Commun. math. Phys. 38, 331—340 (1974)

Differentiability of Spatially Homogeneous Solutions of the Boltzmann Equation in the Non Maxwellian Case

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Received February 23, 1974

Abstract. The non linear Boltzmann equation is studied and differentiable solutions are shown to exist if the initial datum is suitably chosen

1. Introduction

In the present paper we study the initial value problem for the Boltzmann equation in the spatially homogeneous case and obtain an existence and uniqueness theorem which is a generalization of [4] when the extrinsic force is zero.

This space-independent problem has been much studied; for instance in [9] and [1] Povzner and Bodmer obtain a similar result by completely different methods. We note however that our methods are much more simple; moreover Povzner does not investigate the differentiability of the solution while Bodmer considers a somewhat less general equation.

2. Preliminaries

We begin with some basic definitions and results which will be used in the rest of this paper.

Let X be a real Banach space and X^* its dual; by $\| \ \|$ we denote the norm in X and by $\langle x, x^* \rangle$ the value of $x^* \in X^*$ at $x \in X$.

With each $x \in X$ is associated the set $\partial \|x\| = \{x^* \in X^*, \|x+y\| \geqq \|x\| + \langle y, x^* \rangle, \forall y \in X\}$; the application $X \to 2^{X^*}$, $x \to \partial \|x\|$, is called the subdifferential of the norm.

Let $f: D_f \subset X \to X$ and consider the initial value problem

$$\begin{cases} \dfrac{du}{dt} = f(u) \\ u(0) = u_0 \end{cases} \quad t \in [0, T]. \tag{1}$$

A continuous function $u: [0, T] \to X$ is called a solution of (1) if it is differentiable in $[0, T]$ and satisfies (1).

We will use the following lemma (see [7])

Lemma 1. *Let u belong to $C^1(0,T;X)$*[1]*; then $\|u(\cdot)\|$ is differentiable a.e. in $[0,T]$ and we have*

$$\frac{d}{dt}\|u(t)\| = \left\langle \frac{du}{dt}, u^* \right\rangle, \qquad \forall u^* \in \partial\|u(t)\| .$$

3. Properties of the Boltzmann Equation

We shall study the Boltzmann equation in the spatially homogeneous case

$$\frac{\partial u(\xi,t)}{\partial t} = f(u,u), \quad \xi \in \mathbb{R}^3, t \geqq 0 .$$

The bilinear function $(u,v) \to f(u,v)$ is defined as follows[2]

$$f(u,v) = f^+(u,v) - f^-(u,v)$$

where

$$f^+(u,v) = \tfrac{1}{2} \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^{\pi/2} [u(\xi')\,v(\xi_1') + u(\xi_1')\,v(\xi')]\, B(\theta,\xi,\xi_1)\, d\theta\, d\sigma\, d\xi_1$$

and

$$f^-(u,v) = \tfrac{1}{2} \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^{\pi/2} [u(\xi)\,v(\xi_1) + u(\xi_1)\,v(\xi)]\, B(\theta,\xi,\xi_1)\, d\theta\, d\sigma\, d\xi_1$$

(usually the time dependence of u and v will not be written explicitly).

Here the variables ξ', ξ_1' depend on ξ, ξ_1 through

$$\begin{aligned} \xi' &= \xi + \alpha(\alpha \cdot (\xi - \xi_1)), \\ \xi_1' &= \xi_1 - \alpha(\alpha \cdot (\xi - \xi_1)), \end{aligned} \tag{2}$$

where $\alpha = (\cos\theta, \sin\theta\cos\sigma, \sin\theta\sin\sigma)$. For fixed α these formulae represent a unitary transformation in the 6-space ξ, ξ_1 so giving

$$\partial(\xi', \xi_1')/\partial(\xi, \xi_1) = 1 . \tag{3}$$

Equations (2) mean in particular that

$$\begin{aligned} \xi + \xi_1 &= \xi' + \xi_1' , \\ \xi^2 + \xi_1^2 &= \xi'^2 + \xi_1'^2 . \end{aligned} \tag{4}$$

$B(\theta, \xi, \xi_1) : (0, \pi/2)\, x\mathbb{R}^6 \to \mathbb{R}$ is a given measurable function non negative a.e. such that

$$\begin{aligned} B(\theta,\xi,\xi_1) &= B(\theta,\xi_1,\xi) , \\ B(\theta,\xi,\xi_1) &= B(\theta,\xi',\xi_1') . \end{aligned} \tag{5}$$

We recall that owing to (3), (4), and (5) f satisfies the following equalities[3]

$$\int f(u,v)\, d\xi = 0 , \quad \int \xi^2 f(u,v)\, d\xi = 0 \tag{6}$$

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[1] $C^m(0,T;X)$ is the Banach space of the m-times continuously differentiable functions $u:[0,T] \to X$ equipped with the norm $|||u||| = \sup_{t\in[0,T]} \|u(t)\| + \cdots + \sup_{t\in[0,T]} \left\| \frac{d^m}{dt^m} u(t) \right\|$.

[2] Troughout this section by u and v we denote measurable functions.

[3] For a more detailed description and proofs of the properties quoted in this section see [2], [6].

where u and v are any functions such that $f(u, v)$ exists. From now on we assume that the following hypothesis is fulfilled

$$\text{(H)} \quad \sup_{\xi, \xi_1 \in \mathbb{R}^3} \int_0^{\pi/2} \frac{B(\theta, \xi, \xi_1)}{1 + \sqrt{\xi^2 + \xi_1^2}} \, d\theta = B_0 < +\infty$$

An immediate consequence of (H) is the following lemma.

Lemma 2. *We have*

$$\int |f(u, v)| \, d\xi \leqq \gamma \int (1 + \xi^2) \, |u| \, d\xi \int (1 + \xi^2) \, |v| \, d\xi \, ,$$

where $\gamma = 8\pi B_0$.

We now want to establish some basic properties of f which will be used in the following.

Lemma 3. *Let u be non negative a.e. We have*

$$\int (1 + \xi^2)^2 f(u, u) \, d\xi \leqq \gamma \int (1 + \xi^2)^2 \, u \, d\xi \int (1 + \xi^2) \, u \, d\xi \, . \tag{7}$$

Proof. Set $B'(\theta, \xi, \xi_1) = B(\theta, \xi, \xi_1) \; (\xi^2 + \xi_1^2)$; it follows from (4) that B' satisfies (5), then (6) is verified

$$\int \xi^2 \int [u(\xi') \, u(\xi_1') - u(\xi) \, u(\xi_1)] \, B'(\theta, \xi, \xi_1) \, d\theta \, d\sigma \, d\xi_1 \, d\xi = 0$$

(the limits of integration have been omitted for simplicity in notation) and we have

$$\int \xi^4 f(u, u) \, d\xi = - \int \xi^2 \int \xi_1^2 [u(\xi') \, u(\xi_1') - u(\xi) \, u(\xi_1)] \, B(\theta, \xi, \xi_1) \, d\theta \, d\sigma \, d\xi_1 \, d\xi \, .$$

Now (6) implies

$$\begin{aligned}
\int (1 + \xi^2)^2 f(u, u) \, d\xi &= \int \xi^4 f(u, u) \, d\xi \\
&= \int \xi^2 \int \xi_1^2 [u(\xi) \, u(\xi_1) - u(\xi') \, u(\xi_1')] \, B(\theta, \xi, \xi_1) \\
&\quad \cdot d\theta \, d\sigma \, d\xi_1 \, d\xi \\
&= \int (1 + \xi^2) \int (1 + \xi_1^2) \, [u(\xi) \, u(\xi_1) - u(\xi') \, u(\xi_1')] \\
&\quad B(\theta, \xi, \xi_1) \, d\theta \, d\sigma \, d\xi_1 \, d\xi \\
&\leqq \int (1 + \xi^2) \, u(\xi) \int (1 + \xi_1^2) \, u(\xi_1) \, B(\theta, \xi, \xi_1) \quad d\theta \, d\sigma \, d\xi_1 \, d\xi
\end{aligned}$$

and the conclusion follows from (H).

Lemma 4. *If u and v are any functions non negative a.e. then*[4]

$$\begin{aligned}
\int (1 + \xi^2) \operatorname{sgn}(u - v) \, [f(u, u) - f(v, v)] \, d\xi \leqq 2\gamma \int (1 + \xi^2)^2 \, (u + v) \, d\xi& \\
\int (1 + \xi^2) \, |u - v| \, d\xi \, .&
\end{aligned} \tag{8}$$

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[4] sgn u is the function defined as follows

$$\operatorname{sgn} u \begin{cases} = 1 & \text{if} \quad u > 0 \\ = -1 & \text{if} \quad u < 0 \\ = 0 & \text{elsewhere} \end{cases}$$

Proof. We have

$$\begin{aligned}
&\int (1+\xi^2)\,\mathrm{sgn}(u-v)\,[f(u,u)-f(v,v)]\,d\xi \\
&\quad \leqq \int (1+\xi^2)\int |u(\xi_1)-v(\xi_1)|\,[u(\xi)+v(\xi)]\,B(\theta,\xi,\xi_1)\,d\theta d\sigma d\xi_1 d\xi \\
&\qquad -\int (1+\xi^2)\int |u(\xi)-v(\xi)|\,[u(\xi_1)+v(\xi_1)]\,B(\theta,\xi,\xi_1)\,d\theta d\sigma d\xi_1 d\xi \\
&\qquad +\int (1+\xi^2)\int |u(\xi')-v(\xi')|\,[u(\xi_1')+v(\xi_1')]\,B(\theta,\xi,\xi_1)\,d\theta d\sigma d\xi_1 d\xi \\
&\qquad +\int (1+\xi^2)\int |u(\xi_1')-v(\xi_1')|\,[u(\xi')+v(\xi')]\,B(\theta,\xi,\xi_1)\,d\theta d\sigma d\xi_1 d\xi \\
&\quad = 2\int (1+\xi^2)\,[u(\xi)+v(\xi)]\int |u(\xi_1)-v(\xi_1)|\,B(\theta,\xi,\xi_1)\,d\theta d\sigma d\xi_1 d\xi\,,
\end{aligned}$$

where we used (6). Thus the conclusion follows from (H).

For each $\varepsilon\in[0,1]$ let f_ε be the bilinear function $(u,v)\to f_\varepsilon(u,v) = f_\varepsilon^+(u,v)-f_\varepsilon^-(u,v)$, where

$$f_\varepsilon^+(u,v)=\tfrac{1}{2}\int [u(\xi')\,v(\xi_1')+u(\xi_1')\,v(\xi')]\,\frac{B(\theta,\xi,\xi_1)}{1+\varepsilon\sqrt{\xi^2+\xi_1^2}}\,d\theta d\sigma d\xi_1$$

and

$$f_\varepsilon^-(u,v)=\tfrac{1}{2}\int [u(\xi)\,v(\xi_1)+u(\xi_1)\,v(\xi)]\,\frac{B(\theta,\xi,\xi_1)}{1+\varepsilon\sqrt{\xi^2+\xi_1^2}}\,d\theta d\sigma d\xi_1\,.$$

By definition $f_0(u,v)=f(u,v)$; moreover as $B_\varepsilon(\theta,\xi,\xi_1)=B(\theta,\xi,\xi_1)(1+\varepsilon\sqrt{\xi^2+\xi_1^2})^{-1}$ satisfies (5) we have that each f_ε verifies (6), (7), and (8).

The following lemma establishes further properties of f_ε.

Lemma 5. *For each* $\varepsilon,\eta\in[0,1]$ *the following inequalities hold:*

$$\int |f_\varepsilon(u,u)-f_\eta(u,u)|\,d\xi \leqq 4\gamma\Big(\int (1+\xi^2)\,|u|\,d\xi\Big)^2\,|\varepsilon-\eta|$$

$$\int (1+\xi^2)\,|f_\varepsilon(u,u)-f_\eta(u,u)|\,d\xi \leqq 4\gamma|\varepsilon-\eta|\int (1+\xi^2)^2\,|u|\,d\xi \int (1+\xi^2)\,|u|\,d\xi\,.$$

Proof. We have

$$\begin{aligned}
&\int |f_\varepsilon(u,u)-f_\eta(u,u)|\,d\xi \\
&\quad = \int \Big|\int [u(\xi')\,u(\xi_1')-u(\xi)\,u(\xi_1)]\,[B_\varepsilon(\theta,\xi,\xi_1)-B_\eta(\theta,\xi,\xi_1)]\,d\theta d\sigma d\xi_1\Big|\,d\xi \\
&\quad \leqq \int\int [|u(\xi')\,u(\xi_1')|+|u(\xi)\,u(\xi_1)|]\,|B_\varepsilon(\theta,\xi,\xi_1)-B_\eta(\theta,\xi,\xi_1)|\,d\theta d\sigma d\xi_1 d\xi \\
&\quad = 2\int\int |u(\xi)\,u(\xi_1)|\left|\frac{(\varepsilon-\eta)\sqrt{\xi^2+\xi_1^2}\,B(\theta,\xi,\xi_1)}{(1+\varepsilon\sqrt{\xi^2+\xi_1^2})(1+\eta\sqrt{\xi^2+\xi_1^2})}\right| d\theta d\sigma d\xi_1 d\xi \\
&\quad \leqq |\varepsilon-\eta|\,4\pi B_0\int |u(\xi)|\int |u(\xi_1)|\,(\sqrt{1+\xi^2}+\sqrt{1+\xi_1^2})^2\,d\xi_1 d\xi \\
&\quad \leqq |\varepsilon-\eta|\,16\pi B_0\int (1+\xi^2)\,|u(\xi)|\,d\xi\int (1+\xi_1^2)\,|u(\xi_1)|\,d\xi_1\,,
\end{aligned}$$

where we used (6) and (H). The second inequality follows in the same way.

We will be concerned with the following Cauchy problem

$$(\mathrm{I})\begin{cases}\dfrac{\partial u(\xi,t)}{dt}=f(u,u)\\ u(\xi,0)=u_0(\xi)\end{cases}$$

and will prove that, under proper assumptions on the initial datum (I) has a unique solution.

4. The Approaching Problem

Let us denote by $X_r (r=0,1,2)$ the Banach space of the measurable functions $u:\mathbb{R}^3\to\mathbb{R}$ such that

$$\|u\|_r=\int(1+\xi^2)^r\,|u|\,d\xi<+\infty$$

equipped with the norm $\|\ \|_r$ and by X_r^* its dual. By $\partial\|\ \|_r$ we denote the subdifferential of the norm in X_r $(r=0,1,2)$ and by $\langle\ ,\ \rangle_r: X_r\times X_r^*\to\mathbb{R}$ the function $(u,u^*)\to\langle u,u^*\rangle_r=\int(1+\xi^2)^r\,uu^*d\xi$.

Let Q be the closed convex cone $Q=\{u\in X_0, u\geqq 0 \text{ a.e.}\}$. For each $\varepsilon\in(0,1]$ we denote by f_ε the function $f_\varepsilon:Q\to X_0$, $u\to f_\varepsilon(u)=f_\varepsilon(u,u)$ and by f_0 the function $f_0:Q\cap X_1\to X_0$, $u\to f_0(u)=f(u,u)$.

We prove the lemma.

Lemma 6. *Let $u\in X_r$ then $\operatorname{sgn}u\in\partial\|u\|_r$, $(r=0,1,2)$.*

Proof. It is known that $\operatorname{sgn}u\in X_r^*$ (see [5]); moreover for each $v\in X_r$ we have $\|u+v\|_r=\langle u+v,\ \operatorname{sgn}(u+v)\rangle_r\geqq\langle u+v,\operatorname{sgn}u\rangle_r=\|u\|_r+\langle v,\operatorname{sgn}u\rangle_r$.

The following lemmas follow from (H), (6) and Lemmas 3,4 and 5.

Lemma 7. *We have*

(i) *if $u\in Q\cap X_1$ then for each $\varepsilon\in(0,1]$ $f_\varepsilon(u)\in X_1$ and $\langle f_\varepsilon(u),\operatorname{sgn}u\rangle_1=0$,*

(ii) *if $u\in Q\cap X_2$ then for each $\varepsilon\in(0,1]$ $f_\varepsilon(u)\in X_2$ and $\langle f_\varepsilon(u),\operatorname{sgn}u\rangle_2\leqq\gamma\|u\|_1\|u\|_2$,*

(iii) *if $u,v\in Q\cap X_2$ then for each $\varepsilon\in[0,1]$ $f_\varepsilon(u)$, $f_\varepsilon(v)\in X_1$ and $\langle f_\varepsilon(u)-f_\varepsilon(v),\operatorname{sgn}(u-v)\rangle_1\leqq 2\gamma\|u+v\|_2\|u-v\|_1$.*

Lemma 8. *For each $\varepsilon,\eta\in[0,1]$ we have*

(i) $\|f_\varepsilon(u)-f_\eta(u)\|_0\leqq 4\gamma|\varepsilon-\eta|\ \|u\|_1^2,\qquad \forall\, u\in Q\cap X_1,$

(ii) $\|f_\varepsilon(u)-f_\eta(u)\|_1\leqq 4\gamma|\varepsilon-\eta|\ \|u\|_1\|u\|_2,\qquad \forall\, u\in Q\cap X_2.$

Lemma 9. *For each $u,v\in Q\cap X_1$ we have $f_0(u)$, $f_0(v)\in X_0$ and*

$$\|f_0(u)-f_0(v)\|_0\leqq\gamma\|u-v\|_1.$$

Let us consider the initial value problem

$$(\mathrm{I}_\varepsilon)\begin{cases} \dfrac{du}{dt}=f_\varepsilon(u) & \varepsilon>0 \\ & t\in[0,T]\,. \\ u(0)=u_0 & \end{cases}$$

In [4] we proved the following theorem[5]:

Theorem I. *Assume that $u_0\in Q$. Then for each $T>0$ there exists a unique $u\in C^1(0,T;X)$ solution of* (I_ε); *moreover for each $t\in[0,T]$ we have $u(t)\in Q$, $\|u(t)\|_0=\|u_0\|_0$.*

The following theorem establishes further properties of the solution of (I_ε):

Theorem II. *Let $u_0\in Q\cap X_2$ and let u_ε be the solution of* (I_ε); *for each $t\in[0,T]$ we have:*

(i) $$\|u_\varepsilon(t)\|_1=\|u_0\|_1$$

(ii) $$\|u_\varepsilon(t)\|_2\leqq e^{\gamma\|u_0\|_1 t}\|u_0\|_2\,.$$

Proof. (i) and (ii) are an immediate consequence of (i) and (ii) of Lemma 7.

5. Existence of the Differentiable Solution

In this section we will investigate the existence of the solution of the following Cauchy problem

$$(\mathrm{I}_0)\begin{cases} \dfrac{du}{dt}=f_0(u) & \\ & t\in[0,T]\,. \\ u(0)=u_0\,, & \end{cases}$$

We begin with two lemmas.

Lemma 10. *Let $u_0\in Q\cap X_2$ and let u_ε be the solution of (I_ε); then there exists $u\in C(0,T;X_1)$ such that $\lim_{\varepsilon\to0}u_\varepsilon=u$ in $C(0,T;X_1)$.*

Proof. It follows from Lemma 1 that for each $\varepsilon,\eta\in(0,1]$ and a.e. $t\in[0,T]$ we have

$$\begin{aligned}\frac{d}{dt}\|u_\varepsilon(t)-u_\eta(t)\|_1&=\left\langle\frac{d}{dt}u_\varepsilon-\frac{d}{dt}u_\eta,\operatorname{sgn}(u_\varepsilon-u_\eta)\right\rangle_1\\&=\langle f_\varepsilon(u_\varepsilon)-f_\varepsilon(u_\eta),\operatorname{sgn}(u_\varepsilon-u_\eta)\rangle_1+\langle f_\varepsilon(u_\eta)-f_\eta(u_\eta),\operatorname{sgn}(u_\varepsilon-u_\eta)\rangle_1\\&\leqq4\gamma\|u_\varepsilon-u_\eta\|_1\,e^{\gamma\|u_0\|_1t}\|u_0\|_2+\|f_\varepsilon(u_\eta)-f_\eta(u_\eta)\|_1\end{aligned}$$

where we used (iii) of Lemma 7 and Theorem II.

[5] For a sketch of the proof see the appendix.

By Lemma 8 we obtain

$$\frac{d}{dt}\|u_\varepsilon - u_\eta\|_1 \leqq 4\gamma\|u_\varepsilon - u_\eta\|_1 e^{\gamma\|u_0\|_1 t}\|u_0\|_2 + 4\gamma|\varepsilon-\eta|\,\|u_0\|_1 e^{\gamma\|u_0\|_1 t}\|u_0\|_2 .$$

Set $\sigma_{\varepsilon,\eta} = 4\gamma T|\varepsilon-\eta|\,\|u_0\|_1\,\|u_0\|_2\,e^{\gamma\|u_0\|_1 T}$; we have $\lim_{\varepsilon,\eta\to 0}\sigma_{\varepsilon,\eta}=0$ and by Gronwall's lemma

$$\|u_\varepsilon(t)-u_\eta(t)\|_1 \leqq \sigma_{\varepsilon,\eta}\, e^{4\gamma\|u_0\|_2 T e^{\gamma\|u_0\|_1 T}} .$$

The proof is complete.

Lemma 11. *Let u_ε be the solution of* (I_ε) *and* $u=\lim_{\varepsilon\to 0} u_\varepsilon$ *the function given in Lemma 10. Then* $\lim_{\varepsilon\to 0} f_\varepsilon(u_\varepsilon)=f_0(u)$ *in* $C(0,T;X_0)$.

Proof. We have $u(t)\in Q\cap X_1$ for each $t\in[0,T]$ and:

$$\begin{aligned}\|f_\varepsilon(u_\varepsilon)-f_0(u)\|_0 &\leqq \|f_\varepsilon(u_\varepsilon)-f_0(u_\varepsilon)\|_0 + \|f_0(u_\varepsilon)-f_0(u)\|_0\\ &\leqq 4\gamma\varepsilon\|u_\varepsilon\|_1^2+\gamma\|u_\varepsilon-u\|_1\\ &\leqq 4\gamma\varepsilon\|u_0\|_1^2+\gamma\|u_\varepsilon-u\|_1 ,\end{aligned}$$

where we used (i) of Lemma 8, Theorem II and Lemma 9. The conclusion follows from Lemma 10.

We can now prove the following theorem:

Theorem III. *Assume that $u_0\in Q\cap X_2$; then for each $T>0$ the problem* (I_0) *has a solution $u\in C^1(0,T;X_0)$. Moreover for each $t\in[0,T]$ we have $u(t)\in Q$ and:*

$$\|u(t)\|_1=\|u_0\|_1 , \qquad \|u(t)\|_2\leqq e^{\gamma\|u_0\|_1 t}\|u_0\|_2 .$$

Proof. Let u_ε be the solution of (I_ε); by Lemmas 10 and 11 we have $\lim_{\varepsilon\to 0}u_\varepsilon=u$ in $C(0,T;X_0)$ and $\lim_{\varepsilon\to 0}f_\varepsilon(u_\varepsilon)=f_0(u)$ in $C(0,T;X_0)$; thus $\lim_{\varepsilon\to 0}\frac{du_\varepsilon}{dt}=\lim_{\varepsilon\to 0}f_\varepsilon(u_\varepsilon)=f_0(u)$. This implies $u\in C^1(0,T,X_0)$ and $\frac{du}{dt}=f_0(u)$. Moreover we have $u(0)=\lim_{\varepsilon\to 0}u_\varepsilon(0)=u_0$ and $\|u(t)\|_1=\lim_{\varepsilon\to 0}\|u_\varepsilon(t)\|_1=\|u_0\|_1$.

To prove the second estimate we use Fatou's lemma, $\|u_\varepsilon(t)\|_2 \leqq e^{\gamma\|u_0\|_1 t}\|u_0\|_2$ and $\lim_{\varepsilon\to 0}u_\varepsilon=u$ in $C(0,T;X_1)$ to conclude $\|u(t)\|_2\leqq e^{\gamma\|u_0\|_1 t}$ $\|u_1\|_2$.

Lemma 12. *Let $u_0, v_0\in Q\cap X_2$; if $u\in C^1(0,T,X_0)$ (respectively $v\in C^1(0,T;X_0)$) is a solution of problem* (I_0) *with initial datum u_0 (respectively v_0) and $u(t)\in Q$ ($v(t)\in Q$) for each $t\in[0,T]$, then*

(i) $\|u(t)\|_1=\|u_0\|_1 , \qquad \|v(t)\|_1=\|v_0\|_1 ,$

(ii) $\|u(t)\|_2\leqq e^{\gamma t\|u_0\|_1}\|u_0\|_2 , \qquad \|v(t)\|_2\leqq e^{\gamma t\|v_0\|_1}\|v_0\|_2 ,$

(iii) $\|u(t)-v(t)\|_1\leqq\|u_0-v_0\|_1\, e^{2\gamma t[\|u_0\|_2 e^{\gamma t\|u_0\|_1}+\|v_0\|_2 e^{\gamma t\|v_0\|_1}]}$

for each $t\in[0,T]$.

Proof. It follows from (6) and Lemma 3 that (i) and (ii) of Lemma 7 are formally satisfied by f_0; thus the assertions (i) and (ii) have the same proofs of Theorem II.

Concerning (iii) we have for a.e. $t \in [0, T]$:

$$\frac{d}{dt}\|u-v\|_1 = \langle f_0(u) - f_0(v), \operatorname{sgn}(u-v)\rangle_1 \leqq 2\gamma \|u+v\|_2 \, \|u-v\|_1$$

where we used Lemma 1 and (iii) of Lemma 7.

Now
$$\|u(0)-v(0)\|_1 = \|u_0 - v_0\|_1$$

thus the conclusion follows from (i), (ii) and Gronwall's lemma.

As a consequence of Lemma 12 we get the following result:

Theorem IV. *Let $u_0 \in Q \cap X_2$, then for each $T > 0$ the problem* (I_0) *has a unique solution.*

Combining Theorems III and IV we obtain an existence and uniqueness theorem for problem (I).

Theorem V. *Assume that u_0 satisfies the following hypothesis*

$$u_0(\xi) \geqq 0 \text{ a.e.}, \qquad \int (1+\xi^2)^2 u_0 \, d\xi < +\infty .$$

Then for each $T > 0$ there exists a unique $u : [0, T] \to L^1(\mathbb{R}^3)$, $u(t, \xi) \geqq 0$ a.e. $\xi \in \mathbb{R}^3$ for each $t \in [0, T]$, solution of problem (I). *Moreover we have*

$$\int (1+\xi^2)\, u(t,\xi)\, d\xi = \int (1+\xi^2)\, u_0(\xi)\, d\xi ,$$
$$\int (1+\xi^2)^2 u(t,\xi)\, d\xi \leqq \int (1+\xi^2)^2 u_0(\xi)\, d\xi \; e^{\gamma t \int (1+\xi^2) u_0 d\xi} \forall t \in [0, T] .$$

Appendix

For the completeness of our work we shall give an outline of the proof of Theorem I. This result is obtained in a more general case in [4] and requires the results of [7]. Two lemmas precede the main arguments.

Lemma 13. *Set $Q_r = \{u \in Q, \|u\|_0 \leqq r\}$; for each $\varepsilon \in (0, 1]$ there exists $N_\varepsilon(r)$ such that*

$$\|f_\varepsilon(u) - f_\varepsilon(v)\|_0 \leqq N_\varepsilon(r)\, \|u-v\|_0 , \qquad \forall\, u, v \in Q_r . \tag{9}$$

Proof. The estimate (9) is an immediate consequence of assumption (H) of Section 3.

Lemma 14. *For each $r > 0$ there exists $n(\varepsilon, r)$ such that the following equation*

$$nu - f_\varepsilon(u) = v , \qquad \varepsilon \in (0, 1] , \qquad v \in Q_r$$

has a unique solution $u \in Q_r$ for each $n \geqq n(\varepsilon, r)$.

Proof. Set

$$u_1 = \frac{v}{n}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$u_{k+1} = \frac{v + f_\varepsilon^+(u_k)}{n + \frac{1}{u_k} f_\varepsilon^-(u_k)}.$$

It is easy to see that $u_k \in Q_r$ and that for $n \geqq n(\varepsilon, r)$ there exists $\lim_{k \to \infty} u_k = u \in Q_r$. As f_ε is locally lipschitz continuous the conclusion follows.

By Lemma 14 we can define the "local resolvent" function in the following way:

$$R_{Q_r}(n, f_\varepsilon) = (n - f_\varepsilon)^{-1} : Q_r \to Q_r, \qquad n \geqq n(\varepsilon, r)$$

it is not difficult to see that

$$\text{(i)} \qquad \|R_{Q_r}(n, f_\varepsilon)\, u\|_0 \leqq \frac{1}{n} \|u\|_0, \qquad \forall u \in Q_r,$$

$$\text{(ii)} \qquad \|R_{Q_{nr}}(n, f_\varepsilon)\, nu - R_{Q_{nr}}(n, f_\varepsilon)\, nv\|_0 \leqq \frac{n}{n - N_\varepsilon(r)} \|u - v\|_0$$

for each $u, v \in Q_r$ and $n \geqq \max\{n(\varepsilon, r), N_\varepsilon(r)\}$.

For each fixed $r > 0$ and any integer $n \geqq n(\varepsilon, r)$ we define the Yosida's approximation function

$$f_{\varepsilon,n}^r = n R_{Q_{nr}}(n, f_\varepsilon)\, n - n = f_\varepsilon R_{Q_{nr}}(n, f_\varepsilon)\, n.$$

From the local lipschitz continuity of f_ε for each $r > 0$ we have

$$\text{(iii)} \qquad \lim_{n \to \infty} f_{\varepsilon,n}^r(u) = f_\varepsilon(u)$$

uniformly in Q_r.

Now consider the integral equation

$$(\mathrm{I}_n) \qquad u_n(t) = u_0 + \int_0^t f_{\varepsilon,n}^r(u_n)(s)\, ds, \qquad t \in [0, T], u_0 \in Q_r.$$

The following lemma holds:

Lemma 15. *For each* $u_0 \in Q_r$ *and* $n > \max\{n(\varepsilon, r), N_\varepsilon(r)\}$ *there exists a solution* $u_n(t)$ *of problem* (I_n) *such that* $\|u_n(t)\|_0 \leqq \|u_0\|_0$ *for each* $t \in [0, T]$.

Proof. Set $Y = C(0, T; X_0)$, $K = \{u \in Y, u(t) \in Q_{\|u_0\|_0}$ for $t \in [0, T]\}$ and consider the function $\tau_n : K \to Y$ defined by

$$(\tau_n(u))(t) = \exp(-nt)\, u_0 + n \int_0^t \exp(-n(t-s))\, R_{Q_{nr}}(n, f_\varepsilon)\, nu(s)\, ds.$$

Using (i) and (ii) it is not difficult to see that $\tau_n(K) \subset K$ and that for $T = T_0$ sufficiently small (depending on r and n) τ_n is a contraction which maps K into itself. Consequently there is a function $u_n \in K$ such that $\tau_n u_n = u_n$ and it is easy to see that u_n satisfies (I_n). Moreover as we have $\|u_n(T_0)\|_0 \leqq \|u_0\|_0$ we can consider the same problem (I_n) with the initial point u_0 replaced by $u_n(T_0)$ and this allows us to extend to the interval $[0, 2T_0]$ the solution of (I_n) with initial point u_0. Iterating this argument we extend the solution of (I_n) to $[0, T]$ with T arbitrarily given; obviously we have $\|u_n(t)\|_0 \leqq \|u_0\|_0$ for $t \in [0, T]$.

From (ii) and Gronwall's lemma it follows that there exists $u \in K$ such that $\lim_{n \to \infty} u_n = u$ in Y; thus, using (iii), we obtain:

$$(\mathrm{I}) \qquad u(t) = u_0 + \int_0^t f_\varepsilon(u)(s)\, ds\,, \qquad t \in [0, T]\,; \varepsilon \in (0, 1]\,.$$

Finally the uniqueness of the solution of problem (I) and its differentiability are a straightforward consequence of the local lipschitz continuity of f_ε.

The author thanks Profs. G. Da Prato and G. Gallavotti for stimulating discussions.

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Communicated by G. Gallavotti

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