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Parametric Interactions and Scattering

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Abstract. We show how a variety of parametric Hamiltonians arise by a limiting procedure applied to a time-independent Hamiltonian. We then study one such Hamiltonian, that for a parametric frequency converter, in detail and find its associated Raman scattering matrix.

1. Introduction

We study the time evolution of a system with the Hamiltonian

$$H = H_0 + \omega J_3 + \frac{\lambda}{N}(J_- X + J_+ X^*) \tag{1.1}$$

defined on $\mathbb{C}^N \otimes \mathscr{F}$ where

(i) the operators J_3, $J_\pm$ on $\mathbb{C}^N$ satisfy the commutation relations

$$[J_3, J_\pm] = \pm J_\pm ; \quad [J_+, J_-] = 2J_3 . \tag{1.2}$$

(ii) H_0 is a self-adjoint operator on the Hilbert space $\mathscr{F}$;

(iii) X is an operator on $\mathscr{F}$ of a suitably regular type.

We take the initial state on $\mathbb{C}^N$ to be a pure superradiant state, that is

$$\varrho_A = |\xi_N\rangle \langle \xi_N| \tag{1.3}$$

where

$$J_3 \xi_N = \gamma_N N \xi_n ; \quad -\tfrac{1}{2} < \lim_{N\to\infty} \gamma_N \equiv \gamma < \tfrac{1}{2} . \tag{1.4}$$

If ϱ is the initial mixed state on $\mathscr{F}$ then the state at time t is defined by

$$T_t^{(N)}(\varrho) = \mathrm{tr}_{\mathbb{C}^N}[e^{-iHt}(\varrho_A \otimes \varrho) e^{iHt}] \tag{1.5}$$

this being a density matrix on $\mathscr{F}$. We are interested in finding the limit of this as $N \to \infty$. The limit if it exists is written as

$$T_t(\varrho) = \lim_{N\to\infty} T_t^{(N)}(\varrho) \tag{1.6}$$

and is for each t a positive trace-preserving linear map on the space of all density matrices on $\mathscr{F}$.

Before proceeding we discuss the situations in which this Hamiltonian arises.

1. The operators J_3, $J_\pm$ can arise through the consideration of a system of $(N-1)$ 2-level atoms. Each atom is described by operators $J_3^{(r)}$, $J_\pm^{(r)}$ acting on the r^{th} component of $\otimes^{N-1}\mathbb{C}^2$. Then

$$J_3 = \sum_{r=1}^{N-1} J_3^{(r)}, \qquad J_\pm = \sum_{r=1}^{N-1} J_\pm^{(r)} \tag{1.7}$$

have the desired commutators. These operators, and therefore the Hamiltonian H, commute with the total angular momentum J so one can restrict attention to a subspace of $\otimes^{N-1}\mathbb{C}^2$ with a particular value of J^2. The most important of these is the symmetric subspace, of dimension N, since it contains the states where all the atoms are excited or all unexcited. See [1, 2].

2. Alternatively one can consider the Hamiltonian

$$H = \omega_1 a_1^* a_1 + \omega_2 a_2^* a_2 + \lambda(a_1^* a_2 X + a_2^* a_1 X^*) \tag{1.8}$$

where $[a_r, a_s^*] = \delta_{rs}$ and all other commutators involving a_1, a_2 and X vanish. Writing

$$\begin{aligned} J_+ &= a_2^* a_1\,; & J_- &= a_1^* a_2 \\ J_3 &= \tfrac{1}{2}(a_2^* a_2 - a_1^* a_1)\,; & J &= \tfrac{1}{2}(a_2^* a_2 + a_1^* a_1) \end{aligned} \tag{1.9}$$

we get

$$H = (\omega_1 + \omega_2)J + (\omega_2 - \omega_1)J_3 + H_0 + \lambda(J_- X + J_+ X^*)\,. \tag{1.10}$$

Now the operator J commutes with the Hamiltonian H so restricting to a subspace where J is constant we obtain the required Hamiltonian up to a scalar. See [3].

3. In many applications one takes $\mathscr{F}$ to be a boson Fock space and H_0 to be a free Hamiltonian on $\mathscr{F}$. If

$$X = \gamma a(f) + \delta a^*(g) \tag{1.11}$$

then one has the Dicke maser model Hamiltonian, which has been intensively studied recently [2–10]. In most work $\mathscr{F}$ has been supposed to have only a finite number of modes, usually one, and the rotating wave approximation, $\gamma = 0$, has been made. In this case

$$H = \upsilon a^* a + \omega J_3 + \lambda(J_- a^* + J_+ a)\,. \tag{1.12}$$

In [5, 6, 8, 10], however, the rotating wave approximation is not needed and in [6, 10] the infinite mode case is studied. For single mode radiation in a cavity the appropriate coupling constant is $\lambda N^{-\frac{1}{2}}$ according to [2, 4] so the choice λN^{-1} of [9] corresponds to taking the weak coupling limit.

4. For a parametric device the choice is

$$X = \gamma a^*(f_1)\, a^*(f_2) + \delta a^*(g_1)\, a(g_2) \tag{1.13}$$

as we shall see. For $\delta = 0$ one has a parametric amplifier and for $\gamma = 0$ a parametric frequency converter [11, 12]. For single mode radiation in a cavity the appropriate coupling constant is indeed λN^{-1}, for the same reasons as in [2, 4].

2. The Limiting Procedure

We let $\mathscr{D} \subseteq \mathscr{F}$ be a common dense domain for H_0, X, X^* on which H_0 is essentially self-adjoint. We also suppose that $\mathscr{D}$ has a topology for which it is a Frechet space and that H_0, X, X^* are continuous from $\mathscr{D}$ to $\mathscr{F}$. In order to get started we suppose that

$$H^{(N)} = H_0 + \omega J_3 + \frac{\lambda}{N}(J_- X + J_+ X^*) \tag{2.1}$$

is essentially self-adjoint on the dense domain $\mathbb{C}^N \otimes \mathscr{D}$. By [13] this would be true if X and X^* were relatively bounded with respect on H_0 with sufficiently small relative bound.

As in [9] we take an orthonormal basis $\phi_1, \ldots, \phi_N$ of $\mathbb{C}^N$ such that

$$J_+ \phi_s = \sqrt{s(N-s)}\, \phi_{s+1} \tag{2.2}$$

$$J_- \phi_s = \sqrt{(s-1)(N-s+1)}\, \phi_{s-1} \tag{2.3}$$

$$J_3 \phi_s = \left(s - \frac{N}{2} - \frac{1}{2}\right) \phi_s\,. \tag{2.4}$$

The superradiant state ξ_N is equal to ϕ_p where

$$p - \frac{N}{2} - \frac{1}{2} = \gamma_N N \tag{2.5}$$

so that

$$\lim_{N \to \infty} (p/N) = \gamma + \tfrac{1}{2}\,. \tag{2.6}$$

Following the method of [9] we define a new basis by $e_r = \phi_{p+r}$ and rewrite

$$H^{(N)} = H_0 + \omega B_3^{(N)} + \lambda(B_-^{(N)} X + B_+^{(N)} X^*) \tag{2.7}$$

where for $1 - p \leqq r \leqq N - p$

$$B_+^{(N)} e_r = N^{-1} \sqrt{(r+p)(N-r-p)}\, e_{r+1} \tag{2.8}$$

$$B_-^{(N)} e_r = N^{-1} \sqrt{(r+p-1)(N-r-p+1)}\, e_{r-1} \tag{2.9}$$

$$B_3^{(N)} e_r = \left(r + p - \frac{N}{2} - \frac{1}{2}\right) e_r\,. \tag{2.10}$$

We now define operators B_3, $B_\pm$ on the space $l^2(\mathbb{Z})$ of square-summable bilateral sequences by

$$B_3 e_r = r e_r; \qquad B_\pm e_r = e_{r\pm 1} \tag{2.11}$$

where $\{e_r\}$ is the natural orthonormal basis of $l^2(\mathbb{Z})$. We let $l^2(\mathbb{Z}, \mathscr{F})$ denote the space of square-summable, $\mathscr{F}$-valued bilateral sequences.

Lemma 2.1. *Let the operator H^∞ on $l^2(\mathbb{Z}, \mathscr{F})$ given by*

$$H^\infty = H_0 + \omega B_3 + \lambda \sqrt{\tfrac{1}{4} - \gamma^2}\,(B_- X + B_+ X^*) \tag{2.12}$$

be essentially self-adjoint on the domain $\mathscr{L}$ of all $\mathscr{D}$-valued sequences of finite support. Then

$$\lim_{N\to\infty} \left\{ H^{(N)} - \left(p - \frac{N}{2} - \frac{1}{2} \right) \right\} = H^\infty \tag{2.13}$$

in the sense of generalised strong convergence.

Proof. We interpret $H^{(N)}$ as self-adjoint operators on $l^2(\mathbb{Z}, \mathscr{F})$ by defining $B_3^{(N)} e_r$ and $B_\pm^{(N)} e_r$ as above for $1 - p \leqq r \leqq N - p$ and as zero otherwise. Then $B_\pm^{(N)}$ are bounded operators and their norms are uniformly bounded with respect to N. This makes it easy to prove that

$$\lim_{N\to\infty} B_\pm^{(N)} = \sqrt{\tfrac{1}{4} - \gamma^2}\, B_\pm \tag{2.14}$$

in the strong operator topology. It is then easy to show that

$$\lim_{N\to\infty} \left\{ H^{(N)} \psi - \left(p - \frac{N}{2} - \frac{1}{2} \right) \psi \right\} = H^\infty \psi \tag{2.15}$$

for all $\psi \in \mathscr{L}$, and generalised strong convergence follows by [13].

We let U be the unitary operator from $L^2(-\pi, \pi)$ to $l^2(\mathbb{Z})$ given by

$$(U f)_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(\theta) e^{-in\theta}\, d\theta \tag{2.16}$$

and consider the operator K, on the space $L^2\{(-\pi, \pi), \mathscr{F}\}$ of square integrable periodic $\mathscr{F}$-valued functions, given by

$$K = (U^* \otimes 1)\, H^\infty (U \otimes 1)\,. \tag{2.17}$$

Formally, one may easily check that

$$\begin{aligned}(K\psi)(\theta) = & -i\omega \frac{\partial \psi}{\partial \theta} + H_0 \psi(\theta) \\ & + \lambda \sqrt{\tfrac{1}{4} - \gamma^2}\,(e^{-i\theta} X + e^{i\theta} X^*)\, \psi(\theta)\,.\end{aligned} \tag{2.18}$$

We take K to be defined on the domain of all continuous periodic functions $\psi:[-\pi,\pi]\to\mathscr{D}$ which have continuous periodic derivatives $\psi':[-\pi,\pi]\to\mathscr{F}$. To integrate K we need to assume the solubility of an associated time-dependent evolution equation on $\mathscr{F}$, namely

$$\begin{aligned}\frac{d\psi}{dt} &= -iH(t,\theta)\psi \\ &= -i\{H_0+\lambda\sqrt{\tfrac{1}{4}-\gamma^2}\,(e^{-i(\theta+\omega t)}X+e^{i(\theta+\omega t)}X^*)\}\,\psi\,. \qquad (2.19)\end{aligned}$$

Lemma 2.2. *Suppose that for all θ, $t\in\mathbb{R}$ there is an operator $W(\theta,t)$ on $\mathscr{F}$ such that*

(i) *$W(\theta,t)$ leaves $\mathscr{D}$ invariant and W is jointly continuous from $\mathbb{R}\times\mathbb{R}\times\mathscr{D}$ to $\mathscr{D}$;*

(ii) *both partial derivatives of W exist on $\mathscr{D}$ and are jointly continuous from $R\times R\times\mathscr{D}$ to $\mathscr{F}$;*

(iii) *for all $\psi\in\mathscr{D}$ and $t,\theta\in\mathbb{R}$*

$$\frac{\partial}{\partial t}W(\theta,t)\psi=-iH(\theta,t)\,W(\theta,t)\psi\,. \qquad (2.20)$$

Then H^∞ is essentially self-adjoint on $\mathscr{L}$, K is essentially self-adjoint on M and

$$(e^{-iKt}\psi)(\theta)=W(\theta-\omega t,t)\,\psi(\theta-\omega t) \qquad (2.21)$$

for all $\psi\in L^2\{[-\pi,\pi],\mathscr{F}\}$.

Proof. Writing $(U_t\psi)(\theta)$ for the right hand side of Eq. (2.21) we see that U_t leaves M invariant and

$$\begin{aligned}\frac{\partial}{\partial t}(U_t\psi)(\theta) &= -\omega\partial_1 W(\theta-\omega t,t)\,\psi(\theta-\omega t)\\ &\quad+\partial_2 W(\theta-\omega t,t)\,\psi(\theta-\omega t)\\ &\quad-\omega\,W(\theta-\omega t,t)\,\psi'(\theta-\omega t)\end{aligned} \qquad (2.22)$$

while

$$\begin{aligned}\frac{\partial}{\partial\theta}(U_t\psi)(\theta) &= \partial_1 W(\theta-\omega t,t)\,\psi(\theta-\omega t)\\ &\quad+W(\theta-\omega t,t)\,\psi'(\theta-\omega t)\,.\end{aligned} \qquad (2.23)$$

Therefore

$$\begin{aligned}\left(\frac{\partial}{\partial t}+\omega\frac{\partial}{\partial\theta}\right)(U_t\psi)(\theta) &= \partial_2 W(\theta-\omega t,t)\,\psi(\theta-\omega t)\\ &= -iH(\theta-\omega t,t)\,W(\theta-\omega t,t)\,\psi(\theta-\omega t) \qquad (2.24)\\ &= -iH(\theta-\omega t,t)\,(U_t\psi)(\theta)\end{aligned}$$

so on substituting the expression for $H(\theta, t)$

$$\frac{\partial}{\partial t} U_t \psi = -iKU_t \psi . \tag{2.25}$$

It follows from this that U_t is a unitary group. For if

$$\phi(t) = U_{t+s}\psi - U_t U_s \psi \tag{2.26}$$

where $\psi \in M$ then $\phi(t) \in M$ for all t and

$$\phi'(t) = -iK\phi(t) . \tag{2.27}$$

Therefore

$$\begin{aligned} \frac{d}{dt}\|\phi(t)\|^2 &= \langle \phi'(t) | \phi(t) \rangle + \langle \phi(t) | \phi'(t) \rangle \\ &= 0 \end{aligned} \tag{2.28}$$

so

$$\|\phi(t)\| = \|\phi(0)\| = 0 . \tag{2.29}$$

The proof that U_t is isometric on M is similar.

Since U_t is a unitary group and M is a dense invariant domain, K is essentially self-adjoint on M by [14] and $U_t = e^{-iKt}$. Finally, the domain $\hat{\mathscr{L}}$ of K corresponding to the domain $\mathscr{L}$ of H^∞ consists of the set of functions of the form

$$\psi(\theta) = \sum_{r+-n}^{n} e^{ir\theta} \psi_r \tag{2.30}$$

where $\psi_r \in \mathscr{D}$ for all n. Giving M its obvious Frechet space topology so that K is continuous from M to $L^2\{[-\pi, \pi], \mathscr{F}\}$ and is dense in M it follows that the closure of K on $\hat{\mathscr{L}}$ contains K restricted to M. Therefore K is essentially self-adjoint on $\hat{\mathscr{L}}$ and H^∞ is essentially self-adjoint on $\mathscr{L}$.

Theorem 2.3. *Under the hypotheses of Lemma 2.2,*

$$\lim_{N\to\infty} T_t^{(N)}(\varrho) = T_t(\varrho) \tag{2.31}$$

exists in the trace norm topology for all density matrices ϱ on $\mathscr{F}$ and

$$T_t(\varrho) = \frac{1}{2\pi} \int_{-\pi}^{\pi} W(\theta, t)\, \varrho\, W(\theta, t)^*\, d\theta . \tag{2.32}$$

Proof. It is sufficient to prove this when ϱ is a pure state by linearity and density arguments. If $\psi \in \mathscr{F}$ and

$$\psi_{N,t}(\theta) = \{(U^* \otimes 1) e^{-iH^{(N)}t}(e_0 \otimes \psi)\}(\theta) \tag{2.33}$$

then

$$\lim_{N\to\infty} \psi_{N,t}(\theta) = W(\theta-\omega t, t)\psi \tag{2.34}$$

because generalised strong convergence of self adjoint operators implies strong convergence of the corresponding unitary groups [13]. Therefore for all bounded operators A on $\mathscr{F}$

$$\begin{aligned} \mathrm{tr}[A T_t(|\psi\rangle\langle\psi|)] &= \lim_{N\to\infty} \mathrm{tr}[A T_t^{(N)}(|\psi\rangle\langle\psi|)] \\ &= \lim_{N\to\infty} \langle\psi_{N,t}|1\otimes A|\psi_{N,t}\rangle \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \langle W(\theta-\omega t, t)\psi|A|W(\theta-\omega t, t)\psi\rangle\, d\theta \\ &= \mathrm{tr}\left[A\frac{1}{2\pi}\int_{-\pi}^{\pi} W(\theta, t)|\psi\rangle\langle\psi| W(\theta, t)^*\, d\theta\right]. \end{aligned} \tag{2.35}$$

This proves weak convergence and convergence in the trace norm follows by Lemma 4.3 of [15].

3. The Scattering Matrix

We have shown that in the limit $N\to\infty$ the Hamiltonian H is equivalent to the time-dependent "parametric" Hamiltonian

$$H(\theta, t) = H_0 + \mu(e^{-i(\theta+\omega t)}X + e^{i(\theta+\omega t)}X^*) \tag{3.1}$$

on $\mathscr{F}$ where $\mu = \lambda\sqrt{\frac{1}{4}-\gamma^2}$ and θ is to be regarded as a random variable uniformly distributed over $[-\pi, \pi]$.

In case $\mathscr{F}$ is a boson Fock space and X is linear or quadratic in creation and annihilation operators, the integration of the evolution equation is easy in the Heisenberg picture [16, 17]. In the Schrödinger picture, however, the attempt to relate the time evolution to Glauber's coherent states and the P-representation causes some difficulties [17]. For the single mode case these were overcome in [18, 19] by the use of a family of generalised coherent states, which have been studied in a much more general, group-theoretical, context in [20]. To our knowledge, however, the infinite mode case has not been studied and in particular the scattering matrix has not been obtained.

The problem of integrating a time-dependent evolution equation is not trivial. In [9] we solved it for the Dicke maser model by writing down explicitly the evolution operators $W(\theta, t)$ of Lemma 2.2 and then verifying that they satisfied the required conditions; of course this approach is rarely possible. General techniques for integrating the evolution equation, at least for "nice" perturbations X, may be found in [21–23]. It is shown there that if, for example, X is bounded then the evolution equation is soluble. Moreover $\mathscr{D}(H_0)$ is a dense invariant domain for $W(\theta, t)$ and all the conditions of Lemma 2.2 are satisfied.

From now on we restrict to this case, which will turn out to be adequate for the treatment of the parametric frequency converter.

We write $X_1 = X^*$ and $X_{-1} = X$ so that

$$\begin{aligned} H(\theta, t) &= H_0 + \mu(e^{-i(\theta+\omega t)} X_{-1} + e^{i(\theta+\omega t)} X_1) \\ &= H_0 + \mu A(\theta, t). \end{aligned} \tag{3.2}$$

By [21] the solution of the evolution equation between times t_1 and t_2 is

$$\begin{aligned} W_\theta(t_2, t_1) &= e^{-iH_0(t_2-t_1)} - i\mu \int_{s=t_1}^{t_2} e^{-iH_0(t_2-s)} A(\theta, s) e^{-iH_0(s-t_1)} ds \\ &\quad + (-i\mu)^2 \int_{s=t_1}^{t_2} \int_{u=t_1}^{s} e^{-iH_0(t_2-s)} A(\theta, s) e^{-iH_0(s-u)} A(\theta, u) e^{-iH_0(u-t_1)} du\, ds \\ &\quad + \cdots \end{aligned} \tag{3.3}$$

where this series converges in norm for all t_1, t_2 and μ. Since $A(\theta, t)$ is the sum of two terms we can write the n^{th} term of the series as the sum of 2^n parts. For notational convenience if E is a sequence $E = \{\pm 1, \pm 1, \ldots, \pm 1\}$ we write $|E|$ for the length of the sequence and $\bar{E}$ for the sum of the terms of the sequence. Then

$$e^{iH_0 t} W_\theta(t, -t) e^{iH_0 t} = \sum_E (-i\mu)^{|E|} e^{i\bar{E}\theta} S_E(t) \tag{3.4}$$

where

$$\begin{aligned} S_E(t) = \int_{s_1=-t}^{t} \int_{s_2=-t}^{s_1} \cdots \int_{s_n=-t}^{s_{n-1}} & e^{iH_0 s_1} e^{iE_1\omega s_1} X_{E_1} e^{-iH_0(s_1-s_2)} e^{iE_2\omega s_2} X_{E_2} \cdots \\ & e^{-iH_0(s_{n-1}-s_n)} e^{iE_n\omega s_n} X_{E_n} e^{-iH_0 S_n} ds_1 \ldots ds_n \end{aligned} \tag{3.5}$$

and we have put $n = |E|$.

Theorem 3.1. *If*

$$\|S_E(t)\| \leqq \tau K^{|E|} \tag{3.6}$$

for all t and $S_E(t)$ converges strongly as $t \to \infty$ to a limit S_E then for sufficiently small μ the limit

$$T_\infty(\varrho) = \lim_{t\to\infty} e^{iH_0 t}[T_{2t}(e^{iH_0 t} \varrho e^{-iH_0 t})] e^{-iH_0 t} \tag{3.7}$$

exists in the trace norm for all density matrices ϱ on $\mathscr{F}$. $T_\infty(\varrho)$ may be written as a power series involving only even powers of μ.

Proof. We see from Eq. (3.4) that provided $2|\mu|K<1$

$$\sum_E (-i\mu)^{|E|} e^{i\bar{E}\theta} S_E = \lim_{t\to\infty} e^{iH_0 t} W_\theta(t,-t) e^{iH_0 t} \tag{3.8}$$

the limit being taken in the strong operator topology. Writing the left hand side as $S(\theta)$ and putting

$$B_{mn} = \sum_{|E|=m,\bar{E}=n} S_E \tag{3.9}$$

so that $B_{mn}=0$ unless $-m\leqq n\leqq m$ and $(m-n)$ is even, we get

$$S_\theta = \sum_{m=0}^{\infty} \sum_{n=-m}^{m} (-i\mu)^m e^{in\theta} B_{mn}. \tag{3.10}$$

Therefore, for all density matrices ϱ on $\mathscr{F}$

$$\lim_{t\to\infty} e^{iH_0 t}[T_{2t}(e^{iH_0 t}\varrho e^{-iH_0 t})]e^{-iH_0 t}$$

$$= \lim_{t\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \{e^{iH_0 t} W_\theta(t,-t) e^{iH_0 t}\}\, \varrho \{e^{iH_0 t} W_\theta(t,-t) e^{iH_0 t}\}^* \, d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} S_\theta \varrho S_\theta^* \, d\theta \tag{3.11}$$

$$= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{n=-\infty}^{\infty} (-i\mu)^p (i\mu)^q B_{pn} \varrho B_{qn}^*$$

$$= \sum_{m=0}^{\infty} (i\mu)^m \left[\sum_{r=0}^{m} \sum_{n=-m}^{m} (-1)^r B_{rn} \varrho B_{m-r,n}^* \right]. \tag{3.12}$$

It is clear that the sum in square brackets vanishes if m is odd. Therefore

$$T_\infty(\varrho) = \sum_{m=0}^{\infty} (-\mu^2)^m \left[\sum_{r=0}^{2m} \sum_{n=-2m}^{2m} (-1)^r B_{rn} \varrho B_{2m-r,n}^* \right]. \tag{3.13}$$

The above calculations prove the existence of the limit in Eq. (3.7) at least in the weak operator topology. Since S_θ is a strong limit of unitary operators, it is isometric. Therefore by Eq. (3.11)

$$\begin{aligned} \operatorname{tr}[T_\infty(\varrho)] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \operatorname{tr}[S_\theta \varrho S_\theta^*] \, d\theta \\ &= \operatorname{tr}[\varrho] \\ &= \lim_{t\to\infty} \operatorname{tr}[e^{iH_0 t} T_t(e^{iH_0 t}\varrho e^{-iH_0 t}) e^{-iH_0 t}]. \end{aligned} \tag{3.14}$$

It follows by Lemma 4.3 of [15] that convergence is actually in the trace norm.

Corollary 3.2. *Under the conditions of Theorem 3.1, if μ is small then*

$$T_\infty(\varrho)=\varrho+\mu^2\{S_{\{1\}}\varrho S^*_{\{1\}}+S_{\{-1\}}\varrho S^*_{\{-1\}}-S_{\{1,-1\}}\varrho-S_{\{-1,1\}}\varrho-\varrho S^*_{\{1,-1\}} - \varrho S^*_{\{-1,1\}}\}+O(\mu^4). \quad (3.15)$$

Proof. One substitutes into the first two terms of Eq. (3.13) the expressions for B_{mn} given in Eq. (3.9).

We comment that since

$$\mathrm{tr}[T_\infty(\varrho)]=\mathrm{tr}[\varrho] \quad (3.16)$$

this is also true in all orders of perturbation theory. Using Eq. (3.15) it follows that

$$S^*_{\{1\}}S_{\{1\}}+S^*_{\{-1\}}S_{\{-1\}}=S_{\{1,-1\}}+S_{\{-1,1\}}+S^*_{\{1,-1\}}+S^*_{\{-1,1\}} \quad (3.17)$$

a result which could also be proved directly.

4. The Parametric Frequency Converter

The above analysis is very general and subject to a variety of conditions which may or may not be satisfied in a particular case. We illustrate their usefulness now by considering in detail the case of the parametric frequency converter.

We take $\mathscr{F}$ as the boson Fock space with a single particle space $\mathscr{H}$ and let H_0 be a free Hamiltonian on $\mathscr{F}$ with a non-negative absolutely continuous spectrum, apart from the non-degenerate vacuum. We let $f\in\mathscr{H}$ be a test function; considered as localised in the neighbourhood of a microscopic system of N atoms. We take as Hamiltonian for the system of N atoms and the quantised field

$$H=H_0+\omega J_3+\frac{\lambda}{N}(J_++J_-)\,a^*(f)\,a(f). \quad (4.1)$$

After taking the limit $N\to\infty$ the parametric Hamiltonian on $\mathscr{F}$ is

$$H(\theta,t)=H_0+\mu(e^{-i(\theta+\omega t)}+e^{i(\theta+\omega t)})\,a^*(f)\,a(f) \quad (4.2)$$

where

$$\mu=\sqrt{\tfrac{1}{4}-\gamma^2} \quad (4.3)$$

and $(\gamma+\frac{1}{2})$ is the proportion of atoms initially in the excited state. This Hamiltonian is quadratic and commutes with the number operator. The only interesting part of the problem therefore consists of solving the evolution equation on the single particle subspace $\mathscr{H}$. On this subspace

$$H(\theta,t)=H_0+\mu(e^{-i(\theta+\omega t)}+e^{i(\theta+\omega t)})\,|f\rangle\langle f| \quad (4.4)$$

which is of the form discussed in Section 3 with

$$X_{\pm 1} = X = |f\rangle \langle f| . \tag{4.5}$$

In order to verify the conditions of Theorem 3.1 we need to suppose f has regularity properties of a type familiar from scattering theory [13].

Lemma 4.1. *Defining*

$$h(t) = \begin{pmatrix} \langle f | e^{-iH_0 t} f\rangle & \text{if} \quad t \geqq 0 \\ 0 & \text{if} \quad t < 0 \end{pmatrix} \tag{4.6}$$

suppose that

$$\|h\|_1 = \int_{\infty}^{\infty} |h(t)|\, dt < \infty . \tag{4.7}$$

Then for all $\phi \in \mathscr{H}$

$$\int_{\infty}^{\infty} |\langle f | e^{iH_0 t} \phi\rangle|^2\, dt \leqq c \|\phi\|^2 \tag{4.8}$$

and there is a dense subspace $\mathscr{N}$ *of* $\mathscr{H}$ *such that for all* $\phi \in \mathscr{N}$

$$\int_{\infty}^{\infty} |\langle f | e^{-iH_0 t} \phi\rangle|\, dt < \infty . \tag{4.9}$$

Proof. Let $\mathscr{M}$ be the linear subspace of $\mathscr{H}$ spanned by finite linear combinations

$$\phi = \sum_{r=1}^{n} \alpha_r e^{-iH_0 t_r} f \tag{4.10}$$

and let $\mathscr{N} = \mathscr{M} \oplus \mathscr{M}^{\perp}$. Let $\phi \in \mathscr{N}$ and specifically

$$\phi = \sum_{r=1}^{n} \alpha_r e^{-iH_0 t_r} f + g \tag{4.11}$$

where $g \in \mathscr{M}^{\perp}$. Then

$$\begin{aligned} \int_{-\infty}^{\infty} |\langle f | e^{-iH_0 t} \phi\rangle|\, dt &= \int_{-\infty}^{\infty} \left| \sum_{r=1}^{n} \alpha_r \langle f, e^{-iH_0(t+t_r)} f\rangle \right| dt \\ &\leqq 2 \sum_{r=1}^{n} |\alpha_r|\, \|h\|_1 < \infty . \end{aligned} \tag{4.12}$$

For the first inequality we let $\phi \in \mathscr{H}$ be arbitrary and put $\phi = \phi_1 + \phi_2$ where $\phi_1 \in \mathscr{M}^-$ and $\phi_2 \in \mathscr{M}^{\perp}$. In the subspace $\mathscr{M}^-$ we take H_0 in its spectral representation. Then Eq. (4.6) states that the Fourier transform of $|f|^2$ is in $L^1(R)$ and this implies that $|f|^2$ is bounded, so $f \in L^{\infty}(R)$.

By the Plancherel theorem the left hand side of Eq. (4.8) is equal to the square of the L^2 norm of $f \cdot \phi_1$, and this is clearly dominated by $\|f\|_\infty^2 \cdot \|\phi_1\|_2^2$, which is itself smaller than $\|f\|_\infty^2 \cdot \|\phi\|^2$.

Theorem 4.2. *Suppose that Eq.* (4.7) *is valid. Then*

$$\|S_E(t)\| \leqq c\|h\|_1^{n-1} \tag{4.13}$$

for all t, and the limit

$$S_E = \lim_{t\to\infty} S_E(t) \tag{4.14}$$

exists in the strong operator topology, being given by

$$\begin{aligned}\langle\psi|S_E\phi\rangle = \int_{s_1=-\infty}^{\infty} \cdots \int_{s_n=-\infty}^{\infty} &\langle\psi|e^{iH_0s_1}f\rangle e^{iE_1\omega s_1} h(s_1-s_2) e^{iE_2\omega s_2}\dots \\ &\dots h(s_{n-1}-s_n) e^{iE_n\omega s_n} \langle f|e^{-iH_0s_n}\phi\rangle\, ds_1 \dots ds_n\end{aligned} \tag{4.15}$$

where $n=|E|$ *and* $\phi, \psi \in \mathscr{H}$.

Proof. By Eq. (3.5) we have

$$\begin{aligned}\langle\psi|S_E(t)\phi\rangle = \int_{s_1=-t}^{t} \cdots \int_{s_n=-t}^{t} &\langle\psi|e^{iH_0s_1}f\rangle e^{iE_1\omega s_1} h(s_1-s_2)\dots \\ &\dots\langle f|e^{-iH_0s_1}\phi\rangle\, ds_1 \dots ds_n\,.\end{aligned} \tag{4.16}$$

Using Eqs. (4.7) and (4.8) we obtain by standard Fourier analysis that

$$|\langle\psi|S_E(t)\phi\rangle| \leqq \tau\|\phi\|\;\|\psi\|\;\|h\|_1^{n-1}\,. \tag{4.17}$$

Since $\phi, \psi \in \mathscr{H}$ are arbitrary Eq. (4.13) follows.

It is clear from Eqs. (4.15) and (4.16) that $S_E(t)$ converges to S_E at least in the weak operator topology. Since $S_E(t)$ are uniformly bounded in norm as $t\to\infty$ it is sufficient to prove strong convergence for ϕ in the dense subspace $\mathscr{N}$ of Lemma 4.1. If $\phi \in \mathscr{N}$ then

$$\begin{aligned}&\|S_E\phi - S_E(t)\phi\| \\ &\leqq \int_{\max|s_r|>t} \|f\|\;|h(s_1-s_2)\dots h(s_{n-1}-s_n)\langle f|e^{-iH_0s_n}\phi\rangle|\, ds_1\dots ds_n\,.\end{aligned} \tag{4.18}$$

Using Eqs. (4.7) and (4.9) one sees that this converges to zero as $t\to\infty$.

We have now verified for this model all the hypotheses of Theorem 3.1. Therefore the scattering operator T_∞, which describes the processes of coherent Raman scattering, exists and is given by Eqs. (3.13) or (3.15), where the operators S_E are given by Eq. (4.15). This is the main result of the paper, which we illuminate by providing some further information about the operators S_E.

Theorem 4.3.

$$S_E H_0 = (\omega\bar{E} + H_0) S_E \tag{4.19}$$

so S_E *commutes with* H_0 *if and only if* $\bar{E}=0$.

Proof. If $\phi, \psi \in \mathscr{H}$ then by Eq. (4.15)

$$\langle\psi|S_E e^{iH_0 t}\phi\rangle = \int_{s_1=-\infty}^{\infty} \int_{s_n=-\infty}^{\infty} \langle\psi|e^{iH_0 s_1} f\rangle \dots \langle f|e^{-iH_0(s_n-t)}\phi\rangle\, ds_1 \dots ds_n$$

$$= \int_{u_1=-\infty}^{\infty} \int_{u_n=-\infty}^{\infty} e^{i\bar{E}\omega t}\langle\psi|e^{iH_0(u_1+t)} f\rangle \dots \langle f|e^{-iH_0 u_n}\phi\rangle\, du_1 \dots du_n$$

$$= e^{i\bar{E}\omega t}\langle e^{-iH_0 t}\psi|S_E\phi\rangle\,. \tag{4.20}$$

Therefore

$$S_E e^{iH_0 t} = e^{i\bar{E}\omega t} e^{iH_0 t} S_E \tag{4.21}$$

which is equivalent to Eq. (4.19).

For the sake of completeness we write down the form of S_E when the spectral decomposition of H_0 is given. We suppose for simplicity that $\mathscr{H} = L^2(0, \infty)$ and that

$$(H_0\phi)(x) = x\phi(x) \tag{4.22}$$

for all ϕ in the domain of H_0. We also introduce the notation

$$E^r = E_1 + \dots + E_r \tag{4.23}$$

where as before E is a sequence $\{\pm 1, \dots, \pm 1\}$ of length $n = |E|$. We denote by $\hat{h}$ the Fourier transform of h.

Theorem 4.4. *For all* $\phi \in L^2(0, \infty)$

$$\begin{aligned}(S_E\phi)(x) = 4\pi^2 f(x)\,\hat{h}(x+\omega E^1)\,\hat{h}(x+\omega E^2)\dots \\ \dots \hat{h}(x+\omega E^{n-1})\,\overline{f(x+\omega E^n)}\,\phi(x+\omega\bar{E})\,.\end{aligned} \tag{4.24}$$

Proof. By Eq. (4.15)

$$\langle\psi|S_E\phi\rangle = \int_{s_1=-\infty}^{\infty} \int_{s_n=-\infty}^{\infty} \overline{\langle f|e^{-iH_0 s_1}\psi\rangle}\,\{e^{iE^1\omega(s_1-s_2)}h(s_1-s_2)\}\dots$$

$$\{e^{iE^{n-1}(s_{n-1}-s_n)}h(s_{n-1}-s_n)\}\,\{e^{i\bar{E}\omega s_n}\langle f|e^{-iH_0 s_n}\phi\rangle\}\,ds_1 \dots ds_n$$

$$= 4\pi^2 \int_{-\infty}^{\infty} \overline{(\psi\cdot\bar{f})(x)}\,\hat{h}(x+\omega E')\dots\hat{h}(x+\omega E^{n-1})\,(\phi\cdot\bar{f})\,(x+\omega\bar{E})\,dx$$

$$= 4\pi^2 \int_{-\infty}^{\infty} \overline{\psi(x)}\,f(x)\,\hat{h}(x+\omega E')\dots\hat{h}(x+\omega E^{n-1})\,\overline{f(x+\omega E^n)}\,\phi(x+\omega\bar{E})\,dx \tag{4.25}$$

which yields the result.

In order to obtain the first terms in the expansion of $T_\infty(\varrho)$ of Eq. (3.15) we only need a few of the S_E. In the above spectral representation these are

$$(S_{\{\pm 1\}}\phi)(x) = 4\pi^2 f(x)\,\overline{f(x\pm\omega)}\,\phi(x\pm\omega) \tag{4.26}$$

$$(S_{\{\mp 1,\pm 1\}}\phi)(x) = 4\pi^2 f(x)\,\hat{h}(x\mp\omega)\,\overline{f(x)}\,\phi(x)\,. \tag{4.27}$$

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