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A Family of Codes between Some Markov and Bernoulli Schemes

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Abstract. We construct a family of almost continuous codes between a mixing one-step Markov process with two symbols and a Bernoulli scheme.

1. Introduction

It is known (see Ref. [2]) that two "finitely determined" processes with the same entropy are isomorphic. In particular this statement implies that a mixing Markov chain is isomorphic to a Bernoulli scheme with the same entropy ([1, 2]). The isomorphism means, in the language of codings, that one can find invertible and shift-invariant maps between the typical sequences of the two processes. Such "codes", nevertheless, are explicitely known only in a few cases (see [4]); moreover, it is not known whether, endowing the space of sequences with the natural topology defined below, it is possible to construct almost continuous codes.

Here we construct explicitely an uncountable family of almost continuous invertible codes between a mixing one-step Markov process with two symbols and a Bernoulli scheme.

2. Symbols and Definitions

Let $X=\{-1,1\}^{\mathbb{Z}}$ be the set of all doubly infinite sequences of the two numbers $-1, 1$, $\mathscr{B}$ the σ-algebra of the Borel sets in X (endowed with the topology obtained as product of the discrete topologies on the factors); let $T: X\to X$ be the map defined by:

$$(Tx)_i = x_{i+1}\,. \tag{2.1}$$

If μ_S is a T-invariant measure defined on $\mathscr{B}$, we call the triple $S=(X, T, \mu_S)$ a "shift". If, moreover, $\varphi: X\to X$ is a Borel map commuting with T, let us define:

$$\bar{\varphi} S=(X, T, \bar{\varphi}\mu_S)\,,\quad \text{with}\quad \bar{\varphi}\mu_S(Y)=\mu_S(\varphi^{-1}Y)\qquad \forall Y\in\mathscr{B}\,.$$

Let $0<\alpha<1$; call B_α the (Bernoulli) shift whose measure μ_{B_α}, extended to $\mathscr{B}$ by Kolmogorov's theorem, is defined on the cylinders of X by:

$$\mu_{B_\alpha}(\{x\in X\,|\,x_{i+1}=k_1\dots x_{i+l}=k_l\}) = \prod_{\substack{1\leqq r\leqq l\\ k_r=1}} \alpha \prod_{\substack{1\leqq r\leqq l\\ k_r=-1}} (1-\alpha)\,. \tag{2.2}$$

In a similar way we can define the one-step Markov shift M_α by letting:

$$\mu_{M_\alpha}(\{x \in X \mid x_{i+1} = k_1 \ldots x_{i+l} = k_l\}) = \tfrac{1}{2} \prod_{\substack{1 < r \leqq l \\ k_r = k_{r-1}}} \alpha \prod_{\substack{1 < r \leqq l \\ k_r \neq k_{r-1}}} (1-\alpha). \tag{2.3}$$

It is easy to check that B_α and M_α have the same entropy, and thus they are (Refs. [1, 3]) isomorphic. This implies the existence of an invertible measurable map $\psi: X \to X$ commuting with T defined μ_{M_α} – a.e., such that:

$$\bar{\psi} M_\alpha = B_\alpha. \tag{2.4}$$

Using the definitions (2.2) and (2.3) one can immediately check that the map φ defined by letting:

$$(\varphi x)_i = x_{i-1} \cdot x_i \tag{2.5}$$

satisfies (2.4); on the other hand the map φ is clearly not-invertible. It is known, however, that, if we introduce in the set of the measurable mappings commuting with T the metric:

$$\varrho(\varphi, \psi) = \mu_{M_\alpha}(\{x \in X \mid (\varphi x)_0 \neq (\psi x)_0\}). \tag{2.6}$$

φ is an accumulation point of invertible mappings verifying (2.4) (see Ref. [2], Part I, Proposition 11 of Section 4).

In the next section, in effect, we construct an uncountable family of such mappings by modifying slightly φ.

3. The Code

Let $M = \{M_i\}_{i=1,2\ldots}$ be a strictly increasing sequence of positive integers, with $M_1 \geqq 3$, and let $\xi \in X$. We call "n-string" of ξ each k-ple of consecutive indices of ξ, $[i+1, \ldots, i+k]$ such that:

$$\xi_{i+1} = \cdots \xi_{i+k} = 1\,; \quad \xi_i = \xi_{i+k+1} = -1\,; \quad M_n \leqq k < M_{n+1}\,.$$

Then we call the set $[i+1, \ldots, i+k]$ an "n-block" of ξ if it is contained between two consecutive n-strings (i.e. i is the endpoint of an n-string and $i+k+1$ is the starting point of the following one) and there are no strings of higher order contained in it.

Moreover, we call "changeable" an element ξ_i of ξ if by changing its sign the positions and the lenghts of blocks and strings remain unaltered.

Let us consider, now, an n-block $B = [i+1, \ldots, i+k]$ of ξ and let $l \leqq k/2$, be the least positive integer (if it exists) such that:

1) $\xi_{i+1+l} \neq \xi_{i+k-l}$.

2) In the segments $[i+1, \ldots, i+l]$ and $[i+k-l, \ldots, i+k]$ there are not strings.

3) ξ_{i+1+l} and ξ_{i+k-l} are "changeable".

If it is possible to find such an l, we call "conjugate elements associated to B" the pair $\{\xi_{i+1+l}, \xi_{i+k-l}\}$. It is clear that an element ξ_i can belong to at most one such pair.

It is not difficult to check that, μ_{B_α} – a.e., each element of a sequence ξ belongs to infinitely many blocks and is between infinitely many pairs of conjugate elements.

We can now define, μ_{M_α} – a.e., the map $\psi_M: X \to X$ in the following way:

If $(\varphi x)_i$ does not belong to a pair of conjugate elements of the sequence φx, we put:

$$(\psi_M x)_i = (\varphi x)_i \,. \tag{3.1}$$

If, instead, $(\varphi x)_i$ is one of the two conjugate elements associated to a block B of φx, we put:

$$(\psi_M x)_i = (\varphi x)_i \cdot x_{n_B} \tag{3.2}$$

where n_B is the first element of the block B. In other words, the sequence $\psi_M x$ is obtained from the sequence φx by changing only the conjugate elements associated to the blocks (of φx) that "in the sequence x begin with -1" (note that it is surely $(\varphi x)_{n_B} = -1$).

Remark 1. Note that the sequences φx and $\psi_M x$ have blocks, strings and conjugate elements in the same positions.

Remark 2. If we denote $-x$ the sequence defined by $(-x)_i = -x_i$ and we call $\sigma_M \xi$ the sequence obtained from ξ by changing the sign of all the conjugate elements of ξ, we have:

$$\varphi(-x) = \varphi(x)\,; \qquad \psi_M(-x) = \sigma_M \psi_M(x)\,. \tag{3.3}$$

4. Main Results

We collect in the following proposition some statements which easily follow from the construction of ψ_M:

Proposition 1. *For every increasing sequence M of positive integers with $M_1 \geqq 2$ the map ψ_M commutes with T, is measurable with respect to $\mathscr{B}$ and is μ_{M_α}-almost continuous. If M and M' are two different sequences:*

$$\mu_{M_\alpha}(\{x \in X \,|\, (\psi_M x)_0 \neq (\psi_{M'} x)_0\}) > 0\,.$$

Hence by varying M we obtain an uncountable family of essentially different mappings.

Theorem 1. *For every M, ψ_M is μ_{M_α} – a.e. invertible and its inverse is almost continuous.*

Proof. We shall show that for μ_{B_α}-almost every $\xi \in X$, there is a unique $x \in X$ such that:

$$\psi_M x = \xi\,. \tag{4.1}$$

Let $\xi \in X$; for every index k, let $B(k)$ be the least-order block of ξ such that:

1) k belongs to the block $B(k)$.

2) There are, associated to $B(k)$, two conjugate elements, i, j, such that $i \leqq k < j$.

It is clear that for a set of μ_{B_α}-measure 1 such a block exists. Let $\alpha(k)$ be the first element of $B(k)$ [note that 2) implies that k is at least the second element of $B(k)$].

We observe that, for every sequence x:

$$x_k = x_{\alpha(k)} \prod_{l=\alpha(k)+1}^{k} (\varphi x)_l . \tag{4.2}$$

Suppose now that x satisfies (4.1). If l is not a conjugate element of ξ by Remark 1 l cannot be a conjugate element of φx and we have $(\varphi x)_l = \xi_l$; if r, s are two conjugate elements of ξ we get

$$(\varphi x)_r (\varphi x)_s = \xi_r \xi_s .$$

Moreover the only conjugate element which occurs alone in the segment $[\alpha(k), k]$ is i. Hence:

$$\prod_{l=\alpha(k)+1}^{k} (\varphi x)_l = \prod_{\substack{\alpha(k)+1 \leqq l \leqq k \\ l \neq i}} \xi_l (\varphi x)_i .$$

On the other hand, the construction of ψ_M implies that:

$$x_{\alpha(k)} (\varphi x)_i = \xi_i .$$

Using the last two equalities and (4.2) we have:

$$x_k = \prod_{l=\alpha(k)+1}^{k} \xi_l . \tag{4.3}$$

We have proved that the only sequence which can satisfy (4.1) is the one defined in (4.3). To verify that this sequence actually satisfies (4.1) note that

$$(\varphi x)_k = -\xi_k$$

if and only if:

$$B(k-1) \neq B(k); \prod_{l=\alpha(k-1)+1}^{k-1} \xi_l = - \prod_{l=\alpha(k)+1}^{k-1} \xi_l . \tag{4.4}$$

If we put:

$$\alpha = \min[\alpha(k-1), \alpha(k)],$$
$$\beta = \mathrm{Max}[\alpha(k-1), \alpha(k)]$$

(4.4) means:

$$B(k) \neq B(k-1); \prod_{k=\alpha+1}^{\beta} \xi_k = -1 .$$

This occurs if and only if k is a conjugate element of ξ associated to a block starting in a point β such that $x_\beta = -1$. Furthermore the conjugate elements of ξ and $\varphi(x)$ are the same.

Hence the sequence defined in (4.3) satisfies (4.1).

The last part of the theorem easily follows from the construction of ψ_M^{-1}.

Theorem 2. *For every* M, $\bar{\psi}_M M_\alpha = B_\alpha$.

Proof. We take into account the finite sequences of the two numbers $-1, 1$ of the type $\xi_i f_1 B f_2 \xi_l$, where B is an n-block, f_1 and f_2 are n-strings and ξ_i, ξ_l are two negative elements. Let F be such a sequence and C_F the corresponding cylinder:

$$C_F = \{\xi \in X \mid \xi_1 \dots \xi_r = \xi_i f_1 B f_2 \xi_l\} .$$

One can easily realize that

$$\psi_M^{-1}(C_F) = C_G \cup C_{\tilde{G}} ,$$

where G, $\tilde{G}$ are finite sequences such that

$$\varphi(C_{\tilde{G}}) = \sigma_M \varphi(C_G); (\tilde{G})_i = -G_i$$

$$\mu_{B_\alpha}(C_F) = \mu_{B_\alpha}(C_{\varphi G}) .$$

We have

$$\mu_{M_\alpha}(C_G) = \tfrac{1}{2}\mu_{B_\alpha}(C_{\varphi G})$$

$$\begin{aligned}\mu_{M_\alpha}(\psi_M^{-1} C_F) &= \mu_{M_\alpha}(C_G) + \mu_{M_\alpha}(C_{\tilde{G}}) \\ &= \mu_{B_\alpha}(C_{\varphi G}) = \mu_{B_\alpha}(C_F) .\end{aligned}$$

Since the cylinders of the type C_F, with their translates, generate the σ-algebra $\mathscr{B}$, we have

$$\mu_{M_\alpha}(\psi_M^{-1} Y) = \mu_{B_\alpha}(Y) \qquad \forall Y \in \mathscr{B} . \tag{4.5}$$

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