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A Correction to My Paper Spectra of States, and Asymptotically Abelian C^*-Algebras

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Erling Størmer

Department of Mathematics, University of Oslo, Blindern, Oslo, Norway

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It was pointed out to me by Daniel Kastler that I was too careless in the use of the strong-* topology in the proof of Theorem 2.3 in the above paper [1]. As a result it is necessary to change the definition of the spectrum of a state on a C^*-algebra somewhat.

Definition 1. Let $\mathfrak{A}$ be a C^*-algebra and ϱ a state of $\mathfrak{A}$ with GNS representation $(\pi_\varrho, x_\varrho, \mathfrak{X}_\varrho)$. Then the *spectrum* of ϱ, denoted by $\mathrm{Spec}(\varrho)$ is the set of real numbers u such that given $\varepsilon > 0$ there is $A \in \pi_\varrho(\mathfrak{A})''$ for which $\omega_{x_\varrho}(A^*A) = 1$ such that

$$|u(\pi_\varrho(B)\,A x_\varrho, x_\varrho) - (A\pi_\varrho(B)\,x_\varrho, x_\varrho)| < \varepsilon\varrho(B^*B)^{1/2}$$

for all $B \in \mathfrak{A}$.

In the previous definition we asserted that we could choose $A \in \pi_\varrho(\mathfrak{A})$.

Let $\mathfrak{R}_\varrho$ denote the von Neumann algebra $\pi_\varrho(\mathfrak{A})''$ and E_ϱ the projection $[\mathfrak{R}'_\varrho x_\varrho]$, which is the support of ω_{x_ϱ} on $\mathfrak{R}_\varrho$. Let Δ_ϱ be the modular operator of x_ϱ relative to $E_\varrho \mathfrak{R}_\varrho E_\varrho$ acting on $E_\varrho \mathfrak{X}_\varrho$, and consider it as an operator on $\mathfrak{X}_\varrho$ by defining it to be 0 on $(I - E_\varrho)\,\mathfrak{X}_\varrho$.

Definition 2. With the above notation we call Δ_ϱ the *modular operator* of the state ϱ.

Remark 1. $\mathrm{Spec}(\varrho) = \mathrm{Spec}(\omega_{x_\varrho} | \mathfrak{R}_\varrho)$. Indeed, if $u \in \mathrm{Spec}(\varrho)$ and $A \in \mathfrak{R}_\varrho$ satisfies the conditions in Definition 1 then for all $B \in \pi_\varrho(\mathfrak{A})$

$$|u(A x_\varrho, B^* x_\varrho) - (B x_\varrho, A^* x_\varrho)| < \varepsilon \|B x_\varrho\| .$$

Since $\pi_\varrho(\mathfrak{A})$ is strong-* dense in $\mathfrak{R}_\varrho$ the same inequality holds for all $B \in \mathfrak{R}_\varrho$, and thus $u \in \mathrm{Spec}(\omega_{x_\varrho} | \mathfrak{R}_\varrho)$. The converse inclusion is trivial since $\pi_\varrho(\mathfrak{A}) \subset \mathfrak{R}_\varrho$.

Theorem. *Let $\mathfrak{A}$ be a C^*-algebra and ϱ a state of $\mathfrak{A}$ with modular operator Δ_ϱ. Then* $\mathrm{Spec}(\varrho) = \mathrm{Spec}(\Delta_\varrho)$.

Since the proof has to be modified we give a complete proof of the part that $u \neq 0$ in $\mathrm{Spec}(\varrho)$ is contained in $\mathrm{Spec}(\Delta_\varrho)$. In order to simplify notation drop the subscripts ϱ, so $\mathfrak{R} = \mathfrak{R}_\varrho$, $E = E_\varrho$, $x = x_\varrho$, $\Delta = \Delta_\varrho$, etc. By Remark 1 we have to show $\mathrm{Spec}(\omega_x | \mathfrak{R}) = \mathrm{Spec}(\Delta)$.

Let $u \in \mathrm{Spec}(\omega_x | \mathfrak{R})$. Assume $u \neq 0$. Let $\varepsilon > 0$ and choose $A \in \mathfrak{R}$ such that $\|Ax\| = 1$ and such that

$$|u(BAx, x) - (ABx, x)| < \varepsilon \|Bx\| \tag{1}$$

for all $B \in \mathfrak{R}$. Apply (1) to $B(I-E)$. Since $(I-E)x = 0$ we have $u(B(I-E)Ax, x) = 0$, so that $(I-E)Ax = 0$ by cyclicity of x. Thus $Ax = EAx$, and in particular $\|EAEx\| = 1$. Apply next (1) to EB. Then

$$\begin{aligned} &|u(BEAEx, x) - (EAEBx, x)| \\ &\qquad = |u(EBAx, x) - (AEBx, x)| < \varepsilon \|EBx\| \leqq \varepsilon \|Bx\|, \end{aligned}$$

so $u \in \mathrm{Spec}(\omega_x | E\mathfrak{R}E)$. Thus in order to show $u \in \mathrm{Spec}\,\Delta$ we may and do assume x is separating and cyclic for $\mathfrak{R}$, so $E = I$.

Let J be the conjugation on $\mathfrak{X}$ so that $J\Delta^{\frac{1}{2}}Bx = \Delta^{-\frac{1}{2}}JBx = B^*x$ for all $B \in \mathfrak{R}$ [2, Theorem 7.1]. Now the Tomita algebra $\mathfrak{R}_0$ (called modular Hilbert algebra in [2]) is strong-* dense in $\mathfrak{R}$ [2, Theorem 10.1]. (We identify $\mathfrak{R}$ with the Hilbert algebra $\mathfrak{R}x$, and $\mathfrak{R}_0$ with $\mathfrak{R}_0 x$.) Thus in particular (1) holds for all $B \in \mathfrak{R}_0$. Since $\mathfrak{R}_0 x$ is contained in the domain of $\Delta^{-\frac{1}{2}}$, see proof of [2, Theorem 10.1], we have from (1)

$$|u(Ax, \Delta^{-\frac{1}{2}}JBx) - (Bx, J\Delta^{\frac{1}{2}}Ax)| < \varepsilon \|Bx\|$$

or

$$|(Ax, u\Delta^{-\frac{1}{2}}JBx) - (Ax, \Delta^{\frac{1}{2}}JBx)| < \varepsilon \|JBx\|.$$

Let $\tilde{\Delta} = u\Delta^{-\frac{1}{2}} - \Delta^{\frac{1}{2}}$. Then $J\mathfrak{R}_0 x$ belongs to the domain $\mathfrak{D}(\tilde{\Delta})$ of $\tilde{\Delta}$ and we have

$$|(Ax, \tilde{\Delta}y)| < \varepsilon \|y\|$$

for all $y \in J\mathfrak{R}_0 x$ and thus for all $y \in \mathfrak{D}(\tilde{\Delta})$ by proof of [2, Theorem 10.1]. In particular the linear functional $y \to (Ax, \tilde{\Delta}y)$ is bounded on the dense linear subspace $\mathfrak{D}(\tilde{\Delta})$ of $\mathfrak{X}$. Therefore it has an extension to $\mathfrak{X}$ with the same bound. By Riesz representation theorem there is $z \in \mathfrak{X}$ such that $\|z\| < \varepsilon$ and

$$(Ax, \tilde{\Delta}y) = (z, y) \tag{2}$$

for all $y \in \mathfrak{D}(\tilde{\Delta})$.

By definition of the adjoint of an unbounded operator, $Ax \in \mathfrak{D}(\tilde{\Delta})$, and

$$(\tilde{\Delta}Ax, y) = (z, y)$$

for all y in $\mathfrak{D}(\tilde{\Delta})$, and hence for all $y \in \mathfrak{X}$. In particular $\tilde{\Delta} A x = z$, so $\|\tilde{\Delta} A x\| < \varepsilon$. Since the operator $\Delta^{-\frac{1}{2}}(u^{\frac{1}{2}} I + \Delta^{\frac{1}{2}}) \geqq I$ we thus have

$$\begin{aligned} \|(u^{\frac{1}{2}} I - \Delta^{\frac{1}{2}}) A x\| &\leqq \|\Delta^{-\frac{1}{2}}(u^{\frac{1}{2}} I + \Delta^{\frac{1}{2}})(u^{\frac{1}{2}} I - \Delta^{\frac{1}{2}}) A x\| \\ &= \|\Delta^{-\frac{1}{2}}(u I - \Delta) A x\| \\ &= \|\tilde{\Delta} A x\| < \varepsilon . \end{aligned}$$

Since Ax is a unit vector and ε is arbitrary, $u^{\frac{1}{2}} \in \operatorname{Spec}(\Delta^{\frac{1}{2}})$, hence $u \in \operatorname{Spec}(\Delta)$.

The rest of the proof of the theorem should be as before except that in the proof of $u \in \operatorname{Spec}(\Delta)$ implies $u \in \operatorname{Spec}(\varrho)$, we show as before that $u \in \operatorname{Spec}(\Delta)$ implies $u \in \operatorname{Spec}(\omega_x | \mathfrak{R})$, and apply Remark 1 to conclude that $u \in \operatorname{Spec}(\varrho)$.

From the above proof we have

Corollary. *Let notation be as above. If* $u \neq 0$ *belongs to* $\operatorname{Spec}(\varrho)$, *and* A *is as in Definition 1 then* $A x_\varrho$ *belongs to the domain of* $\Delta_\varrho^{-\frac{1}{2}}$.

Proof. With the notation as in the proof we have $A x_\varrho \in \mathfrak{D}(\tilde{\Delta})$. Since $\tilde{\Delta} = u\Delta^{-\frac{1}{2}} - \Delta^{\frac{1}{2}}$ and $A x_\varrho \in \mathfrak{D}(\Delta^{\frac{1}{2}})$ it follows that $A x_\varrho \in \mathfrak{D}(\Delta^{-\frac{1}{2}})$.

References

1. Størmer, E.: Commun. math. Phys. **28**, 279—294 (1972)
2. Takesaki, M.: Tomita's theory of modular hilbert algebras and its applications. Lecture Notes Math. **128**. Berlin-Heidelberg-New York: Springer 1970

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E. Størmer
Department of Mathematics
University of Oslo
Blindern, Oslo 3, Norway