I}=\{p\in\mathbb{R}^{4(n-1)}: l^2m^2\leqq p_I^2<(l+1)^2m^2\}\quad\text{with}\quad l\geqq 2$$

and define:

$$\hat{\Sigma}^l_{I,N\backslash I}=\{p\in\mathbb{R}^{4(n-1)}: l^2m^2\leqq p_I^2<(l+1)^2m^2;\,H^{(2)}(p_I)\neq 0\}$$

that is, we discard those exceptional values corresponding to C.D.D. zeros. $\hat{\Sigma}^l_{I,N\backslash I}$ is a dense open subset of $\Sigma^l_{I,N\backslash I}$ and we have:

$$\hat{\Sigma}_{I,N\backslash I}=\bigcup_{l\geqq 2}\hat{\Sigma}^l_{I,N\backslash I}\,.$$

In this section an *equivalence* will be proved (Theorem 3) between the completeness relations

$$\Delta_{\mathscr{S}_1\mathscr{S}_2} H^{(n)} = \sum_{k=2}^{l} \left(\frac{2i\pi}{Z}\right)^k \frac{1}{k!} \hat{H}^{(n_1+k)}_{K\uparrow\mathscr{S}_1} * \hat{H}^{(n_2+k)}_{\underline{K}\uparrow\mathscr{S}_2} = \sum_{k=2}^{l} \left(\frac{2i\pi}{Z}\right)^k \frac{1}{k!} \hat{H}^{(n_1+k)}_{K\downarrow\mathscr{S}_1} * \hat{H}^{(n_2+k)}_{\underline{K}\downarrow\mathscr{S}_2}$$

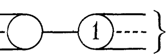
satisfied on the relevant regions $\hat{\Sigma}^l_{I,N\setminus I}$ [here $n_1 = |I|$ and $n_2 = |N\setminus I|$; $\mathscr{S}_1$ (resp. $\mathscr{S}_2$) is any cell of I (resp. $N\setminus I$)] and the following set of relations

$$\Delta_{\mathscr{S}_1\mathscr{S}_2} F^{I,N\setminus I} = \sum_{k=2}^{l} \left(\frac{2i\pi}{Z}\right)^k \frac{1}{k!} \hat{F}^{I,K}_{K\uparrow\mathscr{S}_1} * \hat{F}^{\underline{K},N\setminus I}_{\underline{K}\uparrow\mathscr{S}_2} = \sum_{k=2}^{l} \left(\frac{2i\pi}{Z}\right)^k \frac{1}{k!} \hat{F}^{I,K}_{K\downarrow\mathscr{S}_1} * \hat{F}^{\underline{K},N\setminus I}_{\underline{K}\downarrow\mathscr{S}_2} \quad (27)$$

satisfied on the same regions $\hat{\Sigma}^l_{I,N\setminus I}$.

The non-linear information of general quantum field theory originally known through the completeness relations (11) in terms of the various boundary values and absorptive parts of the n-point functions will then be expressed (up to the manifolds corresponding to C.D.D. zeros) in terms of the one-p.i. objects, a result which will be needed extensively in the following of the program [23].

This equivalence is actually the consequence of an *algebraic* algorithm which will be described below. However at present it is *not* known how to give it a rigorous meaning in the sense of *product of distributions* without making the technical "smoothness postulate" D. Used under the form of Propositions 3 and 7 the latter allows one to derive the following simple results which will be needed for a rigorous proof of (27).

Consider the tree-product $H^\Gamma(k_1, \ldots, k_n)$ which is occuring in the definition (25c) of $F^{I,N\setminus I}$. $H^\Gamma(k)$ can be represented as I  $N\setminus I$ that is:

$$H^\Gamma(k_1, \ldots, k_n) = H^{(n_1+1)}(\{k_j, j\in I\}; k_\lambda) \cdot F^{\{\lambda\},N\setminus I}(k_{\underline{\lambda}}; \{k_j, j\in N\setminus I\}).$$

In the following we shall need to express its absorptive parts $\Delta_{\mathscr{S}_1\mathscr{S}_2} H^\Gamma$ in the channel $(I, N\setminus I)$. We shall prove:

Proposition 8. *The absorptive part $\Delta_{\mathscr{S}_1\mathscr{S}_2} H^\Gamma$ is given (in the sense of distributions) by the following rule:*

$$\Delta_{\mathscr{S}_1\mathscr{S}_2} H^\Gamma = \Delta_{\mathscr{S}_1,\{\lambda\}} H^{(n_1+1)} \cdot F^{\{\underline{\lambda}\},N\setminus I}_{\underline{\lambda}\uparrow\mathscr{S}_2} + H^{(n_1+1)}_{\lambda\uparrow\mathscr{S}_1} \cdot \Delta_{\{\underline{\lambda}\},\mathscr{S}_2} F^{\{\underline{\lambda}\},N\setminus I} \quad (28\text{a})$$

or equivalently

$$\Delta_{\mathscr{S}_1\mathscr{S}_2} H^\Gamma = \Delta_{\mathscr{S}_1,\{\lambda\}} H^{(n_1+1)} \cdot F^{\{\underline{\lambda}\},N\setminus I}_{\underline{\lambda}\downarrow\mathscr{S}_2} + H^{(n_1+1)}_{\lambda\downarrow\mathscr{S}_1} \cdot \Delta_{\{\underline{\lambda}\},\mathscr{S}_2} F^{\{\underline{\lambda}\},N\setminus I}. \quad (28\text{b})$$

Proof. From Propositions 3 and 7 we know that (after testing in the external variables $\{p_i, i\in I\}$, $\{p_j, j\in N\setminus I\}$), $H^{(n+1)}_{\lambda\updownarrow\mathscr{S}}$ is a Hölder-continuous function of p_λ and $F^{\{\lambda\},N}_{\lambda\updownarrow\mathscr{S}}$ a Hölder continuous function of p_λ except at those exceptional values corresponding to C.D.D. zeros. We are then led to compute $\Delta(fg) = f_+ g_+ - f_- g_-$ in the sense of two continuous functions on the open set $\hat{\Sigma}_{I,N\setminus I}$, namely

$$\Delta(fg) = \Delta f \cdot g_+ + f_- \cdot \Delta g = \Delta f \cdot g_- + f_+ \cdot \Delta g$$

which gives sense to (28a) and (28b).

We now need to investigate the properties of the restrictions to the mass-shell of the amputated one-p.i. functions, as it was done in Section III.3 for the original n-point functions $H^{(n)}$.

Indeed, once these restrictions are well-defined as square-integrable functions in the mass-shell variables, the operation $*$ becomes meaningful at the right-hand side of (27) [as an integral of the product of two square integrable functions, after testing in the external variables, i.e. in the same sense as in (11)].

Let us consider for any given cell $K\downarrow\mathscr{S}_2\in S(K\cup(N\backslash I))$ (with $|K|=k\geqq 2$, $|N\backslash I|=n_2$) the real boundary value $F^{K,N\backslash I}_{K\downarrow\mathscr{S}_2}(p_1,\ldots,p_k;\underline{p}')$ of the analytic function $F^{K,N\backslash I}$; here (as in Section III.5) it is more convenient to adopt the notation of independent variables $(p_1,\ldots,p_k;\underline{p}')$ in which $\underline{p}'$ denotes the set of barycentric four-vector variables associated with $(N\backslash I)$, namely:

$$\underline{p}'=\left\{\underline{p}'_i,\, i\in N\backslash I;\ \sum_{i\in N\backslash I}\underline{p}'_i=0\right\}\in\mathbb{R}^{4(n_2-1)}$$

with

$$\underline{p}'_i=p'_i-\frac{p'_{N\backslash I}}{n_2};\qquad p'_{N\backslash I}=-\sum_{j=1}^{k}p_j\,.$$

In view of (25), we have:

— if $N\backslash I=\{\lambda\}$ $(n_2=1)$:

$$F^{K,\{\lambda\}}_{K\downarrow\lambda}(p_1,\ldots,p_k)=H^{(k+1)}_{K\downarrow\lambda}(p_1,\ldots,p_k)\left[H^{(2)}\left(\sum_{j=1}^{k}p_j\right)\right]^{-1}\tag{29a}$$

or in a brief algebraic notation: $F^{K,\{\lambda\}}_{K\downarrow\lambda}=H^{(k+1)}_{K\downarrow\lambda}[H^{(2)}_{\underline{\lambda}\downarrow\lambda}]^{-1}$

— if $n_2>1$:

$$F^{K,N\backslash I}_{K\downarrow\mathscr{S}_2}(p_1,\ldots,p_k;\underline{p}')=H^{(n_2+k)}_{K\downarrow\mathscr{S}_2}(p_1,\ldots,p_k;\underline{p}')-H^{(k+1)}_{K\downarrow\lambda}(p_1,\ldots,p_k)\,F^{\{\lambda\},N\backslash I}_{\underline{\lambda}\downarrow\mathscr{S}_2}(\underline{p}')\tag{29b}$$

or in brief: $F^{K,N\backslash I}_{K\downarrow\mathscr{S}_2}=H^{(n_2+k)}_{K\downarrow\mathscr{S}_2}-H^{(k+1)}_{K\downarrow\lambda}F^{\{\lambda\},N\backslash I}_{\underline{\lambda}\downarrow\mathscr{S}_2}$.

In these formulae, the product of distributions which occurs at the right-hand side is defined as the boundary value of the corresponding product of analytic functions in the tube $\mathscr{T}_{K\downarrow\mathscr{S}_2}$.

We now want to define correctly the mass-shell restrictions $(p^0_j=\omega_j;\, j\in K)$ of the Eqs. (29a), (29b), after multiplication of both sides by $\prod_{j=1}^{k}(p_j^2-m^2)$. In particular, some work will have to be done to prove that the restriction of the products can be computed as products of functions (see Proposition 9 below): here the difficulty arises from the fact that the complex mass-shell is not transverse to the tube $\mathscr{T}_{K\downarrow\mathscr{S}_2}$.

We first have:

Lemma 3. *It is meaningful to define the following distributions in the open set* $\Omega'\times\mathbb{R}^{4(n_2-1)}_{(\underline{p}')}$:

$$\hat{F}^{K,N\backslash I}_{K\uparrow\mathscr{S}_2}(p_1,\ldots,p_k;\underline{p}')=\prod_{j=1}^{k}(p_j^2-m^2)\,F^{K,N\backslash I}_{K\uparrow\mathscr{S}_2}(p_1,\ldots,p_k;\underline{p}')\Big|_{\substack{p^0_j=\omega_j\\1\leqq j\leqq k}}\tag{30}$$

where Ω' *denotes, as in Section III.3 the set of non-overlapping configurations in* $(H^+_m)^k$.

Proof. Let us put: $(F^{K,N\setminus I})^{\mathrm{amp}} = \prod_{j=1}^{k} (k_j^2 - m^2) F^{K,N\setminus I}$. In view of Theorem 2 $(F^{K,N\setminus I})^{\mathrm{amp}}$ is a general $(k+n_2)$-point function; it moreover satisfies the same coincidence relations as $(H^{(n_2+k)})^{\mathrm{amp}}$ in all channels $(\{j\}, [K \cup (N\setminus I)]\setminus\{j\})$ $(\forall j \in K)$. Therefore Proposition B.5 applies to $(F^{K,N\setminus I})^{\mathrm{amp}}$ and yields the announced result. The latter is independent of postulate D.

Let us now define a dense open subset Λ_K of $(H_m^+)^k$ by:

$$\Lambda_K = \{p \in R^{4k} : p_j \in H_m^+ ;\ H^{(2)}(p_K) \neq 0\} .$$

Λ_K is the set of all "non C.D.D." mass-shell configurations. Then on the basis of postulate D, we have.

Proposition 9. i) *On any open subset of* $(H_m^+)^k$ *with compact closure in* Λ_K, *the distributions* $\hat{F}^{K,\{\lambda\}}_{K\downarrow\lambda}(p_1, \dots, p_k)$ *and*

$$^{\varphi}\hat{F}^{K,N\setminus I}_{K\downarrow\mathscr{S}_2}(p_1, \dots, p_k) = \int \hat{F}^{K,N\setminus I}_{K\downarrow\mathscr{S}_2}(p_1, \dots, p_k; \underline{p}')\varphi(-\underline{p}')d\underline{p}'$$

can be identified with square-integrable functions $\left(\text{with respect to the measure } \prod_{j=1}^{k} \frac{dp_j}{2\omega_j}\right)$.

ii) *On* Λ_K *the following relations hold in the sense of products of square integrable functions by bounded continuous functions:*

— *if* $N\setminus I = \{\lambda\}$ $(n_2 = 1)$*:*

$$\hat{F}^{K,\{\lambda\}}_{K\downarrow\lambda}(p_1, \dots, p_k) = \hat{H}^{(k+1)}_{K\downarrow\lambda}(p_1, \dots, p_k) \left[H^{(2)}\left(p_K, \sum_{j=1}^{k} \omega_j\right)\right]^{-1} \tag{31a}$$

or in brief:

$$\hat{F}^{K,\{\lambda\}}_{K\downarrow\lambda} = \hat{H}^{(k+1)}_{K\downarrow\lambda} [H^{(2)}_{\underline{\lambda}\downarrow\lambda}]^{-1}$$

— *if* $n_2 > 1 : \forall \varphi \in \mathscr{S}(\mathbb{R}^{4(n_2-1)}_{(\underline{p}')})$,

$$^{\varphi}\hat{F}^{K,N\setminus I}_{K\downarrow\mathscr{S}_2} = {}^{\varphi}\hat{H}^{(n_2+k)}_{K\downarrow\mathscr{S}_2} - \hat{H}^{(k+1)}_{K\downarrow\lambda}\, {}^{\varphi}\hat{F}^{\{\lambda\},N\setminus I}_{\underline{\lambda}\downarrow\mathscr{S}_2} , \tag{31b}$$

where

$$^{\varphi}\hat{H}^{(n_2+k)}_{K\downarrow\mathscr{S}_2}(p_1, \dots, p_k) = \int \hat{H}^{(n_2+k)}_{K\downarrow\mathscr{S}_2}(p_1, \dots, p_k; \underline{p}')\, \varphi(-\underline{p}')\, d\underline{p}' .$$

Proof. By definition of the one-p.i. function, we have:

$$(F^{K,\{\lambda\}})^{\mathrm{amp}}(p + i\varepsilon) = (H^{(k+1)})^{\mathrm{amp}}(p + i\varepsilon)\, [H^{(2)}(p_K + i\varepsilon_K)]^{-1} , \tag{32}$$

both sides being analytic in the advanced tube $\mathscr{T}_{K\downarrow\lambda} = \{p + i\varepsilon; \varepsilon_j \in V^+, \forall j \in K\}$, and yielding the boundary value:

$$(F^{K,\{\lambda\}}_{K\downarrow\lambda})^{\mathrm{amp}}(p) = \lim_{\varepsilon \to 0} (F^{K,\{\lambda\}})^{\mathrm{amp}}(p + i\varepsilon) . \tag{33}$$

Let now $\hat{p}$ be an arbitrary point in $\Lambda_K \cap \Omega'$, and

$$(H^{(k+1)})^{\mathrm{amp}} = \sum_{\beta} h_\beta \tag{34}$$

be a local decomposition of $(H^{(k+1)})^{\mathrm{amp}}$ in a neighbourhood of $\hat{p}$; according to Proposition B.6, the h'_β's can be chosen to be analytic in local tubes T'_β's such that:

— $\hat{T}_\beta = T_\beta \cap \mathscr{M}_c \neq \emptyset$ [$\mathscr{M}_c$ being the complexified $(H_m^+)^k$],

— $\bigcap_\beta T_\beta$ contains a local restriction at $\hat{p}$ of $\mathscr{T}_{K\downarrow\lambda}$,

— each T_β is contained in the domain of $[H^{(2)}(p_K)]^{-1}$.

Let us then call $\hat{h}_\beta = h_{\beta|\mathscr{M}_c}$, $[H^{(2)}]^{-1} = [H^{(2)}(p_K)]^{-1}{}_{|\mathscr{M}_c}$; in view of (32)—(34), and of the above properties of the h'_β's, we can write, by definition of $F^{K,\{\lambda\}}_{K\downarrow\lambda}$:

$$\hat{F}^{K,\{\lambda\}}_{K\downarrow\lambda} = \sum_\beta \text{b.v.}(\hat{h}_\beta\,[\hat{H}^{(2)}]^{-1}), \tag{35}$$

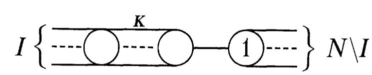
where the boundary values (b.v.) at the right-hand side are taken from the local tubes $\hat{T}_\beta$ (in $\mathscr{M}_c$).

But according to Appendix A, the right-hand side of (35) defines precisely the product of distributions $\hat{H}^{(k+1)}_{K\downarrow\lambda}[H^{(2)}_{\underline{\lambda}\downarrow\lambda}]^{-1}$ in a mass-shell neighbourhood U of p (with $U \subset \Lambda_K \cap \Omega'$).

Now we know from Lemma 1 and Proposition 3 that $\hat{H}^{(k+1)}_{K\downarrow\lambda}$ and $[H^{(2)}_{\underline{\lambda}\downarrow\lambda}]^{-1}$ can be identified respectively with a square integrable function and a bounded continuous function which is the uniform limit of an analytic function (U must be chosen compact in Λ_K, for the latter to hold). Then Proposition A.5 ensures that the product of distributions $\hat{H}^{(k+1)}_{K\downarrow\lambda}\cdot[H^{(2)}_{\underline{\lambda}\downarrow\lambda}]^{-1}$ can be identified on U with the corresponding ordinary product of functions. So this proves (31a) on $\Lambda_K \cap \Omega'$, and therefore on Λ_K since $[H^{(2)}_{\underline{\lambda}\downarrow\lambda}]^{-1}$ is continuous in the whole set Λ_K, and $\Lambda_K \cap \Omega$ is of measure zero in Λ_K. Moreover the latter product is itself a square integrable function on every compact subset of Λ_K, which is the result stated in part i).

A similar argument holds in the case of a general set $N\backslash I$, since in view of Proposition 7, the function $x\hat{F}^{\{\lambda\},N\backslash I}_{\lambda\downarrow\mathscr{S}_2}$ has the same continuity properties as the two-point function $[H^{(2)}]^{-1}$; this ends the proof of Proposition 9.

Finally let us consider the following convolution product



associated with the distribution $[\hat{H}^{(n_1+k)}_{K\downarrow\mathscr{S}_1} * \hat{H}^{(k+1)}_{\underline{K}\downarrow\lambda}]\cdot F^{\{\lambda\},N\backslash I}_{\underline{\lambda}\downarrow\mathscr{S}_2}$ (which has now a precise meaning). We shall make below an extensive use of the following associativity and distributivity properties which are straightforward:

$$\text{i)}\quad [\hat{H}^{(n_1+k)}_{K\downarrow\mathscr{S}_1} * \hat{H}^{(k+1)}_{\underline{K}\downarrow\lambda}]\cdot F^{\{\lambda\},N\backslash I}_{\underline{\lambda}\downarrow\mathscr{S}_2} = \hat{H}^{(n_1+k)}_{K\downarrow\mathscr{S}_1} * [\hat{H}^{(k+1)}_{\underline{K}\downarrow\lambda}\cdot F^{\{\lambda\},N\backslash I}_{\underline{\lambda}\downarrow\mathscr{S}_2}], \tag{36}$$

$$\text{ii)}\quad \hat{H}^{(n_1+k)}_{K\downarrow\mathscr{S}_1} * \hat{H}^{(k+1)}_{\underline{K}\downarrow\lambda}\cdot F^{\{\lambda\},N\backslash I}_{\underline{\lambda}\downarrow\mathscr{S}_2} = \hat{H}^{(n_1+k)}_{K\downarrow\mathscr{S}_1} * \hat{H}^{(n_2+k)}_{\underline{K}\downarrow\mathscr{S}_2} - \hat{H}^{(n_1+k)}_{K\downarrow\mathscr{S}_1} * \hat{F}^{\underline{K},N\backslash I}_{\underline{K}\downarrow\mathscr{S}_2}. \tag{37}$$

Actually, in view of Propositions 3, 7, and 9, these are readily obtained in the sense of products of functions square integrable by bounded continuous ones (after testing in the external variables). Now we are in a position to prove the:

Theorem 3. *The various boundary values and absorptive parts of all the one-p.i. n-point functions* $\{F^{I,N\backslash I}\}$ *constructed in Section IV.2 satisfy (in the sense of distributions) the following set of non-linear relations:*

$$\Delta_{\mathscr{S}_1\mathscr{S}_2}F^{I,N\backslash I} = \sum_{k=2}^{l}\left(\frac{2i\pi}{Z}\right)^k \frac{1}{k!}\hat{F}^{I,K}_{K\uparrow\mathscr{S}_1} * \hat{F}^{\underline{K},N\backslash I}_{\underline{K}\uparrow\mathscr{S}_2} = \sum_{k=2}^{l}\left(\frac{2i\pi}{Z}\right)^k \frac{1}{k!}\hat{F}^{I,K}_{K\downarrow\mathscr{S}_1} * \hat{F}^{\underline{K},N\backslash I}_{\underline{K}\downarrow\mathscr{S}_2} \tag{27}$$

on the relevant l-particle region:

$$\hat{\Sigma}^l_{I,N\backslash I} = \{p \in \mathbb{R}^{4(n-1)} : l^2m^2 \leq p_I^2 < (l+1)^2m^2;\, H^{(2)}(p_I) \neq 0\}.$$

Here $\mathscr{S}_1$ (resp. $\mathscr{S}_2$) is any cell of I (resp. $N\backslash I$); the various distributions occurring on the right-hand side are defined as in Lemma 3, and their product is meaningful in the sense of square-integrable functions (Proposition 9).

Moreover the complete set of these relations (27) *on the relevant sets* $\hat{\Sigma}^l_{I,N\backslash I}$ *is* equivalent *with all the original completeness relations* (11) *expressed in the same regions.*

Proof. We shall stick to the case of "negative" arrows. (The proof would go similarly in the other case.) First we define for all channels $(I, N\backslash I)$:

$$\Theta_{\mathscr{S}_1,\mathscr{S}_2} = \Delta_{\mathscr{S}_1\mathscr{S}_2} H^{(n)} - \sum_{k=2}^{l} C_k \hat{H}^{(n_1+k)}_{K\downarrow\mathscr{S}_1} * \hat{H}^{(n_2+k)}_{\underline{K}\downarrow\mathscr{S}_2}$$

with $C_k = \left(\dfrac{2i\pi}{Z}\right)^k \dfrac{1}{k!}$ and similarly

$$\Theta^1_{\mathscr{S}_1,\mathscr{S}_2} = \Delta_{\mathscr{S}_1\mathscr{S}_2} F^{I,N\backslash I} - \sum_{k=2}^{l} C_k \hat{F}^{I,K}_{K\downarrow\mathscr{S}_1} * \hat{F}^{\underline{K},N\backslash I}_{\underline{K}\downarrow\mathscr{S}_2}.$$

i) Then we start from the definition of the one-p.i. two-point function:

$$F^{\{1\},\{2\}} = -[H^{(2)}]^{-1},$$

which yields (in view of Proposition 3)

$$\Delta_{\{1\},\{2\}} F^{\{1\},\{2\}} = [H^{(2)}_{2\downarrow 1}]^{-1} \Delta_{\{1\},\{2\}} H^{(2)} [H^{(2)}_{1\downarrow 2}]^{-1}.$$

This can be rewritten:

$$\begin{aligned}\Delta_{\{1\},\{2\}} F^{\{1\},\{2\}} = {} & [H^{(2)}_{2\downarrow 1}]^{-1} \left[\Delta_{\{1\},\{2\}} H^{(2)} - \sum_{k=2}^{l} C_k \hat{H}^{(k+1)}_{K\downarrow 1} * \hat{H}^{(k+1)}_{\underline{K}\downarrow 2}\right] [H^{(2)}_{1\downarrow 2}]^{-1} \\ & + [H^{(2)}_{2\downarrow 1}]^{-1} \left[\sum_{k=2}^{l} C_k \hat{H}^{(k+1)}_{K\downarrow 1} * \hat{H}^{(k+1)}_{\underline{K}\downarrow 2}\right] [H^{(2)}_{1\downarrow 2}]^{-1}.\end{aligned}$$

Then in view of Proposition 9 and (36), the last bracket can be rewritten:

$$\sum_{k=2}^{l} C_k \hat{F}^{\{1\},K}_{K\downarrow 1} * \hat{F}^{\underline{K},\{2\}}_{\underline{K}\downarrow 2}$$

which yields

$$\Theta^1_{\{1\},\{2\}} = [H^{(2)}_{2\downarrow 1}]^{-1} \Theta_{\{1\},\{2\}} [H^{(2)}_{1\downarrow 2}]^{-1}. \tag{38a}$$

ii) When (for instance) I is reduced to one element $\{i\}$, we start from the definition of $F^{\{i\},N\backslash\{i\}}$ to derive:

$$F^{\{i\},N\backslash\{i\}}_{i\downarrow\mathscr{S}_2} = -F^{\{i\},\{\underline{i}\}}_{i\downarrow\underline{i}} H^{(n)}_{\underline{i}\downarrow\mathscr{S}_2}.$$

Taking into account Proposition 8, this yields

$$\Delta_{\{i\},\mathscr{S}_2} F^{\{i\},N\backslash\{i\}} = [H^{(2)}_{\underline{i}\downarrow i}]^{-1} \Delta_{(i),\mathscr{S}_2} H^{(n)} - \Delta_{(i),\{\underline{i}\}} F^{(i),\{\underline{i}\}} H^{(n)}_{\underline{i}\downarrow\mathscr{S}_2}.$$

This can be rewritten

$$\begin{aligned}\Delta_{\{i\},\mathscr{S}_2} F^{\{i\},N\backslash\{i\}} = {} & [H^{(2)}_{\underline{i}\downarrow i}]^{-1} \Theta_{\{i\},\mathscr{S}_2} - \Theta^1_{\{i\},\{\underline{i}\}} H^{(n)}_{\underline{i}\downarrow\mathscr{S}_2} \\ & + [H^{(2)}_{\underline{i}\downarrow i}]^{-1} \left[\sum_{k=2}^{l} C_k \hat{H}^{(k+1)}_{K\downarrow i} * \hat{H}^{(k+n_2)}_{\underline{K}\downarrow\mathscr{S}_2}\right] - \left[\sum_{k=2}^{l} C_k \hat{F}^{\{i\},K}_{K\downarrow i} * \hat{F}^{\underline{K},\{\underline{i}\}}_{\underline{K}\downarrow\underline{i}}\right] H^{(n)}_{\underline{i}\downarrow\mathscr{S}_2}.\end{aligned}$$

But the last two terms of the right-hand side can be rewritten using relations (31) and (36):

$$\sum_{k=2}^{l} C_k \hat{F}^{\{i\},K}_{K\downarrow i} * \hat{F}^{\underline{K},N\setminus\{i\}}_{\underline{K}\downarrow\mathscr{S}_2}$$

and finally we get

$$\Theta^1_{\{i\},\mathscr{S}_2} = [H^{(2)}_{\underline{i}\downarrow i}]^{-1}\,\Theta_{\{i\},\mathscr{S}_2} - \Theta^1_{\{i\},\{\underline{i}\}} H^{(n)}_{i\downarrow\mathscr{S}_2}\,. \tag{38 b}$$

iii) We then turn to the general case when $|I|$ (resp. $|N\setminus I|$) is different from one. From the definition (25c) and from Proposition 8, we get:

$$\Delta_{\mathscr{S}_1\mathscr{S}_2} F^{I,N\setminus I} = \Delta_{\mathscr{S}_1\mathscr{S}_2} H^{(n)} - \Delta_{\mathscr{S}_1,\{\lambda\}} H^{(n_1+1)} F^{(\underline{\lambda}),N\setminus I}_{\underline{\lambda}\downarrow\mathscr{S}_2} - H^{(n_1+1)}_{\lambda\downarrow\mathscr{S}_1} \Delta_{\{\underline{\lambda}\},\mathscr{S}_2} F^{\{\underline{\lambda}\},N\setminus I}\,.$$

This can be rewritten:

$$\begin{aligned}\Delta_{\mathscr{S}_1\mathscr{S}_2} F^{I,N\setminus I} &= \Theta_{\mathscr{S}_1,\mathscr{S}_2} - \Theta_{\mathscr{S}_1,\{\lambda\}}\; F^{\{\underline{\lambda}\},N\setminus I}_{\underline{\lambda}\downarrow\mathscr{S}_2} - H^{(n_1+1)}_{\lambda\downarrow\mathscr{S}_2}\;\Theta^1_{\{\underline{\lambda}\},\mathscr{S}_2}\\ &+ \sum_{k=2}^{l} C_k\big[\hat{H}^{(n_1+k)}_{K\downarrow\mathscr{S}_1} * \hat{H}^{(n_2+k)}_{\underline{K}\downarrow\mathscr{S}_2} - (\hat{H}^{(n_1+k)}_{K\downarrow\mathscr{S}_1} * \hat{H}^{(k+1)}_{\underline{K}\downarrow\lambda})\;F^{\{\underline{\lambda}\},N\setminus I}_{\underline{\lambda}\downarrow\mathscr{S}_2}\\ &- H^{(n_1+1)}_{\lambda\downarrow\mathscr{S}_1}\;(\hat{F}^{\{\underline{\lambda}\},K}_{K\downarrow\underline{\lambda}} * \hat{F}^{\underline{K},N\setminus I}_{\underline{K}\downarrow\mathscr{S}_2})\big]\,.\end{aligned}$$

Now using relations (36) and (37), the last summation can be rewritten:

$$\sum_{k=2}^{l} C_k \hat{F}^{I,K}_{K\downarrow\mathscr{S}_1} * \hat{F}^{\underline{K},N\setminus I}_{\underline{K}\downarrow\mathscr{S}_2}$$

which allows to write:

$$\Theta^1_{\mathscr{S}_1,\mathscr{S}_2} = \Theta_{\mathscr{S}_1,\mathscr{S}_2} - \Theta_{\mathscr{S}_1,\{\lambda\}}\; F^{\{\underline{\lambda}\},N\setminus I}_{\underline{\lambda}\downarrow\mathscr{S}_2} - H^{(n+1)}_{\lambda\downarrow\mathscr{S}_1}\;\Theta^1_{\{\underline{\lambda}\},\mathscr{S}_2}\,. \tag{38 c}$$

Putting together relations (38a–38c) it is then straightforward to check that they form a homogeneous linear system which can be solved by recursion either in the $\Theta_{\mathscr{S}_1\mathscr{S}_2}$ or in the $\Theta^1_{\mathscr{S}_1\mathscr{S}_2}$. The vanishing of the whole set of the $\Theta_{\mathscr{S}_1\mathscr{S}_2}$ (expressing asymptotic completeness) is therefore equivalent with the vanishing of all the $\Theta^1_{\mathscr{S}_1\mathscr{S}_2}$, which achieves the proof of the theorem.

VI. One-Particle Irreducibility in All Channels

Finally we turn to the definition of a general n-point function which should be one-particle irreducible in *all channels* $(I, N\setminus I)$, that is to the possibility of a simultaneous extraction of one-particle singularities on the various manifolds

$$\mathscr{M}_{I,\mu} = \{k_I^2 = m_\mu^2, I \in \mathscr{P}^*(N)\}.$$

This will be done in two steps.

A. First we shall define a general n-point function $F^{(n)}$ which is one-particle irreducible in all channels $(I, N\setminus I)$ with $2 \leqq |I| \leqq n-2$. This will be done by recursion over the number n of external variables. First we notice that for $n \leqq 3$ we can trivially define: $F^{(n)} = H^{(n)}$. In a second step, we assume that for any $n \leqq n_0 - 1$, it is possible to define a general n-point function $F^{(n)}$ which is one-particle irreducible in all channels $2 \leqq |I| \leqq n-2$. Then we define $F^{(n_0)}$ by the following recursion formula

$$F^{(n_0)} = H^{(n_0)} - \sum_{T\in\mathscr{T}^{(n_0)}} H^T\,. \tag{39}$$

Here some explanation of the notation is needed: $\mathscr{T}^{(n_0)}$ denotes the set of *trees* with n_0 external lines and any vertex at least three-lined. With any such tree T we associate a *convolution-product* H^T thus defined [1]:

$$H^T(k) = \prod_v F^{(n_v)}(\{k_j, j \in \mathscr{S}_v\}) \prod_j [H^{(2)}(k_j)]^{-1} . \tag{40}$$

Here the first product is taken over the *vertices* of T: with any vertex v incident with n_v lines $\{j \in \mathscr{S}_v\}$ we associate the function $F^{(n_v)}$. Indeed it can be checked that for any v, $n_v < n_0$ so that (39) provides a good recursive definition of $F^{(n_0)}$. The second product is taken over *internal* lines. Then we can state:

Proposition 10. *$F^{(n_0)}$ is a general n_0-point function.*

Proof. Straightforward in view of the recursion hypothesis, once taken into account the conservation of the primitive structure by tree-convolution ([1], Section 4).

Proposition 11. *$F^{(n_0)}$ is analytic on any manifold $\mathscr{M}_{I,\mu}$ with $2 \leqq |I| \leqq n_0 - 2$.*

Proof. Let us make choice of any such channel $(I, N_0 \backslash I)$. From the study of tree-products made in [1], it is clear that for any tree $T \in \mathscr{T}^{(n_0)}$ three situations may occur:

i) $(I, N_0 \backslash I)$ is not a *vertex partition* for T: then it is known ([1], p. 200) that H^T is analytic on $k_I^2 = m_\mu^2$. We call $F_1(k)$ the contribution of all such trees to (39).

ii) $(I, N_0 \backslash I)$ is a *vertex partition* for T, but not a *line partition*: that is, there exists *one* vertex v_0 of T where it can be realized as a partition $(\mathscr{I}, \mathscr{S}_{v_0} \backslash \mathscr{I})$ of the lines incident to v_0, with $2 \leqq |\mathscr{I}| \leqq n_{v_0} - 2$. Then H^T is analytic on any manifold $k_I^2 = m_\mu^2$ since the function $F^{(n_{v_0})}$ associated with v_0 in (40) is one-particle irreducible in the channel $(\mathscr{I}, \mathscr{S}_{v_0} \backslash \mathscr{I})$. We call $F_2(k)$ the contribution of all such trees to (39).

iii) $(I, N_0 \backslash I)$ is associated with an *internal line* i of T. Then it is easy to check that the summation over this third family of trees yields the tree-product H^Γ introduced in (24). Indeed we must sum on both sides of i and apply the recursion formula (39) with the order $|I| + 1$ (resp. $|N_0 \backslash I| + 1$) at its two endpoints.

Finally we have:

$$F^{(n_0)} = H^{(n_0)} - F_1 - F_2 - H^\Gamma .$$

Applying (25c) then yields

$$F^{(n_0)} = F^{I, N_0 \backslash I} - F_1 - F_2 .$$

Now in view of Proposition 5, any function in the right-hand side is known to be regular on $\mathscr{M}_{I,\mu}$; which ends the proof of the recursion property.

B. In a second step we obtain a general n-point function $F_{ir}^{(n)}$ one-particle irreducible in *all* channels by amputation in all external variables:

$$F_{ir}^{(n)}(k) = \prod_{i=1}^{n} [H^{(2)}(k_i)]^{-1} F^{(n)}(k) .$$

Indeed, as seen in Section IV.1, the amputation in k_i ensures regularity on $k_i^2 = m_\mu^2$. Then $F_{ir}^{(n)}$ is analytic on any manifold $\mathscr{M}_{I,\mu}$, which is the desired result.

VII. Final Remarks

In this first step of the analytic M.P.S.A. program, we have been led to add a technical postulate (namely postulate D) to the usual axioms of the L.S.Z. field theory. Although this postulate was not necessary to construct one-particle irreducible functions, it played a crucial role in the rigorous derivation of non-linear properties of the latter; we shall therefore have to keep this postulate (or an equivalent one) in the following of the program.

Concerning the existence of sharp retarded operators, such a postulate was not crucial for the present study, since the mass-shell restrictions which occur in writing completeness relations have only to do with *local* analyticity properties in momentum space. Actually the sharp character of retarded operators is only linked with *increase properties at infinity* and is therefore irrelevant in the framework of the local analytic structure of Green's functions. So it is only for convenience that this postulate was assumed here: it allowed us to write completeness relations in their usual form, namely without regularizing factors for the fields.

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Appendix A. Essential Supports and Local Analytic Structure of Distributions

We recall very briefly some mathematical facts which are exposed in detail in [26, 28].

The essential support of a distribution $T(p)$ in $R^n_{(p)}$ at a point $\hat{p}$ is a certain closed cone $\Sigma_{\hat{p}}(T)$ with apex at the origin in $R^n_{(x)}$ (the dual space of $R^n_{(p)}$) which can be defined in terms of the exponential decrease properties of a certain "generalized Fourier transform" of T; the interest of this notion is to allow a simple and intrinsic description of the local analytic structure of T in the neighbourhood of $\hat{p}$.

Here it is sufficient to know that if $\Sigma_{\hat{p}}(T)$ is contained in the union of a finite family of open convex cones $\tilde{\mathscr{C}}_\beta$, then there exists a neighbourhood $\mathscr{U}(\hat{p})$ of $\hat{p}$ in which:

$$T(p) = \sum_\beta T_\beta(p) \tag{41}$$

with

$$T_\beta(p) = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon \in \mathscr{C}_\beta}} F_\beta(p + i\varepsilon)\,.$$

Here F_β denotes an analytic function in a local tube $\mathscr{T}_\beta = \mathscr{U}(\hat{p}) + i\mathscr{C}_\beta \cap B_\varrho$; $\mathscr{C}_\beta$ is the interior of the dual cone of $\tilde{\mathscr{C}}_\beta$, and B_ϱ is a ball with radius ϱ centered at the origin. The decomposition (41) is highly non-unique, when $\sum_\beta$ is not reduced to a single term.

In order to determine $\Sigma_{\hat{p}}(T)$ – or at least a "majorization" of it [i.e. a cone which contains $\Sigma_{\hat{p}}(T)$] – one can make use of the following facts:

Proposition A1. *For any distribution T' which coincides with T in a neighbourhood of $\hat{p}$, one has*

$$\Sigma_{\hat{p}}(T') = \Sigma_{\hat{p}}(T)\,.$$

Proposition A2. *If the Fourier transform $\tilde{T}$ of T has its support contained in a closed cone Γ (with apex at the origin, or more generally in $a+\Gamma$, a being a fixed vector), then $\Sigma_{\hat{p}}(T)\subset\Gamma$.*

The possibility of defining the local restriction of T to an arbitrary submanifold $\mathscr{M}$ containing the point $\hat{p}$ is ruled by the following

Proposition A3. *Let $\mathscr{N}_{\hat{p}}$ be the normal submanifold to $\mathscr{M}$ at $\hat{p}$ ($\mathscr{N}_{\hat{p}}$ is a certain vector subspace of R^n_x). The following properties are equivalent:*

1. $\Sigma_{\hat{p}}\cap\mathscr{N}_{\hat{p}}=\{0\}$.
2. *The above decomposition* (41) *of $T(p)$ can always be performed in such a way that, at $\hat{p}$, $\mathscr{M}$ be transverse to all cones $\mathscr{C}_\beta$; this entails that $T(p)$ admits a restriction to $\mathscr{M}$ in a neighbourhood of $\hat{p}$ (this restriction having a C^∞-dependence with respect to the variables which are transverse to $\mathscr{M}$).*

A general sufficient condition for the product of two distributions T_1, T_2 to make sense is given by:

Proposition A4. *$T_1\cdot T_2$ is well defined in a neighbourhood of $\hat{p}$ if $\Sigma_{\hat{p}}(T_1)\cap(-\Sigma_{\hat{p}}(T_2))=\emptyset$.*

This amounts to say that under this condition there always exist decompositions of the type (41):

$$T_1=\sum_{\beta_1}T_{\beta_1},\qquad T_2=\sum_{\beta_2}T_{\beta_2}\tag{42}$$

such that all couples of corresponding cones $\mathscr{C}_{\beta_1}$, $\mathscr{C}_{\beta_2}$ have non-empty intersections $\mathscr{C}_{\beta_1\beta_2}=\mathscr{C}_{\beta_1}\cap\mathscr{C}_{\beta_2}\neq\emptyset$. Then, one has:

$$T_1\cdot T_2=\sum_\beta\lim_{\substack{\varepsilon_{\beta_1\beta_2}\to 0\\ \varepsilon_{\beta_1\beta_2}\in\mathscr{C}_{\beta_1\beta_2}}}F_{\beta_1}\cdot F_{\beta_2}(p+i\varepsilon_{\beta_1\beta_2})$$

and the sum is independent of the decomposition (42).

We also need (in Section V) the following result:

Proposition A5. *If T_1, T_2 are as in Proposition A 4, and if moreover:*

i) *T_1 is represented by a square-integrable function on $\mathscr{U}(\hat{p})$.*

ii) *T_2 is represented by a bounded function on $\mathscr{U}(\hat{p})$,*

then the product of distributions $T_1\cdot T_2$ is represented by the usual product of the corresponding functions, which is itself square-integrable on $\mathscr{U}(\hat{p})$.

This study of products of distributions is treated in the second paper under reference [28].

Appendix B. Local Analytic Structure of the G.R.D.'s $F^{\mathrm{amp}}_{K\updownarrow\mathscr{S}}$ in the Neighbourhood of the Mass-Shell Manifold $\mathscr{M}_K$

Since this appendix is used both in the case of the physical $(n+k)$-point function $H^{(n+k)}(k=|K|, n=|N|)$ and in the case of the one-p.i. function $F^{K,N}$ we shall work with a general $(n+k)$-point function $F(p)$, where $p=\left[p_m;\ m\in K\cup N;\ \sum_m p_m=0\right]$, and put: $F^{\mathrm{amp}}(p)=\prod_{j\in K}(p_j^2-m^2)\,F(p)$.

The boundary values of F^{amp} in which we are interested are the $F^{\mathrm{amp}}_{K\updownarrow\mathscr{S}}$, $\mathscr{S}$ being an arbitrary cell in $S(N)$.

B.1. Support in x-Space of $F_{K\updownarrow\mathscr{S}}$

The Fourier transform of $F_{K\updownarrow\mathscr{S}}(p)$ $(p\in\mathbb{R}^{4(n+k-1)}_{(p)})$ is actually a distribution $F_{K\updownarrow\mathscr{S}}(x)$ of the variables $\underline{x}=\{\underline{x}_m=x_m-x_{m_0};\ m\in K\cup N\backslash\{m_0\}\}$ $(\underline{x}\in\mathbb{R}^{4(n+k-1)}_{(\underline{x})}\equiv\mathbb{R}^{4(n+k)}_x/\mathbb{R})$ and – as it was proved in [20] – $F_{K\updownarrow\mathscr{S}}(x)$ has the same support[6] $\Gamma_{K\updownarrow\mathscr{S}}$ as the physical $(n+k)$-point function $H_{K\updownarrow\mathscr{S}}(x)$; but $H_{K\updownarrow\mathscr{S}}(x)=r_{K\updownarrow\mathscr{S}}(x)=\langle\Omega, R_{K\updownarrow\mathscr{S}}(x)\Omega\rangle$, and it is more convenient to describe $\Gamma_{K\updownarrow\mathscr{S}}$ through its canonical inverse image in x-space $\mathbb{R}^{4(n+k)}_{(x)}$.

Now let us recall that the support $\Gamma_{\mathscr{S}}$ of every g.r.p. $R_{\mathscr{S}}(\mathscr{S}\in S(N))$ is a finite union of closed convex cones Γ_t which are all defined by n independent conditions of the type $x_i-x_j\in\bar{V}^+$ $(i,j\in N)$ [19]. It turns out that any such set of n conditions is conveniently represented by a tree-graph t whose n vertices are the points $x_i(i\in N)$ and whose branches $[x_i-x_j]$ are time-like vectors with end-points x_i, x_j associated with the conditions $x_i-x_j\in\bar{V}^+$ mentioned above. Here it is irrelevant to describe in detail the set $\{t\}_{\mathscr{S}}$ of trees t which are associated with a given cell $\mathscr{S}$, but we need to know the following fact.

Proposition B1. *For every $\mathscr{S}$ in $S(N)$, one has:*

$$\Gamma_{\{l\}\uparrow\mathscr{S}}\subset\bigcup_{i\in N}\{\{x_r;r\in N\}\in\Gamma_{\mathscr{S}};x_l-x_i\in\bar{V}^+\}$$

$$\Gamma_{\{l\}\downarrow\mathscr{S}}\subset\bigcup_{i\in N}\{\{x_r;r\in N\}\in\Gamma_{\mathscr{S}};x_l-x_i\in\bar{V}^-\}\,.$$

Let us now recall that $K\uparrow\mathscr{S}=\{l_1\}\uparrow(\dots(\{l_k\}\uparrow\mathscr{S})\dots)$ $(l_1,\dots,l_k$ being the elements of $K)$; thus by recursion over the number of elements of K, it is a corollary of Proposition B.1 that $\Gamma_{K\uparrow\mathscr{S}}$ is a union of convex cones Γ_b associated with tree-graphs b (with $n+k$ vertices) of the type described above.

Now every tree b is a "bush" over a certain tree t associated with $\mathscr{S}$; we say that a tree b is a "bush over t" if it enjoys the following properties with respect to t:

i) For every $j\in K$, there exists an element $m(j)$ in $K\cup N\backslash\{j\}$, and a branch $[j,m(j)]$ with end points x_j, $x_{m(j)}$.

ii) All branches of this type are either time-like vectors, or zero vectors (case of coinciding points, or "clusters").

In a bush b, a vertex x_j is called "*maximal*" (resp. "minimal") if no branch of b links x_j with any other vertex x_m in the strict future (resp. past) of x_j. The closed cone Γ_b which is associated with such a bush b over t is defined by:

$$\Gamma_b=\{x\colon x_j-x_{m(j)}\in\bar{V}^{b(j)},j\in K;\{x_r;r\in N\}\in\Gamma_t\}\,.$$

$b(j)$ is a sign function associated with the bush b, and which can take the values $+$, $-$, 0; $\bar{V}^0=\{0\}$.

[6] More precisely: $\Gamma_{K\updownarrow\mathscr{S}}$ is a set which contains the actual (unknown) support of $F_{K\updownarrow\mathscr{S}}$; this is a loose but convenient way of speaking.

We then have:

Proposition B2. $\Gamma_{K\uparrow\mathscr{S}} \subset \bigcup_{b\in B^+} \Gamma_b$, *where* B^+ *is the set of all bushes b over all trees t in* $\{t\}_{\mathscr{S}}$ *such that:* $b(j) = +, \forall j\in K$.

Similarly one obtains the supports of the g.r.d.'s $r_{\{l\}\downarrow(K\setminus\{l\})\uparrow\mathscr{S}}$:

Proposition B3. $\forall l \in K: \Gamma_{\{l\}\downarrow(K\setminus\{l\})\uparrow\mathscr{S}} \subset \bigcup_{b\in B_l} \Gamma_b$, *where* B_l *is the set of all bushes b over all trees t in* $\{t\}_{\mathscr{S}}$ *such that:* $b(l) = -$, *and* $b(j) = +, \forall j \neq l, j\in K$.

B.2. A Majorization of $\Sigma_{\hat{p}}(F^{\mathrm{amp}}_{K\uparrow\mathscr{S}})$ in the Neighbourhood of the Mass-shell Manifold of K

Let us put $\mathscr{M}_K = \left\{p: \sum_{m\in K\cup N} p_m = 0;\, p_j \in H^+_m, \forall j\in K\right\}$ and choose $\hat{p}\in\mathscr{M}_K$. For every cell $\mathscr{S}$ in $S(N)$, we have, in view of the spectral conditions (see Sections II.2 and IV.1):

$$\forall l\in K: F^{\mathrm{amp}}_{\{l\}\downarrow(K\setminus\{l\})\uparrow\mathscr{S}} = F^{\mathrm{amp}}_{K\uparrow\mathscr{S}},$$

in a neighbourhood of $\hat{p}$.

Therefore, Proposition A.2 and A.1 yield:

$$\Sigma_{\hat{p}}(F^{\mathrm{amp}}_{K\uparrow\mathscr{S}}) \subset \Gamma_{K\uparrow\mathscr{S}} \cap \left(\bigcap_l \Gamma_{\{l\}\downarrow(K\setminus\{l\})\uparrow\mathscr{S}}\right).$$

But in view of Propositions B.2 and B.3, we deduce:

$$\Sigma_{\hat{p}}(F^{\mathrm{amp}}_{K\uparrow\mathscr{S}}) \subset \left[\bigcap_{l\in K}\left(\bigcup_{b_l\in B_l}\Gamma_{b_l}\right)\right] \cap \left(\bigcup_{b\in B^+}\Gamma_b\right). \tag{43}$$

Now in any convex component $\Gamma_{b_{l_1}} \cap\cdots\cap \Gamma_{b_{l_\nu}} \cap \Gamma_b$ of the cone at the right-hand side of (43), we can see that *all* points $x_j (j\in K)$ which are maximal (namely simultaneously maximal in $b, b_{l_1}, \ldots, b_{l_\nu}$) must certainly form clusters with other points (at least one) either in K or in N; this is because such a maximal point x_l must satisfy together with the other x_m's the conditions which express that $\{x_m\}\in\Gamma_{b_l}$, and one of this condition is $x_l - x_{m(l)} \in \bar{V}^-$, which implies $x_l = x_{m(l)}$ [for a certain $m(l) \neq l$].

Let us now call a "budded bush" β any bush for which $\beta(j) =$ either $+$ or 0, and whose *all* maximal points x_l with $l\in K$ are clusters, and let Γ_β be the corresponding cone. Then from the above analysis of the right hand-side of (43) we obtain:

Proposition B4.

$$\Sigma_{\hat{p}}(F^{\mathrm{amp}}_{K\uparrow\mathscr{S}}) \subset \bigcup_{\beta\in B_0}\Gamma_\beta,$$

where B_0 *is the set of all budded bushes over all trees t in* $\{t\}_{\mathscr{S}}$.

B.3. Restriction of $F^{\mathrm{amp}}_{K\uparrow\mathscr{S}}$ to $\mathscr{M}_K$ in the Neighbourhood of a "Non-Overlapping" Configuration $\hat{p}$

We shall apply the result of Proposition A.3 to the study of the restriction of $F^{\mathrm{amp}}_{K\uparrow\mathscr{S}}(p)$ to the manifold $\mathscr{M}_K$ in the neighbourhood of a point $\hat{p}$ such that

$\{\hat{p}_j, j \in K\} \in \Omega'$. The normal manifold[7] $\mathcal{N}_{\hat{p}}$ to $\mathcal{M}_K$ at $\hat{p}$ is described conveniently in x-space as follows:

$$\mathcal{N}_{\hat{p}} = \left\{ \underline{x}_m = x_m - X, m \in K \cup N;\ X = \frac{\sum_m x_m}{(n+k)};\ \underline{x}_j = \lambda_j \hat{p}_j, \forall j \in K;\ \lambda_j \in \mathbb{R};\ \underline{x}_i = 0, i \in N \right\}.$$

Then it is easy to check that all the intersections of $\mathcal{N}_{\hat{p}}$ with the cones Γ_β of Proposition B.4 are reduced to $\{0\}$.

In fact, since $\{\hat{p}_j, j \in K\} \in \Omega'$, all the trajectories $\underline{x}_j = \lambda_j \hat{p}_j$ are distinct; now if $x \in \Gamma_\beta \cap \mathcal{N}_{\hat{p}}$, it must satisfy $\underline{x}_j \in \bar{V}^+$, $\forall j \in K$ (since $\beta \in B^+$); but if such a point $\{\underline{x}\}$ were $\neq 0$, then one four-vector $\underline{x}_j$ at least would be maximal and single, which would contradict the fact that $x \in \Gamma_\beta$ (β being a budded bush). So we have proved:

Proposition B5. *At every point $\hat{p}$ such that $\{\hat{p}_j, j \in K\} \in \Omega'$ we have: $\Sigma_p(F^{\mathrm{amp}}_{K\uparrow\mathcal{S}}) \cap \mathcal{N}_{\hat{p}} = \{0\}$; equivalently the mass shell restriction $\hat{F}_{K\uparrow\mathcal{S}}$ of $F^{\mathrm{amp}}_{K\uparrow\mathcal{S}}$ is well-defined on the open set $\{p \in \mathcal{M}_K; \{p_j, j \in K\} \in \Omega'\}$.*

A similar result holds for $\Sigma_{\hat{p}}(F^{\mathrm{amp}}_{K\downarrow\mathcal{S}})$.

Special Case: $N = \{\lambda\}$ $(n = 1)$.

In this case $F_{K\uparrow\lambda}(x)$ is an ordinary retarded function, and $\Gamma_{K\uparrow\lambda} = \{x : \underline{x}_j \in V^+, \forall j \in K\}$. In p-space, $\Gamma_{K\uparrow\lambda} = \{p_j : p_j \in V^+\}$ is contained in the open dual cone $\hat{\Gamma}_\beta$ of Γ_β, for all β's. In fact an arbitrary $\hat{\Gamma}_\beta$ is defined by a set of conditions $\{p : p_{J_1} \in V^+, \dots p_{J_\nu} \in V^+;\ |J_1| > 1, \dots |J_\nu| > 1,\ (J_1 \dots J_\nu) = \text{partition of } K\}$ (here $J_1, \dots, J_\nu$ correspond to the clusters of the budded bush β); we then note that $\forall \beta, \Gamma_\beta \subset \mathscr{C}^+_K = \{p : p_K \in V^+\}$.

We thus have:

Proposition B6. *In the neighbourhood of every point $\hat{p}$ in Ω', the retarded distribution $F_{K\uparrow\lambda}(p_j, j \in K)$ can be decomposed as a sum of boundary values of functions F_β, which are analytic in local tubes T_β with the following properties:*

i) each T_β has a non-empty intersection $\hat{T}_\beta$ with the complexified manifold $(H^+_m)^k_c$ ($\hat{T}_\beta$ being a local tube in $(H^+_m)^k_c$).

ii) $\bigcap_\beta T_\beta$ contains the intersection of the retarded tube $\mathcal{T}_{K\uparrow\lambda}$ (with imaginary basis $\Gamma_{K\uparrow\lambda}$) with a suitable complex neighbourhood of $\hat{p}$.

iii) each T_β is contained in the tube $\mathcal{T}^+_K$, with imaginary basis $\mathscr{C}^+_K$.

The same results are valid for the advanced distribution $F_{K\downarrow\lambda}$, and in both cases ($K\uparrow$ and $K\downarrow$), the point $\hat{p}$ can also be chosen in the subset $(-\Omega')$ of $(H^-_m)^k$.

References

1. Lassalle, M.: Commun. math. Phys. **36**, 185 (1974)
2. Bros, J.: Analytic methods in mathematical physics. New York: Gordon and Breach 1970
3. Symanzik, K.: J. Math. Phys. **1**, 249 (1960) and in Symposium on theoretical physics, Vol. **3**. New York: Plenum Press 1967
4. Zimmermann, W.: Nuovo Cimento **13**, 503 (1959); **16**, 690 (1960)
5. Lehmann, H., Symanzik, K., Zimmermann, W.: Nuovo Cimento **1**, 205 (1955); **6**, 319 (1957)
6. Hepp, K.: Commun. math. Phys. **1**, 95 (1965)
7. Steinmann, O.: Commun. math. Phys. **10**, 245 (1968)

[7] With respect to the Minkowski metric, by convenience.

8. Glaser, V., Lehmann, H., Zimmermann, W.: Nuovo Cimento **6**, 1122 (1957)
9. Völkel, A. H.: Commun. math. Phys. **2**, 176 (1966)
10. Streater, R. F., Wightman, A. S.: PCT, spin-statistics and all that. New York: Benjamin 1964
11. Jost, R.: The general theory of quantized fields. Am. Math. Soc. Providence 1965
12. Castillejo, L., Dalitz, R., Dyson, F. J.: Phys. Rev. **101**, 453 (1966)
13. Haag, R.: Phys. Rev. **112**, 669 (1958)
14. Ruelle, D.: Helv. Phys. Acta **35**, 147 (1962)
15. Martin, A.: In: Problems of theoretical physics, p. 113. Moscow: Bauka 1969
16. Bros, J., Epstein, H., Glaser, V., Stora, R.: To appear
17. Epstein, H., Glaser, V., Stora, R.: Cern preprint TH. 1719 (1973)
18. Steinmann, O.: Helv. Phys. Acta **33**, 257, 347 (1960)
19. Bros, J.: Thesis (Paris 1970)
20. Lassalle, M.: Thesis (Orsay 1975)
21. Ruelle, D.: Nuovo Cimento **19**, 356 (1961)
    Araki, H.: J. Math. Phys. **2**, 163 (1961)
22. Pohlmeyer, K.: Report Desy 66/36 (unpublished), Hamburg (1966)
23. Bros, J., Lassalle, M.: III. Two-particle irreducibility properties (to appear)
24. Titchmarsh, E. C.: Theory of Fourier integrals, p. 145. Oxford: Clarendon Press 1959
25. Courant, R., Hilbert, D.: Methods of mathematical physics, Vol. II, p. 248. New York: Interscience Publ. 1962
26. Bros, J., Epstein, H., Glaser, V.: Helv. Phys. Acta **45**, 149 (1972); see also Bros, J., Iagolnitzer, D. In: Proceedings of the International Moscow 1972 Conference on Quantum Field Theory and Quantum Statistics
27. Schwartz, L.: Théorie des distributions. Paris: Hermann 1959
28. Bros, J., Iagolnitzer, D.: Ann. Inst. Henri Poincaré, vol. XVIII n^0 2, 147 (1973) and Essential supports and local analytic structure of distributions (to appear)
29. Williams, D. N.: Lectures in Theoretical Physics, Vol. **10**. B, 357 (1967)

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