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The Structure Theorem in S-Matrix Theory

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Abstract. A basic tool in the derivation of multiparticle discontinuity formulae in S-matrix theory is a "structure theorem" which proves analyticity properties for integrals of products of scattering functions [1, 5, 7].

We present here some recent mathematical results and show how they provide directly a general form of this theorem. This new proof, which removes an unnecessary technical assumption of the previous ones, is a development of a method proposed by Pham [8].

I. Introduction

The basic quantities of interest in the relativistic quantum physics of systems of massive particles with short-range interactions are the scattering functionals S_{IJ} between sets I and J of initial and final particles. From general quantum principles, each S_{IJ}, or its "connected part" S^c_{IJ}, is known [1, 2] to be a tempered distribution, which is defined on the space of all real on-mass-shell initial and final energy-momentum 4-vectors $p_k(p_k^2 = p_{k0}^2 - \boldsymbol{p}_k^2 = m_k^2,\ (p_k)_0 > 0)$ and contains an energy-momentum conservation δ-function:

$$S^c_{IJ} = T_{IJ} \times \delta^{(4)}\Big(\sum_{i\in I} p_i - \sum_{j\in J} p_j\Big)\,. \tag{1}$$

The distribution T_{IJ} is defined on the physical-region $\mathscr{M}_{IJ}$ of the process $I \to J$ (i.e. the set of all real 4-momenta p_k satisfying the above mentioned mass-shell constraints and the further condition $\Sigma p_i = \Sigma p_j$).

Decisive advances have been made at the end of the sixties in the general derivation and understanding of the physical-region analytic structure of the distributions T_{IJ}. On the one hand, a macroscopic causality property has been stated and proved to be equivalent to some basic analytic properties of T_{IJ} [3, 4]. These properties ensure in particular that for each process $I \to J$, there is a unique analytic function F_{IJ} (defined in a domain of the complexified mass-shell $\mathscr{M}^c_{IJ}$) to which T_{IJ} is equal at all points which do not lie on $+\alpha$-Landau surfaces of connected graphs, and from which it is a "plus $i\varepsilon$" boundary value at almost all $+\alpha$-Landau points.

On the other hand, a general form has been derived from unitarity for the discontinuities of the scattering functions around the $+\alpha$-Landau singularities [5, 6]. The usefulness of this result in various contexts is described elsewhere (see for instance [1, 2] and the original references quoted therein). Its derivation makes use of various relations derived from unitarity among integrals (over internal on mass shell 4-momenta) of products of scattering functions S^c_{IJ} or $(S^c_{IJ})^-$ $(=S^{c*}_{JI})$.

The present paper is concerned with a basic preliminary result in this study, which provides analyticity properties of these integrals, called "bubble diagram functions", from the analyticity properties of the "bubbles" T_{IJ} or T^-_{IJ} involved. This type of result has first been proved in Ref. [1] (see also [5]) and is called structure theorem. It has also been used more recently for the derivation of generalized optical theorems [7]. The proof of [1] is obtained by considering appropriate distortions of the integration domain into complex regions where all functions F or F^-, associated with the scattering functions T or T^- involved, are analytic. Complications arise because T (or T^-) is *not* everywhere equal to, or a boundary value of, F (or F^-). The points where this is no longer true are:

i) Among those that lie in the intersection of several $+\alpha$-Landau surfaces with no common "parent". In the neighborhood of a point $p=\{p_k\}$ of this type, T is in general a *sum* of appropriate boundary values of analytic functions.

ii) The points $p=\{p_k\}$ such that two initial, or two final, 4-momenta are colinear. This set of points is called $\mathscr{M}_0$.

Although these points are exceptional, i.e. lie in lower-dimensional submanifolds, they can indeed occur in the integration domains. A regularity property is assumed at Points (ii) in [1, 7]. We shall not discuss it here. For Points (i) an important technical assumption, called "patching property" was used in the references quoted above. More recent mathematical results allow one to show that this property can, as a matter of fact, be *proved* (see [4]). However, although this fact has its own interest, our main purpose is not to study it in more detail, but rather to present an alternative derivation, which uses general and simple results on products and integrals of distributions, and directly provides a general form of the structure theorem, from the very statement of the macrocausality property. This method is a development of an idea presented in [8].

The basic mathematical tools [9–11] (whose usefulness goes beyond the problem of the structure theorem in S-matrix theory) are described in Part I. The notion of essential support of a distribution is presented in Subsection a), at the end of which the "decomposition theorems" (which express a distribution f with given essential support, as a corresponding boundary value, or sum of boundary values, of analytic functions) are recalled. We then present in Subsection b) two theorems on products and integrals of distributions. These theorems were first proved in [10], in a very different mathematical framework. New, simple and direct proofs which are well adapted to the physical context of Part II are presented here in Subsection b), and with more details in [11].

For physical reasons presented in [2–4], macrocausality is in fact an exponential fall-off property of transition amplitudes in appropriate situations, which in view of the very definitions of Part I a), is a condition on the essential support of S^c_{IJ} (or T_{IJ}). The methods and results of Part I b) are then used in Part II to obtain directly corresponding fall-off or essential support properties for the "bubble diagram functions". This is the general form of the structure theorem and the analyticity properties of these functions then follow (as those of the scattering functions) from the decomposition theorems recalled in Part I a).

II. Mathematical Results [9–11]

a) Essential Support of a Distribution

Let $R^n_{(p)}$ be the n-dimensional real vector space of a variable $p=(p_1 \dots p_n)$ and let $R^n_{(v)}$ be its dual space ($v = v_1 \dots v_n$).

The generalized Fourier transform at a point P of a distribution f (defined in $R^n_{(p)}$), is defined on the $n+1$ real dimensional space of the variables v and of a supplementary real variable t by the formula:

$$F(v, t; P) = \int f(p)\, e^{-iv\cdot p - t(p-P)^2}\, dp \tag{2}$$

where

$$v \cdot p = \sum_{i=1}^{n} v_i p_i \quad \text{and} \quad p^2 = \sum_{i=1}^{n} p_i^{\,2}\,.$$

We assume for simplicity that f has a compact support; F is therefore a well defined function for all values of v and t, whose value at $t=0$ is the usual Fourier transform of f.

A direction $\hat{v}_0$ in $R^n_{(v)}$ is said to be *outside the essential support* $\hat{S}_P(f)$ *of* f *at* P if there exist an open cone $\mathscr{V}$ in $R^n_{(v)}$ (with apex at the origin) containing the direction $\hat{v}_0$, a polynomial $\mathscr{P}$ and constants $\alpha > 0$, $\gamma_0 > 0$ such that:

$$|F(v, \gamma |v|; P)| < \mathscr{P}(|v|)\, e^{-\alpha\gamma|v|} \tag{3}$$

for all v in $\mathscr{V}$ and all positive γ less than $\gamma_0 (0 < \gamma < \gamma_0)$.

In other words, $\hat{S}_P(f)$ is the set of directions in v-space along which the generalized Fourier transform of f at P does *not* decrease exponentially in the sense of Eq. (3).

$\hat{S}_P(f)$ is a closed subset of the unit sphere $S^{n-1}_{(v)}$ (with center at the origin) in $R^n_{(v)}$ if each direction $\hat{v}$ is represented by a point of this sphere.

The *essential support* $\hat{S}_{\mathscr{D}}(f)$ of f over a domain $\mathscr{D}$ of $R^n_{(p)}$ is the subset of $\mathscr{D} \times S^{n-1}_{(v)}$ defined as $\bigcup_{P\in\mathscr{D}} (P, \hat{S}_P(f))$.

We shall denote by $S_P(f)$, resp. $S_{\mathscr{D}}(f)$, the cone in $R^n_{(v)}$ with apex at the origin whose basis on $S^{n-1}_{(v)}$ is $\hat{S}_P(f)$, resp. the union $\bigcup_{P\in\mathscr{D}} (P, S_P(f))$. For convenience, we shall also refer to $S_P(f)$, resp. $S_{\mathscr{D}}(f)$, as the essential support of f at P, resp. over $\mathscr{D}$.

Remark 1. $S_P(f) = S_P(\chi f)$ whenever χ is a C^∞ function, locally analytic and different from zero at P.

Remark 2. If $\hat{v}_0$ is outside $\hat{S}_P(f)$, there moreover always exist a neighborhood $\mathscr{N}$ of P in $R^n_{(p)}$, together with an open cone $\mathscr{V}'$ containing $\hat{v}_0$, $\alpha' > 0$, $\gamma'_0 > 0$ and $\mathscr{P}'$ such that the following bounds of Type (3) be satisfied uniformly for all Points P' in $\mathscr{N}$.

$$|F(v, \gamma |v|; P')| < \mathscr{P}'(|v|)\, e^{-\alpha'\gamma|v|} \tag{3'}$$

in the region $v \in \mathscr{V}'$, $0 < \gamma < \gamma'_0$.

As a consequence, $\hat{S}_{\mathscr{D}}(f)$ is a *closed* subset of $\mathscr{D} \times S^{n-1}_{(v)}$.

Remark 3. Being given any closed set Σ of $S^{n-1}_{(v)}$ whose intersection with $\hat{S}_P(f)$ is empty, there always exists a neighborhood $\mathscr{N}$ of P (in $R^n_{(p)}$) such that the generalized Fourier transform F_χ of χf at P satisfies bounds of the Form (3) for all

$\hat{v}$ in Σ, *with* $\mathscr{P}$ *replaced by a rapid decrease factor whenever* χ *has its support in* $\mathscr{N}$. Namely, there exist $\mathscr{V}$, $\alpha > 0$, $\gamma_0 > 0$ and $C_N < \infty$ for all positive integers N such that:

$$|F_\chi(v, \gamma|v|; P) < -\frac{C_N}{1+|v|^N} e^{-\alpha\gamma|v|} \tag{4}$$

for all v in $\mathscr{V}$ and $0 \leqq \gamma < \gamma_0$.

Distributions Defined on Manifolds

Consider now a distribution f defined on a real analytic manifold $\mathscr{M}$. Being given a Point P of $\mathscr{M}$, one may consider a system of real analytic local coordinates of $\mathscr{M}$ at P, and define $\hat{S}_P(f)$ as above in this system. [One may for instance consider C^∞ functions χ, locally analytic and different from zero at P, with a sufficiently small support with respect to the coordinate system considered; $\hat{S}_P(\chi f)$ is independent of χ in view of Remark 1, and defines $\hat{S}_P(f)$.]

It is then possible to show that $\hat{S}_P(f)$ is a well defined subset of directions of the cotangent vector space $T_P^* \mathscr{M}$ at P to $\mathscr{M}$, *independent of the local coordinate system.*

The essential support $\hat{S}_{\mathscr{D}}(f)$ of f over a domain $\mathscr{D}$ of $\mathscr{M}$ is as above defined as $\bigcup_{P \in \mathscr{D}} (P, \hat{S}_P(f))$ and is now a well defined closed subset of the sphere cotangent bundle $\bigcup_{P \in \mathscr{D}} (P, S_P^* \mathscr{M})$ (where $S_P^* \mathscr{M}$ is the unit sphere in $T_P^* \mathscr{M}$).

We shall only consider in Part II submanifolds $\mathscr{M}$ of $R^n_{(p)}$ of dimension $n-l$, defined by a set of l equations $L_j(p) = 0$ $(j = 1, \ldots, l)$, where each L_j is a real analytic function of p.

We below denote by $\delta(\mathscr{M})$ the product $\prod_{j=1}^{l} \delta(L_j(p))$, which is a well defined distribution on $R^n_{(p)}$. Being given a distribution f defined on $\mathscr{M}$, the product $f \times \delta(\mathscr{M})$ is also clearly a well defined distribution on $R^n_{(p)}$.

Let $N(P)$ be the l-dimensional real subspace of $R^n_{(v)}$ conormal at P to $\mathscr{M}$ $\left(\text{i.e. } N(P) \text{ is the set of vectors of the form } \sum_{j=1}^{l} \lambda_j \nabla L_j(P)\right)$.

The cotangent vector space $T_P^* \mathscr{M}$ at P to $\mathscr{M}$ can be identified with the quotient subspace $R^n_{(v)}/N(P)$, and the following result holds:

Lemma 1. i) $S_P(\delta(\mathscr{M}))$ *is the set* $N(P)$.
ii) $S_P(f \times \delta(\mathscr{M}))$ *is invariant by addition of vectors in* $N(P)$.
iii) $S_P(f) = S_P(f \times \delta(\mathscr{M}))/N(P)$.

Decomposition Theorems

We recall the following results.

Lemma 2. f *is analytic at* P *if and only if* $\hat{S}_P(f)$ *is empty (i.e. the generalized Fourier transform of* f *decreases exponentially in all directions in the sense of* (3)).

It is well known that the *local analyticity* of f at P cannot be expressed in terms of an exponential decrease of the usual Fourier transform. Lemma 2 tells us that it can be characterized in terms of the above exponential decrease of the generalized Fourier transform.

We next state:

Theorem 1 (decomposition theorem at P). *The following properties of a distribution f defined on $R^n_{(p)}$ are equivalent:*

i) *$S_P(f)$ is contained in a (finite) union of closed convex salient cones C_j (with apex at the origin).*

ii) *Being given any family $\{C'_j\}$ of (closed convex salient) cones C'_j with apex at the origin, such that for each j, $\hat{C}'_j$ contains $\hat{C}_j$ in its interior*[1]*, there always exist a (real) neighborhood $\mathcal{N}$ of P and distributions f_j, each of which is the boundary value in $\mathcal{N}$ of an analytic function F_j from the directions $q = \operatorname{Im} p$ of the open dual cone Γ'_j of C'_j, such that:*

$$f = \sum_j f_j \quad \text{in} \quad \mathcal{N}.$$

This theorem also holds for distributions f defined on a real analytic manifold $\mathcal{M}$; F_j is then an analytic function defined in a (non specified) domain of the complexified manifold $\mathcal{M}^c$, and Property ii) still makes sense for instance in any local coordinate system.

The following theorems, which generalize Theorem 1 when P varies in a domain $\mathcal{D}$ of $R^n_{(p)}$, or of $\mathcal{M}$, are also proved:

Theorem 2. *The two following properties are equivalent:*

i) *$S_{\mathcal{D}}(f)$ is contained in a closed subset $\Sigma = \bigcup_{P \in \mathcal{D}} (P, \Sigma_P)$ of $T^*\mathcal{D}$, whose each fiber Σ_P is a closed convex salient cone.*

ii) *f is, in $\mathcal{D}$, the boundary value of a unique analytic function F; at each Point P of $\mathcal{D}$, this boundary value is obtained from the directions of the open dual cone $\tilde{\Sigma}_P$ of Σ_P*[2].

Theorem 3. *Let $S_{\mathcal{D}}(f)$ be contained in a (finite) union of closed subsets Σ_j of $T^*\mathcal{D}$ such that, for each j, the fibers $(\Sigma'_j)_P$ of Σ_j be as above closed convex salient cones for all values of P in $\mathcal{D}$.*

Then there exist distributions f_j, such that $S_{\mathcal{D}}(f_j)$ is contained in Σ_j for each j, and $f = \sum_j f_j$ in $\mathcal{D}$.

According to Theorem 2, each f_j is the corresponding boundary value of an analytic function F_j.

b) Products and Integrals of Distributions

In this subsection we only consider, for simplicity, distributions f defined on $R^n_{(p)}$. (This is sufficient for the needs of Part II.)

[1] $\hat{C}_j$ denotes the basis of C_j on $S^{n-1}_{(v)}$.

[2] When $\mathcal{D}$ is a domain of $R^n_{(p)}$, this means that for any given (closed convex salient) Σ'_P whose basis on $S^{n-1}_{(v)}$ contains the basis of Σ_P in its interior, there always exists a neighborhood $\mathcal{N}$ of P such that f is, in $\mathcal{N}$, the boundary value of F from the directions $q = \operatorname{Im} p$ of the open dual cone $\tilde{\Sigma}'_P$ of Σ'_P.

In the case of a manifold $\mathcal{M}$, this also makes sense, for instance in any local coordinate system at P.

Products of Distributions

Theorem 4. *A sufficient condition for the product of two distributions f_1, f_2 to be well defined over a domain $\mathscr{D}$ is:*

$$v_1 + v_2 \neq 0$$

wherever v_1 and v_2 are non zero vectors belonging respectively to $S_P(f_1)$ and $S_P(f_2)$, where P is any common point in $\mathscr{D}$.

$S_P(f_1 f_2)$ is then contained in the set $S_P(f_1) + S_P(f_2)$ of vectors of the form $v_1 + v_2$ with v_1 in $S_P(f_1)$ and v_2 in $S_P(f_2)$ [3].

Proof. We below define $f_1 \times f_2$ in the neighborhood of any given Point P in $\mathscr{D}$. To that purpose, we consider closed cones $\Sigma_P^{(1)}$ and $\Sigma_P^{(2)}$ with apex at the origin in $R^n_{(v)}$, whose bases on the unit sphere contain respectively $\hat{S}_P(f_1)$ and $\hat{S}_P(f_2)$ in their interiors, and such that $v_1 + v_2$ is still non zero wherever v_1 and v_2 are non zero vectors in these sets.

We then consider C^∞ functions χ_1, χ_2 locally analytic and different from zero at P, and with a (sufficiently small) support around P chosen such that the Bounds (4) be satisfied outside $\Sigma_P^{(1)}$ and $\Sigma_P^{(2)}$ respectively, by the generalized Fourier transforms $F^{(1)}_{\chi_1}$ and $F^{(2)}_{\chi_2}$ at P of $\chi_1 f_1$ and $\chi_2 f_2$.

The product $\chi_1 f_1 \times \chi_2 f_2$ is defined in a standard way as the inverse Fourier transform of:

$$\int (\widetilde{\chi_1 f_1})(v') \times (\widetilde{\chi_2 f_2})(v - v')\, dv' \tag{5}$$

where $\chi_i f_i$ is the Fourier transform of $\widetilde{\chi_i f_i}$.

In fact, in view of the rapid decrease of $\widetilde{\chi_1 f_1}$ and $\widetilde{\chi_2 f_2}$ outside $\Sigma_P^{(1)}$ and $\Sigma_P^{(2)}$ respectively [Bounds (4) at $\gamma = 0$], and of their slow increase elsewhere (tempered distributions), this integral is convergent for all values of v, and defines a slowly increasing function, whose Fourier transform is therefore a well defined distribution.

By using the bounds (4) for $F^{(1)}_{\chi_1}$ and $F^{(2)}_{\chi_2}$ respectively, it is then not difficult to show that the generalized Fourier transform F_{12} at P of the product $\chi_1 f_1 \times \chi_2 f_2$ does satisfy analogous bounds in all directions $\hat{v}$ which do not belong to the set $\Sigma_P^{(1)} + \Sigma_P^{(2)}$. Write for instance F_{12} in the form

$$F_{12}(v, t) = \int dv'\, F^{(1)}_{\chi_1}(v', (1-\eta)\, t)\, F^{(2)}_{\chi_2}(v - v', \eta t), \tag{6}$$

where $0 < \eta < 1$. The result is ensured, as easily seen, by the fact that the intersection of the cone $\Sigma_P^{(1)}$, and of the cone $v - \Sigma_P^{(2)}$ (with apex at v), is empty, and that their distance is propertional to $|v|$.

The announced property of $S_P(f_1 f_2)$ follows from the fact that χ_1 and χ_2 can be chosen with arbitrarily small supports. $\left(\text{One checks that the definition of } f_1 f_2 = \dfrac{\chi_1 f_1 \times \chi_2 f_2}{\chi_1 \chi_2} \text{ “at } P\text{” does not depend on the choice of } \chi_1, \chi_2.\right)$

[3] v_1 and v_2 are allowed to be zero in this last statement.

Remark. An alternative proof of Theorem 4 can be obtained by making use of the decomposition Theorem 1 of Subsection a), together with the natural definition of the product of two distributions which are boundary values of analytic functions from *common* directions. The definition of $f_1 \times f_2$ obtained by that procedure also coincides with that given above.

For details, see [11].

The following corollary readily follows from Theorem 4:

Theorem 4'. *A sufficient condition for the product of r distributions $f_1, \ldots, f_r$ to be well defined over a domain $\mathscr{D}$ is:*

$$\sum_{i=1}^{r} v_i \neq 0$$

wherever

(i) each v_i lies in $S_P(f_i)$, $i = 1 \ldots r$, $P \in \mathscr{D}$, *and*

(ii) *at least one v_i is different from zero.*

$S_P(f_1 \times \cdots \times f_r)$ is then contained in the set $S_P(f_1) + \cdots + S_P(f_r)$ of vectors of the form $\sum_{i=1,r} v_i$ with v_i in $S_P(f_i)$ $(i = 1, \ldots, r)$[4].

Integrals of Distributions

Let $p = (p_1 \ldots p_n)$ and $p' = (p'_1 \ldots p'_{n'})$ be two sets of independent variables whose dual variables will be denoted by $v = v_1 \ldots v_n$ and $v' = v'_1 \ldots v'_{n'}$.

We consider a distribution f defined on $R^n_{(p)} \times R^{n'}_{(p')}$ whose support with respect to the variables p' is contained in a compact set K of $R^{n'}_{(p')}$ when p lies in some neighborhood $\mathscr{N}$ of a Point P.

Let g be the distribution on $R^n_{(p)}$ which is well defined (in $\mathscr{N}$) as the integral of f over $R^{n'}_{(p')}$:

$$g(p) = \int f(p, p')\, dp' . \tag{7}$$

The following result holds:

Theorem 5. *If a direction $(\hat{v}_0, 0)$ is outside $\hat{S}_{P,P'}(f)$ for all Points P' in K, then $\hat{v}_0$ is outside $\hat{S}_P(g)$.*

Proof. In view of Remark 1 of Subsection a), it is clearly sufficient to prove the theorem for distributions f whose support is contained in $\mathscr{N} \times K$ (if not, replace f by χf with a suitable χ). We assume below that this condition is satisfied. Let $F(v, v', t; P, P')$ be the generalized Fourier transform of f at P, P':

$$F(v, v', t; P, E') = \int f(p, p')\, e^{-ipv - ip'v' - t(p-P)^2 - t(p'-P')^2}\, dp\, dp' . \tag{8}$$

From the assumption of the theorem ($(\hat{v}_0, 0) \notin \hat{S}_{P,P'}(f)$) and Remark 2 of Subsection a), it is easily seen that there exist a neighboring cone $\mathscr{V}$ of $\hat{v}_0$, a polynomial $\mathscr{P}$, $\alpha > 0$, $\gamma_0 > 0$ such that the following bounds hold *uniformly with respect to P'* in any given compact set[5] (when $v \in \mathscr{V}$, and $0 \leqq \gamma < \gamma_0$):

$$|F(v, 0, \gamma|v|; P, P')| < \mathscr{P}(|v|)\, e^{-\alpha\gamma|v|} . \tag{9}$$

[4] v_i is allowed to be zero in this last statement.

[5] To see this, consider an appropriate finite covering of this set, in each part of which bounds of this form hold [see (3')]. (The essential support of f at P, P' is empty if $P' \notin K$ [see for instance Eq. (10)].)

For later purposes, we also note the following bounds, which are derived (in the region $t>0$) from the fact that f has its support in K:

$$|F(v,0,t;P,P')| < Q(|v|,t)\, e^{-t(d(P',K))^2}, \tag{10}$$

where $d(P', K)$ is the distance of P' to K and Q is a polynomial which depends on the order of f.

The generalized Fourier transform of g at P can be written (in the region $t>0$) in the form:

$$\begin{aligned} G(v,t;P) &= \int g(p)\, e^{-ipv-t(p-P)^2}\, dp \\ &= L t^{n'/2} \int dP'\, F(v,0,t;P,P'), \end{aligned} \tag{11}$$

where $L = [\int e^{-P'^2}\, dP']^{-1}$. [This is seen by using the Expression (8) of F in the right-hand side of (11): one may interchange the order of integration and first integrate over P', since f has a compact support with respect to the variables p, p' and $e^{-t(p'-P')^2}$ is a function of the Schwarz space $\mathscr{D}$.]

To derive bounds of the Form (3) on G when $v \in \mathscr{V}$, $0<\gamma<\gamma_0$), it is then sufficient to divide the integration domain in the right-hand side of (11), i.e. P'-space, into two parts:

i) the set K_1 of Points P' whose distance to K is at most $\sqrt{\alpha}$ and

ii) its complement. The Bounds (9) and (10) then readily provide the needed bounds for the first and second contributions respectively. (Note that K_1 is a compact set and that $t<\gamma_0|v|$.)

Remark. An alternative proof using the decomposition Theorem 3 of Subsection a) is also presented in [11].

III. Macrocausality and the Structure Theorem

Macrocausality [2–4] is an appropriate mathematical expression of a certain classical limit, in terms of particles, of quantum theory; namely of the principle that any energy-momentum transfer over large distances which cannot be attributed to (stable) physical particles according to classical ideas, gives effects that are damped exponentially with distance. (Shortrange of the interactions.)

Let us fix some notations.

The space $R^n_{(p)}$ of Part I is here the space R^{4N} of all initial and final 4-momenta p_k (N is the total number of initial and final particles). Its dual space is physically the space of the variables $u=\{u_k\}$ where each u_k is a *space-time displacement* of particle k. It is convenient to define now the scalar product $u\cdot p$ through the formula:

$$u\cdot p = -\sum_{i\in I} p_i u_i + \sum_{j\in J} p_j u_j, \tag{12}$$

where

$$p_k\cdot u_k = (p_k)_0 (u_k)_0 - \sum_{\nu=1}^{3} (p_k)_\nu (u_k)_\nu \qquad (\boldsymbol{p}_k = (p_k)_1, (p_k)_2, (p_k)_3).$$

The submanifold $\mathscr{M}_{IJ}$ of $R^{4N}_{(p)}$ is defined (see the Introduction) by the conditions $p_k^2=m_k^2$, $(p_k)_0>0$ and $\sum_{i\in I} p_i = \sum_{j\in J} p_j$, and the space $N(P)$ of vectors $u=\{u_k\}$ conormal at a Point $P=\{P_k\}$ to $\mathscr{M}_{IJ}$ is the set of vectors of the form $u_k=\lambda_k P_k + a$, where λ_k is an arbitrary real scalar and a is independent of k.

A classical trajectory in space-time, which is the line parallel to p_k and passing through u_k, is associated with each given set (p_k, u_k).

A connected classical multiple scattering Diagram $\mathcal{D}$ in space-time is a connected net in space-time with external and internal oriented lines. Each external line k arrives at, or is issued from one vertex. Each internal line l is issued from one vertex and ends at another vertex. There are at least two incoming and two outgoing (external or internal) lines at each vertex. Finally, each line possesses a 4-momenta p_k, resp. p_l and the following properties are satisfied:

i) mass-shell constraints:

$$p_k^2 = m_k^2\,, \qquad (p_k)_0 > 0$$

$$p_l^2 = m_l^2\,, \qquad (p_l)_0 > 0\,,$$

where m_k, m_l belong to a (finite) set of physical (strictly positive) masses.

ii) Energy-momentum conservation at each vertex (the sum of the incoming 4-momenta at any given vertex equals that of the outgoing 4-momenta at this vertex).

iii) Propagation law: each line is parallel to its 4-momentum (with a positive sign). In particular, if we denote by $(v_l)_{in}$ and $(v_l)_f$ the space-time vertices from which Line l is issued, resp. to which it ends, there must exist $\alpha_l \geqq 0$ such that:

$$(v_l)_f - (v_l)_{in} = \alpha_l p_l\,. \tag{13}$$

For later purposes, we note that (13) clearly implies the loop equations:

$$\sum_l z(l)\,\alpha_l p_l = 0 \tag{14}$$

for each closed loop z of $\mathcal{D}$ where $z(l) = 0$ if z does not contain Line l, $z(l) = +1$, resp. -1 if it contains it with the correct, resp. opposite orientation.

Being given a set of initial and final trajectories (p_k, u_k), $u = \{u_k\}$ is said to be causal at $p = \{p_k\}$ (in a connected way) if it is possible to construct at least one $\mathcal{D}$ whose external incoming and outgoing lines "match" the given initial and final trajectories (for details, see [2, 3]).

One checks easily that the set of causal u at a given Point p is invariant under dilation by $\lambda > 0$ (λu is causal if u is), and under addition of vectors in $N(p)$. We shall denote by $\mathscr{C}(p)$ the cone (with apex at the origin) of causal u at p, and by $C(p)$ the quotient subspace $\mathscr{C}(p)/N(p)$.

Macrocausality. For physical reasons explained in [2, 3], macrocausality can be stated (if p does not belong to $\mathscr{M}_0$ [6]) in the form:

"The essential support of T_{IJ} at a Point p of $\mathscr{M}_{IJ}$ is contained in $C(p)$."

Alternatively (see Lemma 1 in Part I), the essential support of $T_{IJ} \times \delta(\mathscr{M}_{IJ})$ at p is (contained in) $\mathscr{C}(p)$.

For simplicity, it will also be convenient later to consider S^c_{IJ} as defined on the space of all initial and final 3-momenta $\boldsymbol{p}_k$. Then, macrocausality can also be stated in the form:

"The essential support of S^c_{IJ} at p is (contained in) the set of $\boldsymbol{u} = \{\boldsymbol{u}_k\}$ for which $\{(\boldsymbol{u}_k, (u_k)_0 = 0)\}$ is causal at p."

Finally, it is not difficult to check that the essential support of $T_{IJ} = T^*_{JI}$ is opposite to that of T_{IJ} itself [and similarly for $(S^c_{IJ})^-$].

[6] We recall that $\mathscr{M}_0$ is the set of Points $p = p_k$ such that two initial or two final, 4-momenta are colinear.

Analytic Properties of T_{IJ}

For the convenience of the reader, we briefly recall the following facts (for details see [2, 3]).

A Point $p=\{p_k\}$ is said to belong to the Landau surface $L(G)$, resp. to its $+\alpha$-landau part $L^+(G)$, of a connected graph G of the process $I \to J$[7], if it is possible to find at least one set of internal 4-momenta p_l and of real scalars α_l, resp. of positive α_l, one of which at least is non zero, such that the above Conditions i), ii) of a diagram $\mathscr{D}$ be satisfied, together with the loop Eq. (14). (Each external line has the corresponding 4-momentum p_k.)

It is clear that $C(p)$ is empty (apart from the origin) if p is not a $+\alpha$-Landau point of a surface $L^+(G)$. There is in general one causal direction in $C(p)$ if p lies on one $+\alpha$-Landau surface. This direction is replaced by those of a convex salient cone at the points which lie on several $+\alpha$-landau surfaces $L^+(G')$, $L^+(G''), \ldots$ with a common "parent" (i.e. G', $G'', \ldots$ are various contractions of a common graph G). Finally, if p lies in the intersection of several $+\alpha$-landau surfaces with no common parent, $C(p)$ is the union of the cones associated with each parent.

The $+\alpha$-landau surfaces are at most of Codimension 1 in $\mathscr{M}_{IJ}$ (and are not dense in $\mathscr{M}_{IJ}$). Lemma 1 and Theorems 1 and 2 of Part I therefore provide the analytic properties of T_{IJ} announced in the introduction, and Theorem 3 provides the "patching property".

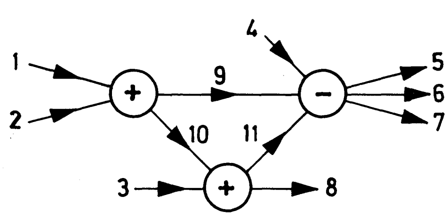
Bubble Diagram Functions

We consider a (topological) connected graph with external and internal oriented lines; I, J, will here denote the sets of incoming, resp. outgoing external lines. Each line is again associated with a physical particle.

A "bubble diagram" B is obtained by associating a bubble $+$ or $-$ with each vertex. Each $+$ bubble, resp. $-$ bubble, is the functional $S^c_{I_b J_b}$, resp. $(S^c_{I_b J_b})^-$ of the process whose initial and final particles I_b, J_b are the incoming and outgoing lines at the vertex considered.

The "bubble diagram function" G^B_{IJ} associated with B is then the distribution (when it is well defined: see below) obtained by integrating the product of all $+$ and $-$ bubbles, over all internal on-mass-shell 4-momenta. What is meant here will become clear on the following example.

Example. Consider the bubble diagram:



whose external lines are numbered from 1 to 8 and internal lines from 9 to 11.

[7] G has, as above, external and internal oriented lines, the external ones being associated with the initial and final Particles I and J. But all mention of space-time is now removed.

It defines the bubble diagram function:

$$\begin{aligned} G^B(p_1, p_2, p_3, p_4; p_5, p_6, p_7, p_8) \\ = \int S^c_{1,2\to 9,10}(p_1, p_2; p_9, p_{10}) \times S^c_{3,10\to 8,11}(p_3, p_{10}; p_8, p_{11}) \\ \cdot (S^c_{4,9,11\to 5,6,7})^-\ (p_4, p_9, p_{11}; p_5, p_6, p_7) \\ \prod_{l=9,10,11} \cdot\ \delta(p_l^2 - m_l^2)\, \theta((p_l)_0)\, d^4 p_l\,. \end{aligned} \tag{15}$$

If G^B and all S^c or $(S^c)^-$ involved are expressed in terms of the 3-momenta variables $\boldsymbol{p}_k$, $\boldsymbol{p}_l$, (15) can also be written in the form:

$$\begin{aligned} G^B(\boldsymbol{p}_1, \dots, \boldsymbol{p}_8) = \int S^c_{1,2\to 9,10}(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{p}_9, \boldsymbol{p}_{10}) \times S^c_{3,10\to 8,11}(\boldsymbol{p}_3, \dots, \boldsymbol{p}_{11}) \\ \cdot (S^c_{4,9,11\to 5,6,7})^-(\boldsymbol{p}_4, \dots, \boldsymbol{p}_7) \prod_{l=9,10,11} \frac{d^3 \boldsymbol{p}_l}{2\omega(\boldsymbol{p}_l)}\,, \end{aligned} \tag{16}$$

where $\omega(\boldsymbol{p}_l) = (\boldsymbol{p}_l^2 + m_l^2)^{1/2}$.

Since each S^c or $(S^c)^-$ contains an energy-momentum conservation δ-function, it is clear that a global δ-function can be factored out from the integral, namely:

$$G^B_{IJ} = F^B_{IJ} \times \delta^4\left(\sum_{i\in I} p_i - \sum_{j\in J} p_j\right), \tag{17}$$

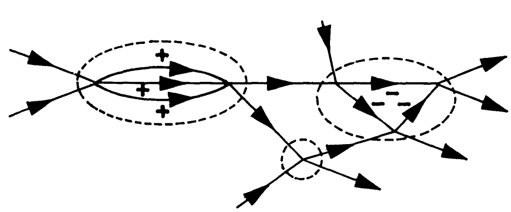
where F^B_{IJ} is defined on $\mathscr{M}_{IJ}$.

A connected multiple scattering diagram associated with B is a connected Diagram $\mathscr{D}_B$ in space-time which includes external and internal lines associated with the external and internal lines of B and may also include supplementary internal lines arising from the replacement of each bubble b by a Subdiagram $\mathscr{D}_b$, whose external lines match the incoming and outgoing lines of b: see example below. The Diagram $\mathscr{D}_B$ must satisfy Properties i) and ii) of a Diagram $\mathscr{D}$ stated above, but Property iii) is replaced by:

iii)′ "The coefficients α_l in (13) associated with the original internal lines of B may have *arbitrary* positive or negative values.

The coefficients α_l associated with the internal lines of a Subdiagram $\mathscr{D}_b$ must be positive, resp. negative, if b is *a* $+$, resp. $-$, bubble."

Example. An example of a $\mathscr{D}_B$ associated with the bubble Diagram B quoted above is:



Here the Subdiagrams $\mathscr{D}_b$ are shown inside circles. The 4-momenta of all lines and the space-time positions of the vertices have to be specified and the Conditions i)–iii)′ must be satisfied. The $+$ or $-$ signs above a line mean that the corresponding α_l is to be positive or negative.

Being given a set $p=\{p_k\}$ of external (initial and final) 4-momenta, we denote by $\mathscr{C}^B(p)$ the cone of vectors $u=\{u_k\}$ which ensure the existence of at least one connected multiple scattering diagram associated with B, and by $C^B(p)$ the quotient subspace $\mathscr{C}^B(p)/N(p)$.

The structure theorem will be proved below under the following constraints: we shall not treat the points of the set $\mathscr{M}_0$ (which may occur in the integration domain for some of the bubbles) and the situations (referred to, below, as "$u=0$ points") for which a non trivial $\mathscr{D}_B$ exists when $u=\{u_k\}=0$ (i.e. when the displacements u_k of all *external* particles are fixed at zero). We shall *assume* that they do not modify the results (we do not know so far how to remove this technical assumption in general).

We then show below how Theorems 3 and 4 of Part I directly provide the following general form of the structure theorem from the very statement of macrocausality given above:

Structure Theorem

"*F_{IJ}^B is a well defined distribution whose essential support at a Point p of $\mathscr{M}_{IJ}$ is contained in the set $C^B(p)$.*"

Before giving the proof, we first outline the main analyticity properties of F_{IJ}^B associated with this general form of the structure theorem. (The structure theorem was first proved in the form of these properties in [1, 5, 7].)

Analytic Properties of Bubble Diagram Functions

Property 1.

"F_{IJ}^B is analytic at all points which lie on no Landau surface $L^\sigma(G_B)$." Here G_B is any topological graph associated with a Diagram $\mathscr{D}_B$ as before, and $L^\sigma(G_B)$ is the part of the Landau surface $L(G_B)$ subject to the constraint that each line l of G_B that is an internal line of some subgraph G_b has a coefficient α_l which is positive, resp. negative, if b a $+$, resp. $-$, bubble.

Property 1) directly follows from Lemma 2 of Part I and from the fact that $C^B(p)$ is clearly empty (apart from the origin) if p lies on no $L^\sigma(G_B)$.

Theorem 1 of Part I also provides the following property:

Property 2.

"If p lies on one surface $L^\sigma(G_B)$, and if there is only one direction in $C^B(p)$, or several directions all contained in a unique closed convex salient cone, then F_{IJ}^B is at p the boundary value of an analytic function from the directions of the dual cone."

More general properties, which we shall not state here in detail, are also derived from Theorems 2 and 3 of Part I. For a detailed study of the Landau surfaces and of the sets $C^B(p)$ at Landau points, see [1, 5, 7].

Proof of the Structure Theorem

For simplicity, we consider all distributions $S_{I_bJ_b}^c$ [or $(S_{I_bJ_b}^c)^-$] involved in the integrand, as well as G_{IJ}^B, as defined on the space of the *3-momentum* variables.

The integral over the internal momenta is therefore an integral with respect to the measure $\prod_l \frac{d^3 \boldsymbol{p}_l}{2\omega(p_l)}$ where $\omega(p_l) = (\boldsymbol{p}_l^2 + m_l^2)^{1/2}$ [see the example of Eq. (16)]. We shall show that the essential support of G_{IJ}^B at p is contained in the set of directions $\boldsymbol{u} = \{\boldsymbol{u}_k\}$ for which $u = \{\boldsymbol{u}_k, (u_k)_0 = 0\}$ is in $\mathscr{C}^B(p)$. The announced result on F_{IJ}^B then follows from Lemma 1 of Part I.

The dual variables of the variables $\boldsymbol{p}_k$, $\boldsymbol{p}_l$ where k labels as before the external lines of B and l its internal lines, will be denoted $\boldsymbol{u}_k$, $\boldsymbol{u}_l$ and it is here convenient to define the scalar product through the formula:

$$\langle \{\boldsymbol{u}_k\}, \{\boldsymbol{u}_l\}; \{\boldsymbol{p}_k\}, \{\boldsymbol{p}_l\} \rangle = \sum_{i \in I} \boldsymbol{u}_i \boldsymbol{p}_i - \sum_{j \in J} \boldsymbol{u}_j \boldsymbol{p}_j - \sum \boldsymbol{u}_l \boldsymbol{p}_l . \tag{18}$$

Consider one of the scattering functions $S_{J_b J_b}^c$ involved in the integrand of a bubble diagram function. It defines a corresponding distribution in the space of all external and internal 3-momenta $\boldsymbol{p}_k$, $\boldsymbol{p}_l$ whose essential support at a Point $P = \{\boldsymbol{P}_k\}, \{\boldsymbol{P}_l\}$ is known, from macrocausality, to be contained in the set of vectors $\{\boldsymbol{u}_k\}, \{\boldsymbol{u}_l\}$ of the form:

$$\boldsymbol{u}_r = 0 \quad \text{if } r \text{ is not involved in } b ,$$

$$\boldsymbol{u}_s = \varepsilon_s \boldsymbol{v}_s \quad \text{if } s \text{ is involved in } b ,$$

where $\{(\boldsymbol{v}_s, 0)\}_{s \in I_b, J_b}$ is in $\mathscr{C}(\{P_k\}, \{P_l\}, k, l \in I_b, J_b)$ and $\varepsilon_s = +1$, resp. $\varepsilon_s = -1$, if s is an external line of B, or an outgoing internal line of B, resp. if s is an incoming internal line of B. [This sign arises because of the definition (18) of the scalar product which introduces a-sign for the incoming internal lines of B.]

The essential support of a term $(S_{I_b J_b}^c)^-$ is contained in the set of vectors of the form $\boldsymbol{u}_r = 0$, $\boldsymbol{u}_s = -\varepsilon_s \boldsymbol{v}_s$.

Let $\mathscr{S}(P)$ denote the set of vectors $\{\boldsymbol{u}_k\}, \{\boldsymbol{u}_l\}$ obtained by summation of one vector of this type for each bubble b involved in the integrand, one of them at least being non zero.

If a vector in $\mathscr{S}(P)$ is zero, then necessarily $\boldsymbol{v}_k = 0$ for all external Lines k of B (since each external line appears in one bubble at most) and:

$$\boldsymbol{v}_{l,in} - \boldsymbol{v}_{l,f} = 0 \tag{19}$$

for each internal Line l of B, where $\boldsymbol{v}_{l,in}$ and $\boldsymbol{v}_{l,f}$ denote the displacements of particle l associated with the respective bubbles for which l is an incoming, resp. outgoing particle.

This last condition ensures that the various causal (or anticausal) space-time diagrams $\mathscr{D}_b$ associated with the sets $\{\boldsymbol{v}_s\}$ (or $-\{\boldsymbol{v}_s\}$) for each bubble b, fit together to form a non trivial $\mathscr{D}_B$.

If, as mentioned above the statement of the structure theorem, we exclude here "$u = 0$ points", there is no zero vector in $\mathscr{S}(P)$. According to Theorem 4′, the product of the $S_{I_b J_b}^c$ [and $(S_{I_b J_b}^c)^-$] in the integrand is then a well defined distribution, whose essential support[8] at P is contained in $\mathscr{S}(P)$. [The presence of the factors $\omega(p_l)^{-1}$ does not alter this result since these factors are analytic for all real values.]

Finally, consider a given Point $\{P_k\}$. If a vector $\{u_k\}$, $\{u_l=0\}$ is in $\mathscr{S}(\{P_k\}, \{P_l\})$ for some value of $\{P_l\}$, then the conditions $u_l=0$ ensure, for the same reasons as above, the existence of a non trivial $\mathscr{D}_B$. According to Theorem 5[9], $\{u_k\}$ is outside $S_{(\{P_k\})}(G^B_{IJ})$ if $\{u_k\}$, $\{u_l=0\}$ is outside $\mathscr{S}(\{P_k\}, \{P_l\})$ for all values of $\{P_l\}$, and the theorem is therefore proved.

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[8] Apart from the origin.

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Note Added in Proof. After receiving our present work, Professors T. Kawaï and H. P. Stapp have also carried out some further work on the structure theorem. Although it is presented in a very different mathematical language, their proof for "$u \neq 0$ points" does not really differ from ours. On the other hand, they have made some progress on the problem of "$u = 0$ points" which is not treated here. We think however that the new results they have obtained in this connection can also be proved in the mathematical framework of the present paper.