Commun. math. Phys. 34, 223—228 (1973)
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Non Quasi-free Classes of Product States of the C.C.R.-Algebra

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Received May 16. 1973

Abstract. Two exemples of pure states of Van Hove's Universal Receptacle in the boson case are presented with are not unitarily equivalent to any quasi-free state. In particular, it is shown that a discrete state is unitarily equivalent to some quasi-free state if and only if it is equivalent to the Fock state related to the chosen decomposition of the test function space.

I. Introduction

This paper is a continuation of a previous one [1] in which we showed that non discrete pure states of the Van Hove's Universal Receptacle in the fermion case are not unitarily equivalent to any quasi-free state. The situation in the boson case is a little more complicated. Indeed, the quasi-free states of the C.C.R.-algebra are of discrete and non discrete type [2]. If we restrict ourselves to a fixed basis of the test function space H_0 the discrete states are equivalent to a class of states which we called "physically pure" ones. Those "physically pure" states are different from the quasi-free states except for the Fock state, moreover there exist[1] non discrete states which are disjoint from every quasi-free state of the decomposition of H_0 we consider. But the question remained open as if we can state the same assertions considering all quasi-free states issued from any possible decomposition of H_0.

I.1. Notations

Let $(H_k)_{k\in\mathbb{N}}$ a countable family of two-dimensional real vector spaces, and $H=\bigoplus_{k\in\mathbb{N}} H_k$ the weak sum of the H_k's. ($H=\{\varphi\in H_0 \mid P_k\varphi=0$ for a finite number of k's$\}$, $H_0=\bigoplus_{k\in\mathbb{N}} H_k$ denoting the Hilbert sum).

Equipped with σ, a regular, antisymmetric, real bilinear form (H_0,σ) is a separable symplectic space.

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[1] The Klauder-McKenna-Woods criterion [3] provides examples of this, as $\Omega_k=1/\sqrt{2}\,\xi_k^1+1/\sqrt{2}\,\xi_k^2$. See notation further.

Let $\Delta(H_0, \sigma)$ denote the algebra generated by finite linear combinations of δ_ψ's, $\psi \in H_0$, such that:

$$\delta_\psi(\varphi) = 0 \quad \text{if} \quad \psi \neq \varphi$$
$$\delta_\psi(\psi) = 1$$

with the product law:

$$\delta_\psi \delta_\varphi = e^{-i\sigma(\psi,\varphi)} \delta_{\psi+\varphi}$$

and the involution:

$$\delta_\psi \mapsto \delta_\psi^* = \delta_{-\psi}.$$

Let $\mathscr{F}(H_0, \sigma)$ the set of states of $\Delta(H_0, \sigma)$. We define a norm on $\Delta(H_0, \sigma)$ by:

$$x \in \Delta(H_0, \sigma), \qquad \|x\| = \sup_{\omega \in \mathscr{F}(H_0, \sigma)} \sqrt{\omega(x^* x)}.$$

It is a C^*-algebra norm [4]. The closure of $\Delta(H_0, \sigma)$ will be denoted by $\Delta_0 \equiv \overline{\Delta(H_0, \sigma)}$ ($\Delta \equiv \overline{\Delta(H, \sigma)}$) and we shall call Δ_0 the C.C.R.-algebra and Δ the local C.C.R.-algebra.

For more details see [5] and [4]. Let $\mathscr{R}(H, \sigma)$ the set of non-degenerated representations π of $\Delta(H, \sigma)$ such that the mapping $\lambda \in \mathbb{R}$, $\lambda \mapsto \pi(\delta_{\lambda\psi})$ is strongly continuous. Let $\pi_k \in \mathscr{R}(H_k, \sigma)$ be an irreducible representation of $\Delta(H_k, \sigma)$ into the separable Hilbert space $\mathscr{H}_k$. There is only one complex structure J such that $JH_k = H_k$, $\forall k \in \mathbb{N}$, which defines a σ-permitted hilbertian form s on H. Let ω_k be such that $\omega_k(\delta_\psi) = \exp(-\frac{1}{2}s(\psi, \psi))$ with $\delta_\psi \in \Delta(H_k, \sigma)$. ω_k is a pure state of $\Delta(H_k, \sigma)$ ([5], (3.2.1) and (3.2.2)) to which corresponds, in the G.N.S. construction, the representation π_k, called the Schrödinger representation, and the cyclic vector $\xi_k \in \mathscr{H}_k$. Let $\pi = \bigotimes_{k \in \mathbb{N}} \pi_k$ and recall that each $\Omega = \bigotimes_{k \in \mathbb{N}} \Omega_k$, Ω_k being a unitary vector of $\mathscr{H}_k$, determines an incomplete tensor product $\mathscr{H}^\Omega = \bigotimes_{k \in \mathbb{N}}^{\mathscr{C}(\Omega)} \mathscr{H}_k$, with $\mathscr{C}(\Omega)$ the equivalence class of Ω for the relation $\approx$ ($\Omega \approx \Omega'$ iff $\sum_{k \in \mathbb{N}} |1 - (\Omega_k | \Omega'_k)| < +\infty$). Let π_Ω the irreducible representation such that $x \in \Delta$, $\pi_\Omega(x) = \pi(x) | \mathscr{H}^\Omega$.

I.2. Definitions

Definition I.2.1. The state $\omega_\Omega \equiv (\Omega | \pi_\Omega(\cdot)\Omega)$ will be called a state of Van Hove's Universal Receptacle *(V.H.U.R.-state)* relating to the decomposition $(H_k)_{k \in \mathbb{N}}$.

Let us denote by A_k the field operator, defined by

$$\pi_k(\delta_{\psi_k}) = e^{iA_k(\psi_k)}, \qquad \psi_k \in H_k.$$

We shall write the corresponding creation and annihilation operators, as:

$$a^+(\psi_k) = \tfrac{1}{2}(A_k(\psi_k) - iA_k(J\psi_k)),$$
$$a^-(\psi_k) = \tfrac{1}{2}(A_k(\psi_k) + iA_k(J\psi_k)).$$

We choose $\psi_k^1 \in H_k$, $\|\psi_k^1\|^2 = s(\psi_k^1, \psi_k^1) = 1$ and we shall use $a_k^+ = a^+(\psi_k^1)$, $a_k^- = a^-(\psi_k^1)$.

Recall that ξ_k is a cyclic vector corresponding to the state ω_k, and that $(\xi_k^n)_{n\in\mathbb{N}}$ with $\xi_k^n = 1/\sqrt{n!}(a_k^+)^n \xi_k$ is an orthonormal basis of $\mathscr{H}_k$. Any unitary Ω_k of $\mathscr{H}_k$ can be written $\Omega_k = \sum_{n\in\mathbb{N}} \alpha_k^n \xi_k^n \left(\sum_{n\in\mathbb{N}} |\alpha_k^n|^2 = 1 \ \forall\, k \in \mathbb{N}\right)$. From now we shall denote $\beta_k^n = |\alpha_k^n|^2$.

Definition I.2.2. A representation π_Ω (a state ω_Ω) is a *discrete* one if and only if $\sum_{\substack{(k,j,l)\in\mathbb{N}^3 \\ j\neq l}} \beta_k^j \beta_k^l < +\infty$. If this series does not converge $\pi_\Omega(\omega_\Omega)$ is called a *continuous* representation (state).

This is the terminology of [6].

Definition I.2.3. A state ω_Ω will be called a *"physically pure"* one if $\alpha_k^n = 0$, $\forall\, n \neq m(k)$.

Corollary I.2.4. [2, Proposition 4.2]. *There exists a physically pure state $\omega_{\Omega'}$ unitarily equivalent to ω_Ω iff ω_Ω is a discrete state.*

Definition I.2.5. A *quasi-free state* on Δ is a state ω for which $\omega(\delta_\varphi) = \exp(-\tfrac{1}{2}s'(\varphi, \varphi) + i\chi(\varphi))$, $\forall\, \varphi \in H$ with s' a σ-allowed hilbertian structure on H and χ in the algebraic dual of H. Cf. [7–9].

There is only one Fock state ω_J among the V.H.U.R.-states related to the decomposition $(H_k)_{k\in\mathbb{N}}$. The discrete quasi-free states are all unitarily equivalent to this Fock state, and they have χ continuous [2, (4.3) and (4.6)].

II. Characterization of the Discrete States and an Example of a Class of non Quasi Free Continuous States

II.1. Discrete Case

Recalling that every-discrete V.H.U.R.-state is unitarily equivalent to a "physically pure" state, we can restrict ourself to consider the "physically pure" states.

Let ω_Ω a "physically pure" state which is disjoint from the Fock state ω_J related to the decomposition $(H_k)_{k\in\mathbb{N}}$ of H_0 that we fixed. Then $\Omega = \bigotimes_{k\in\mathbb{N}} \xi_k^{m(k)}$.

Let $\omega_{s',\chi}$ a pure quasi-free state on Δ, i.e. $\omega_{s',\chi}$ is such that $\omega_{s',k}(\delta_\varphi) = e^{i\chi(\varphi)} e^{-\frac{1}{2}s'(\varphi,\varphi)}$ with $\varphi \in H$ and s' a σ-allowed hilbertian structure on H ($s' = -\sigma \circ J'$, J' a complex structure on H). Via G.N.S. we obtain from $\omega_{s',\chi}$ the Gelfand troïka $(\mathscr{H}_{s'}, \pi_{s'}, \Xi_{s'})$, such that $\forall x \in \Delta$, $\omega_{s',\chi}(x) = (\Xi_{s'} | \pi_{s'}(x) \Xi_{s'})$, $\Xi_{s'} = \bigotimes_{k \in \mathbb{N}} \Xi_k$, $\Xi_k = \sum_{n \in \mathbb{N}} \alpha_k'^n \xi_k^n$, $\alpha_k'^n = \exp\left(-\frac{|c_k|^2}{2}\right) \frac{c_k^n}{\sqrt{n!}}$, $c_k \in \mathbb{C}$, $\beta_k'^n = |\alpha_k'^n|^2$ [2]. To the representation $\pi_{s'}$ corresponds in the Gårding-Wightman classification [10] the measure ν_χ on $\mathbb{N}^{\mathbb{N}}$. If $\omega_{s',\chi}$ is unitarily equivalent to ω_Ω, $\omega_{s',\chi}$ is a discrete state and therefore it is unitarily equivalent to the Fock state of the decomposition of H related to s'. We can choose $c_k \neq 0$, $\forall k \in \mathbb{N}$. The measure ν_χ can be described as $\nu_\chi = \bigotimes_{k \in \mathbb{N}} \nu_k$ with ν_k a measure on $\mathbb{N}$ and $\nu_k(\{n\}) = \beta_k'^n = \exp(-|c_k|^2)|c_k|^{2n}/n!$ $c_k \neq 0$ $\forall k \in \mathbb{N}$, thus we have a quasi-invariant measure.

Let

$$L_{k,n} = \{m \in \mathbb{N}^{\mathbb{N}} | m(k) = n\}$$

$$\nu_\chi(L_{k,n}) = \beta_k'^n = \exp(-|c_k|^2) \frac{|c_k|^{2n}}{n!}$$

$m(k) \geqq 1$ for an infinite collection of M of k's, thus:

$$\nu_\chi(L_{k,m(k)}) < \frac{1}{\sqrt{2\pi}} < 1$$

for those k's. Let

$$L^m = \bigcap_{k \in M} L_{k,m(k)} . \qquad M_p = M \cap \{1, \ldots, p\}$$

$$\nu_\chi(L^m) = \inf_{p \in \mathbb{N}} \nu_\chi\left(\bigcap_{k \in M_p} L_{k,m(k)}\right) = 0 .$$

Yet, let π_Ω be the representation constructed via G.N.S. from ω_Ω and μ_Ω the measure on $\mathbb{N}^{\mathbb{N}}$ corresponding to π_Ω in the Gårding-Wightman classification. We can choose $\Omega'' \sim \Omega$ [2] with $\Omega'' = \bigotimes_{k \in \mathbb{N}} \Omega_k''$,

$$\Omega_k'' = \sum_{n \in \mathbb{N}} \gamma_k^n \xi_k^n \quad \text{and} \quad \gamma_k^n \neq 0 \ \ \forall (n,k) \in \mathbb{N}^2 ,$$

$$\gamma_k^n = \varepsilon_{kn} \quad \text{if} \quad n \neq m(k), \qquad \varepsilon_k = \sum_n^\infty \varepsilon_{kn} ,$$

$$\sum_{k \in \mathbb{N}} \varepsilon_k < +\infty , \quad \text{and} \quad \gamma_k^{m(k)} = 1 - \varepsilon_k .$$

[2] $\sim$ is the weak equivalence of C_0-vectors defined by von Neumann [11].

Then $\mu_\Omega = \bigotimes_{k\in\mathbb{N}} \mu_k$, μ_k a measure on $\mathbb{N}$ and $\mu_k(\{n\}) = |\gamma_k^n|^2$

$$\mu_\Omega(L_{k,m(k)}) = 1 - \varepsilon_k$$

and

$$\mu_\Omega(L^m) = \inf_{p\in\mathbb{N}} \mu_\Omega\Big(\bigcap_{k\in M_p} L_{k,m(k)}\Big) = \Big(\prod_{k\in\mathbb{N}} (1-\varepsilon_k)\Big)^2 > 0 .$$

Therefore ν_χ and π_Ω cannot be equivalent. From ([12], Theorem 1.3, quoted by [10]) we can conclude ω_Ω is not unitarily equivalent to $\omega_{s',\chi}$ and summarize:

Proposition II.1.1. *The discrete* V.H.U.R.-*states on* Δ *related to a decomposition* $(H_k)_{k\in\mathbb{N}}$ *of* H *are either equivalent to the Fock state* ω_J *of this decomposition, or disjoint from any quasi-free state on* Δ.

Example. $\omega_{\bigotimes_{k\in\mathbb{N}} \xi_k^1}$ (one particle in each mode) is not unitarily equivalent to any quasi-free state of Δ.

II.2. Continuous Case

Consider a non discrete state ω_Ω such that:

$$\exists l_0 \in \mathbb{N} \quad \forall k \in \mathbb{N} \quad \alpha_k^n = 0 \quad \text{if} \quad n \notin [1, l_0] .$$

Let $L_k = \{m \in \mathbb{N}^{\mathbb{N}} \mid 1 \leqq m(k) \leqq l_0\}$ and $L = \bigcap_{k\in\mathbb{N}} L_k$. We can choose $\Omega'' \sim \Omega$ where

$$\Omega'' = \bigotimes_{k\in\mathbb{N}} \Omega_k'', \qquad \Omega_k'' = \sum_{n\in\mathbb{N}} \gamma_k^n \xi_k^n, \qquad \gamma_k^n \neq 0 \;\; \forall (n,k) \in \mathbb{N}^2 ,$$

and

$$|\gamma_k^0|^2 + \sum_{n}{}_{l_0+1}^{\infty} |\gamma_k^n|^2 = \varepsilon_k, \qquad \sum_{k\in\mathbb{N}} \varepsilon_k < +\infty ,$$

μ_Ω is as in (II.1) such that

$$\mu_\Omega = \bigotimes_{k\in\mathbb{N}} \mu_k, \qquad \mu_k(\{n\}) = |\gamma_k^n|^2 ,$$

$$\mu_\Omega(L_k) = 1 - \varepsilon_k ,$$

$$\mu_\Omega(L) = \inf_{p\in\mathbb{N}} \mu_\Omega\Big(\bigcap_{k^1}^{p} L_k\Big) = \prod_{k\in\mathbb{N}} (1-\varepsilon_k) > 0 .$$

Let $\omega_{s',\chi}$ a pure quasi-free state on Δ. Let $(\mathscr{H}_{s'}, \pi_{s'}, \Xi_{s'})$ and ν_χ its corresponding Gelfand troïka and Gårding-Wightman measure. $\Xi_{s'} = \bigotimes_{k\in\mathbb{N}} \Xi_k$,

$$\Xi_k = \sum_{n\in\mathbb{N}} \alpha_k'^n \xi_k^n, \qquad \alpha_k'^n = \exp(-|c_k|^2/2)\, c_k^n/\sqrt{n!} ,$$

$$c_k \neq 0, \qquad \forall k \in \mathbb{N}, \qquad \beta_k'^n = |\alpha_k'^n|^2 .$$

$$\forall k \in \mathbb{N} \qquad \nu_\chi(L_k) = \sum_{n}{}_{1}^{l_0} \beta_k'^n \leqq 1 - e^{-u_0} < 1$$

where $u_0 = (l_0!)^{1/l_0}$. Hence:

$$v_\chi(L) = \inf_{p \in \mathbb{N}} v_\chi\left(\bigcap_{k^1}^{p} L_k\right) = 0 .$$

We can state:

Proposition II.2.1. *The non-discrete* V.H.U.R.*-states* ω_Ω *on* Δ *such that*

$$\exists\, l_0 \in \mathbb{N}, \quad \forall\, k \in \mathbb{N}, \quad \forall\, n \in \mathbb{N} - [1, l_0], \quad \alpha_k^n = 0$$

are disjoint from any quasi-free state on Δ.

Example.

$$\omega_{\underset{k \in \mathbb{N}}{\otimes} \Omega_k}, \qquad \Omega_k = 1/\sqrt{2}\,\xi_k^1 + 1/\sqrt{2}\,\xi_k^2 .$$

Conclusion

We have stated that unitary equivalence to the quasi-free states is not typical for product states of the C.C.R.-algebra.

Acknowledgements. The author wishes to thank J. Manuceau, M. Sirugue, D. Testard, A. Verbeure and E. J. Woods for fruitful discussions.

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