Commun. math. Phys. 38, 103—110 (1974)

Faraday Transport in a Curved Space-Time

J. Madore

Laboratoire de Physique Théorique, Institut Henri Poincaré, Paris, France

Received October 30, 1973; in revised form May 17, 1974

Abstract. A covariant expression is given for the Faraday transport of electromagnetic radiation in a curved space-time.

I. Introduction and Notation

It is our purpose here to derive a covariant expression for the propagation of the polarization vector of a linearly polarized electromagnetic wave through a magnetic interstellar or intergalactic plasma (Faraday transport).

We shall consider the wave in the eikonal approximation and the plasma in the 3-fluid approximation.

Let ϱ, p, n, u^μ be the density, pressure, number density and 4-velocity of the electron component and let n_i, u_i^μ be the number density and 4-velocity of the ion component. We suppose that the electron component is a perfect fluid. Its energy-momentum tensor is therefore of the form $t_{\mu\nu} = (\varrho c^2 + p) u_\mu u_\nu - p g_{\mu\nu}$. We neglect collisions between the electrons and the ions and the molecules and we suppose that the degree of ionization remains constant.

We shall consider the wave to be a perturbation of the background potential A_μ of the form

$$\tilde{A}_\mu = A_\mu + \varepsilon A'_\mu . \tag{1.1}$$

The parameter ε characterizes the order of magnitude of the amplitude of the wave. We shall suppose that $\varepsilon \ll 1$ and we shall neglect quantities quadratic in ε.

In the presence of the wave all of the quantities which describe the electron fluid will be perturbed from their former value by a small amount of the order of ε. We shall write all perturbed quantities in the form (1.1).

Because of the relatively large mass of the proton, we shall suppose that the ionic component remains unperturbed in the presence of the wave:

$$\tilde{n}_i = n_i, \qquad \tilde{u}_i^\mu = u_i^\mu . \tag{1.2}$$

We consider the wave in the eikonal approximation and we write A'_μ as the product of an amplitude a_μ and a phase factor $e^{i\frac{\omega}{c}\phi}$:

$$A'_\mu = a_\mu e^{i\frac{\omega}{c}\phi} . \tag{1.3}$$

ω is a constant with the dimension of inverse time. a_μ and ϕ are functions of the point in space-time x^α; we suppose they are relatively slowly varying functions with a characteristic length L which we shall assume to be also the characteristic length of the background plasma. The derivative of A'_μ given by

$$\partial_\lambda A'_\mu = i\frac{\omega}{c}\xi_\lambda A'_\mu + A'_{\mu,\lambda}, \tag{1.4}$$

where

$$\xi_\lambda = \partial_\lambda \phi, \qquad A'_{\mu,\lambda} = (\partial_\lambda A'_\mu)_{\phi=\text{const}} .$$

We set

$$A'_{\mu;\lambda} = (D_\lambda A'_\mu)_{\phi=\text{const}} .$$

We define r by $\dfrac{dx^\alpha}{dr} = \xi^\alpha$ and we set $_{;\alpha}\xi^\alpha = \delta/\delta r$.

We set

$$\xi^2 = \xi \cdot \xi = \xi_\alpha \xi^\alpha \quad \text{etc.}$$

The perturbation of the fluid ϱ' etc. will also be products of an amplitude and a phase factor with the same phase as in (1.3). We shall use only the following property which follows from this form:

$$\frac{d\varrho'}{d\phi} = i\frac{\omega}{c}\varrho' \quad \text{etc.}$$

The frequency of the wave in the rest frame of the electron component is $\omega u \cdot \xi$. We can choose ϕ such that at a given point $u \cdot \xi = 1$. ω becomes then the frequency at that point. At any other point on the same ray $u \cdot \xi$ will in general not be equal to one because of the gravitational and kinematical red-shift corrections.

Define $\delta = c/\omega L$. The eikonal approximation is $\delta \ll 1$.

We shall impose the Lorentz gauge condition:

$$A^{\mu}{}_{;\mu}=0, \qquad D_{\mu}\tilde{A}^{\mu}=0\,.$$

Therefore we have

$$\xi \cdot a = O(\delta)\,. \tag{1.5}$$

At each point introduce two vectors k^{α} and l^{α} in the plane defined by u^{α}, ξ^{α}, such that

$$k \cdot u = 0, \;\; k^2 = -1, \;\; \xi \cdot l = 0, \;\; u \cdot l = 1\,. \tag{1.6}$$

Let $\hat{a}^{\mu}$ be the component of a^{μ} normal to u^{μ}. Then because of (1.5) we have

$$a^{\mu} = \hat{a}^{\mu} + a \cdot u\, l^{\mu} + O(\delta)\,. \tag{1.7}$$

The phase velocity v_p of the wave with respect to the electron fluid is given by

$$c^2/v_p^2 = 1 - \xi^2/(\xi \cdot u)^2\,. \tag{1.8}$$

Let $F_{\mu\nu}$ be the Maxwell tensor.

Because of the large value of the conductivity σ at low frequencies, the unperturbed Maxwell tensor may be written in the form

$$F_{\alpha\beta} = \sqrt{-g}\, \varepsilon_{\alpha\beta\gamma\delta} B^{\gamma} u^{\delta} \equiv B f_{\alpha\beta}\,, \tag{1.9}$$

where

$$B^{\alpha} \equiv B m^{\alpha}, \qquad m^2 = -1\,, \tag{1.10}$$

is the magnetic field in the rest frame of the electron component.

The specific enthalpy f of the electron fluid is defined by

$$\varrho c^2 + p = m c^2 n f\,. \tag{1.11}$$

m is the electron mass. $\omega_B = eB/mc$ is the Larmor frequency. We shall suppose that

$$\omega_B/\omega \ll 1 \tag{1.12}$$

and we shall neglect terms quadratic in ω_B/ω. The plasma frequency ω_p is defined by

$$\omega_p^2 = \frac{4\pi e^2 n}{m}\,. \tag{1.13}$$

II. Faraday Transport

The basic equations are Maxwell's equations:

$$D_{\nu} F^{\mu\nu} = 4\pi e (n u^{\mu} - n_i u_i^{\mu})\,, \tag{2.1 a}$$

the equations of motion of the electron component:

$$D_\nu t^{\mu\nu} = -enF^{\mu\nu}u_\nu\,, \tag{2.1b}$$

and the electron-number conservation equation:

$$D_\nu(nu^\nu) = 0\,. \tag{2.1c}$$

From these three equations, one derives, using (1.2), (1.9), the following equations for the first order perturbations:

$$D_\nu F'^{\mu\nu} = 4\pi e(n'u^\mu + nu'^\mu)\,, \tag{2.2a}$$

$$D_\nu t'^{\mu\nu} = -en(F'^{\mu\nu}u_\nu + F^{\mu\nu}u'_\nu)\,, \tag{2.2b}$$

$$D_\nu(n'u^\nu + nu'^\nu) = 0\,. \tag{2.2c}$$

The problem is to solve (2.2b, c) for n' and u'^μ and to place the solution in (2.2a). (2.2a) then becomes an equation for A'_μ.

It is straightforward to calculate the following expressions in the Lorentz gauge:

$$F'^{\mu\nu} = i\frac{\omega}{c}(\xi^{[\mu}a^{\nu]} + O(\delta))e^{i\frac{\omega}{c}\phi}\,, \tag{2.3}$$

$$D_\nu F'^{\mu\nu} = \left[\frac{\omega^2}{c^2}(\xi^2 a^\mu + O(\delta^2)) - i\frac{\omega}{c}\left(2\frac{\delta a^\mu}{\delta r} + a^\mu\xi^\lambda{}_{;\lambda}\right)\right]e^{i\frac{\omega}{c}\phi}\,, \tag{2.4}$$

$$\begin{aligned} D_\nu t'^{\mu\nu} = i\frac{\omega}{c}\,&[(\varrho' c^2 + p')u\cdot\xi u^\mu + (\varrho c^2 + p)(u'\cdot\xi u^\mu + u\cdot\xi u'^\mu) \\ &- p'\xi^\mu + O(\delta)]\,. \end{aligned} \tag{2.5}$$

Neglecting terms of order δ and using (1.11) we find then the following equations

$$\left[\frac{\omega^2}{c^2}\xi^2 a^\mu - i\frac{\omega}{c}\left(2\frac{\delta a^\mu}{\delta r} + a^\mu\xi^\lambda{}_{;\lambda}\right)\right]e^{i\frac{\omega}{c}\phi} = 4\pi e(nu'^\mu + n'u^\mu)\,, \tag{2.6}$$

$$\varrho' + mnf\frac{u'\cdot\xi}{u\cdot\xi} = 0\,, \tag{2.7}$$

$$\begin{aligned} mnfu'^\mu + \frac{p'}{c^2}\left(u^\mu - \frac{\xi^\mu}{\xi\cdot u}\right) + \frac{en}{c^2}\left(\frac{\xi^\mu}{\xi\cdot u}a\cdot u - a^\mu\right)e^{i\frac{\omega}{c}\phi} \\ = inm\frac{\omega_B}{\omega u\cdot\xi}f^{\mu\nu}u'_\nu\,, \end{aligned} \tag{2.8}$$

$$n'u\cdot\xi + nu'\cdot\xi = 0\,. \tag{2.9}$$

From (2.7) and (2.9), we find $p' = mnc^2 f'$; that is, the perturbation is isentropic.

Let v_s be the velocity of sound in the electron fluid:

$$p' = v_s^2 \varrho' . \tag{2.10}$$

Using (1.12) and (2.10), the system of Eqs. (2.6)–(2.9) yields in a straightforward manner the following equations for the transverse and longitudinal components of a^μ:

$$\begin{aligned}\left(\xi^2 - \frac{\omega_p^2}{f\omega^2}\right)\hat{a}^\mu - i\frac{c}{\omega}\left[2\frac{\delta \hat{a}^\mu}{\delta r} + \hat{a}^\mu \xi^\lambda{}_{;\lambda} + 2\hat{a}^\alpha \frac{\delta u_\alpha}{\delta r} l^\mu \right.& \\ \left. + \frac{\omega_B}{fcu\cdot\xi}\,\frac{\omega_p^2}{f\omega^2}(f^\mu{}_\nu + k^\mu k^\alpha f_{\alpha\nu})\hat{a}^\nu\right]& \\ = \frac{i\omega_p^2}{f\omega^2}\,\frac{\omega_B}{f\omega u\cdot\xi}\,\frac{v_p}{c}\,\frac{v_p^2 - c^2}{v_p^2 - v_s^2} a\cdot u f^{\mu\nu}k_\nu + \frac{2ic}{\omega} a\cdot u\left(\frac{\delta l^\mu}{\delta r} + l^\alpha \frac{\delta u_\alpha}{\delta r} l\mu\right),&\end{aligned} \tag{2.11}$$

$$\begin{aligned}\left(\xi^2 - \frac{\omega_p^2}{f\omega^2}\,\frac{v_p^2 - c^2}{v_p^2 - v_s^2}\right) a\cdot u - \frac{ic}{\omega}\left(2\frac{da\cdot u}{dr} + a\cdot u\,\xi^\lambda{}_{;\lambda} - 2a\cdot u\, l^\alpha \frac{\delta u_\alpha}{\delta r}\right)& \\ = -\frac{2ic}{\omega}\hat{a}^\alpha \frac{\delta u^\alpha}{\delta r} - \frac{i\omega_p^2}{f\omega^2}\,\frac{v_p}{c}\,\frac{\omega_B}{f\omega\xi\cdot u}\,\frac{c^2}{v_p^2 - v_s^2} f_{\alpha\beta}k^\alpha \hat{a}^\beta .&\end{aligned} \tag{2.12}$$

We now use explicitly the assumption that the wave is linearly polarized. We assume that the transverse part $\hat{a}^\mu$ of a^μ is real. Because of the hypothesis that $\delta \ll 1$, the Eqs. (2.11) and (2.12) must be valid as complex equations. If we equate therefore the real and imaginary parts to zero, use (1.12) and neglect again terms of order δ, we find the following equations:

$$\xi^2 = \omega_p^2 / f\omega^2 , \tag{2.13}$$

$$2\frac{\delta \hat{a}^\mu}{\delta r} + \hat{a}^\mu \xi^\lambda{}_{;\lambda} + 2\hat{a}^\alpha \frac{\delta u_\alpha}{\delta r} l^\mu = -\frac{\omega_B}{c}\,\frac{\omega_p^2}{f^2\omega^2 u\cdot\xi}(f^\mu{}_\nu + k^\mu k^\alpha f_{\alpha\nu})\hat{a}^\nu , \tag{2.14}$$

$$a\cdot u = -\frac{i\omega_B}{f\omega\xi\cdot u}\,\frac{v_p}{c}\,\frac{c^2}{c^2 - v_s^2} f_{\alpha\beta}k^\alpha \hat{a}^\beta . \tag{2.15}$$

Equation (2.13) is the dispersion relation for linearly polarized transverse radiation. The ξ-lines are time-like and describe a motion with group velocity $v_g = v_p^{-1}$. Thus the amplitude propagates, according to (2.14), with $v_g < c$.

The equation which interests us here is (2.14). If we write $\hat{a}^\mu$ in the form

$$\hat{a}^\mu = \hat{a} n^\mu, \; n^2 = -1, \tag{2.16}$$

then we find from (2.14) a conservation equation for the amplitude:

$$(\hat{a}^2 \xi^\lambda)_{;\lambda} = 0, \tag{2.17}$$

and a transport equation (Faraday transport) for the polarization vector n^μ:

$$\frac{\delta n^\mu}{\delta r} = -l^\mu n_\alpha \frac{\delta u^\alpha}{\delta r} - \frac{\omega_B}{2c} \frac{\omega_p^2}{f^2 \omega^2 u \cdot \xi} (f^\mu{}_\nu + k^\mu k^\alpha f_{\alpha\nu}) n^\nu . \tag{2.18}$$

In deriving this equation we have neglected terms of order $\omega_p^2 \delta/\omega^2$. Therefore, for it to be significant it is necessary that

$$\delta \ll \omega_B/\omega . \tag{2.19}$$

Also we obtained Eq. (2.15) by neglecting terms of order $\omega^2 \delta/\omega_p^2$ which would contribute, through the last term on the right-hand side of (2.11), a term of order $\omega^2 \delta^2/\omega_p^2$ to (2.14). Therefore, for (2.14) to be significant it is necessary that

$$\frac{\omega_B}{\omega} \frac{\omega_p^2}{\omega^2} \gg \frac{\omega^2}{\omega_p^2} \delta^2 . \tag{2.20}$$

However, considering the definition of δ, we have also from (2.18)

$$\frac{\omega_B}{\omega} \frac{\omega_p^2}{\omega^2} \gtrsim \delta . \tag{2.21}$$

Therefore, we must have

$$\delta \ll \frac{\omega_p^2}{\omega^2} \ll 1 . \tag{2.22}$$

The three inequalities which lead to Eq. (2.18) are (1.12), (2.19), and (2.22).

Consider the limit $\omega_p \to 0$. This is equivalent to the vacuum case since we have neglected the effect of the neutral matter. (We have set the dielectric constant and the permeability equal to one.) Equation (2.14) is no longer valid since we can no longer neglect the last term on the right-hand side of Eq. (2.11). However Eq. (2.13) yields

$$\xi^2 = 0, \tag{2.23}$$

and from (2.11) and (2.12) [or directly from (2.6)] we obtain

$$2\frac{\delta a^{\mu}}{\delta r}+a^{\mu}\xi^{\lambda}{}_{;\lambda}=0\,. \tag{2.24}$$

The decomposition of a^{μ} into transverse and longitudinal parts is no longer necessary since there is no dispersion. It is also no longer possible since we have no vector field u^{μ}. However, if we write $a^{\mu}=a\check{n}^{\mu}$, with $\check{n}^2=-1$, then we find that $a=\hat{a}$ and (2.17) remains valid. We find also from (2.24) that $\check{n}^{\mu}$ is transported parallelly along the ξ-lines.

Equation (2.23) was first obtained in a curved space-time by von Laue [1]. Equation (2.24) was obtained by Misner (see, for example [2]). Ehlers [3] has made a systematic study of the WKB approximation as applied to Maxwell's equations in a curved background metric.

III. Faraday Rotation

Introduce a unit vector field p^{μ} normal to k^{μ} and u^{μ} and such that $\delta p^{\mu}/\delta r$ is normal to n^{μ}. Let ψ be the angle between n^{μ} and p^{μ}; let α be the angle between B^{μ} and k^{μ}. Then

$$f_{\alpha\beta}n^{\alpha}p^{\beta}=\sin\psi\cos\alpha\,, \tag{3.1}$$

and (2.18) yields the following equation for ψ:

$$\frac{d\psi}{dr}=\frac{\omega_B u\cdot\xi}{2c}\;\frac{\omega_p^2}{f^2\omega^2(u\cdot\xi)^2}\cos\alpha\,. \tag{3.2}$$

This is the Faraday rotation formula (see for example [4]). The extra red-shift factor $u\cdot\xi$ in the numerator may be understood in the following way.

Let γ be the curve along which the light travels, from the point of emission to the point of absorption. Let s be the arc length along the electron-fluid flow lines. Then the total angle through which the polarization vector rotates is given by

$$\Delta\psi=\int_{\gamma}d\psi=\int_{\gamma}\frac{d\psi}{dr}dr=\int_{\gamma}\frac{d\psi}{ds}\,\frac{ds}{dr}dr\,. \tag{3.3}$$

In the last expression on the right-hand side of this equality, $d\psi/ds$ is the rate of Faraday rotation as seen by an observer at rest with respect to the electron fluid. Since we are neglecting curvature this must be equal to the corresponding expression in a flat space-time. The extra red-shift factor comes from $ds/dr=u\cdot\xi$.

The author would like to thank the referee for his critical comments.

References

1. von Laue, M.: Physik. Z. **XXI**, 659 (1920)
2. Isaacson, R. A.: Phys. Rev. **166**, 1263 (1968)
3. Ehlers, J.: Z. Naturforsch. **22**, 1328 (1967)
4. Ginzburg, V. L.: Propagation of electromagnetic waves in plasmas. Amsterdam (North Holland): 1961

Communicated by J. Ehlers

J. Madore
Lab. de Physique Théorique
Institut Henri Poincaré
11, rue Pierre and Marie Curie
F-75231 Paris Cedex 05, France