Commun. math. Phys. 34, 179—192 (1973)

Lorentz Covariance of the $P(\varphi)_2$ Quantum Field Theory without Higher Order Estimates

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Received October 25, 1972; in revised form April 18, 1973

Abstract. We give a simple proof of Lorentz covariance for the $P(\varphi)_2$ model without using the higher order estimates: For each Poincaré transformation $\{a, \Lambda\}$ and each bounded region B of Minkowski space there exists a unitary operator U which correctly transforms the Heisenberg picture field operator: $U\varphi(f)\, U^* = \varphi(f_{\{a,\Lambda\}})$, $f \in C_o^\infty(B)$.

I. Introduction

The Lorentz covariance of boson field theories in two dimensional space-time was first studied by Cannon and Jaffe [1] for the $(\varphi^4)_2$ model in the sense of Haag-Kastler axioms [4]. Their results were extended to the $P(\varphi)_2$ by Rosen [9]. In each case, higher order estimates were used to study the corresponding models. It is well known that most of the results for the $P(\varphi)_2$ model can be obtained by using the hypercontractive property of the semi-group e^{-tH_0} [2, 3, 5, 11]. Recent results by Klein have shown the self-adjointness of the locally correct generator of Lorentz transformation for the $P(\varphi)_2$ interaction by introducing the $L_2(Q, d\mu)$ representation of Fock space $\mathscr{F}$ [7].

The main purpose of this paper is to simplify the proof of Lorentz covariance for the $P(\varphi)_2$ interaction by using the hypercontractive properties of the semigroups generated by the locally correct Hamiltonian and Lorentzian. We shall follow the method developed by Cannon and Jaffe [1]. However, we are able to prove the main theorems of references [1] and [9] using only hypercontractive semi-groups; we don't use the higher order estimates.

The locally correct Hamiltonian we shall consider has the form

$$H(g) = H_0 + H_I(g) \tag{1.1}$$

with

$$H_I(g) = \int :P(\varphi(x)): g(x)\, dx\,, \tag{1.2}$$

where H_0 is the free boson Hamiltonian, $P(\alpha)$ a polynomial of degree $2n$ with positive leading coefficient, $\varphi(x)$ the free boson field at time $t=0$ and $g(x)\in L_1(R)\cap L_2(R)$ is a positive function. Then $H(g)$ is essentially self-adjoint on $D(H_0)\cap D(H_I(g))$ and bounded below [2, 3, 5]. The Heisenberg picture field operators $\varphi(f)$ formally given by

$$\varphi(f)=\int e^{itH(g)}\varphi(x)\,e^{-itH(g)}f(x,t)\,dx\,dt \tag{1.3}$$

are essentially self-adjoint on any core for $(H(g)+b)^{\frac{1}{2}}$ [3], provided $f=\bar{f}\in C_0^\infty(B)$ for B any bounded open subset in space-time.

Let $\mathscr{A}(B)$ be the von Neumann algebra generated by the operators

$$\{e^{i\varphi(f)}: f=\bar{f}\in C_0^\infty(B)\}\,.$$

Let $\{a, \Lambda_\beta\}$ be a Poincaré transformation of two space-time dimensions defined by

$$\{a,\Lambda_\beta\}(x,t)=(x\cosh\beta+t\sinh\beta+\alpha,\, x\sinh\beta+t\cosh\beta+\tau) \tag{1.4}$$

where $a=(\alpha,\tau)$. For functions $f(x,t)$

$$f_{\{a,\Lambda_\beta\}}(x,t)=f(\{a,\Lambda_\beta\}^{-1}(x,t))\,. \tag{1.5}$$

The main result [1, Theorem 2.1.1 and 9, Theorem 1.1] is

Theorem 1.1. *Let $\{a,\Lambda\}$ be a Poincaré transformation. The transformation*

$$\varphi(x,t)\to\varphi(\{a,\Lambda\}(x,t)) \tag{1.6}$$

is locally unitarilly implemented in $\mathscr{F}$. That is, for every bounded set $R\subset R^2$, there exists a unitary operator U_B such that, for all $f\in C_0^\infty(B)$,

$$U_B\varphi(f)\,U_B^*=\varphi(f_{\{a,\Lambda\}})\,. \tag{1.7}$$

The Lorentz covariance of the field operators can be extended to the case of the algebra $\mathscr{A}(B)$. According to Cannon and Jaffe [1, Section 2.2] the problem reduces to the case of pure Lorentz rotations for the bounded region B' such that $B'\cup\Lambda B'\subset B_I$, where B_I is the causal shadow of any interval $I=[a,b]$ in $R^+=\{x>0\}$. This is, for $f\in C_0^\infty(B_I)$, supp $f\cup$ supp $f_{\Lambda_\beta}\subset B_I$, there is a unitary operator U such that

$$U\varphi U^*=\varphi(f_{\Lambda_\beta})\,. \tag{1.8}$$

The above reduction is a consequence of the space-time covariance of the field operators. For more detailed discussion of the connection between the above statement and Theorem 1.1 we refer the reader to Cannon and Jaffe [1]. In proving the theorem, we follow closely the notation used in references [1, 2, 7].

II. The Locally Correct Hamiltonian and Lorentzian

In this section we summarize some well-known results given in references [3, 4, 5] on the locally correct Hamiltonian and the generator of Lorentz rotations (Lorentzian), and we also prove some useful relations between $H(g)$ and the Lorentzian which we use in the following section.

We introduce the spectral representation of Fock space $\mathscr{F}$ with respect to the maximal abelian algebra generated by the spectral projections for free field operators. $\mathscr{F}$ is then represented as $L_2(Q, dq)$ with probability measure dq. In this space the Fock vacuum Ω_0 is represented by the function 1 and the algebra generated by the spectral projections of free field operators is the algebra of bounded multiplication operators $L_\infty(Q, dq)$.

We first state the known results for H_0 and $H(g)$. The reader may find these results in the references (See Proposition II.2, Proposition II.17, and Theorem II.16 in Ref. [5] and Lemma A.2 in Ref. [10]).

Lemma 2.1. (a) *e^{-tH_0} is a contractive semi-group in $L_p(Q, dq)$ for all $p \geqq 1$ and $t \geqq 0$.*

(b) *e^{-tH_0} is a strongly continuous semi-group for $1 \leqq p < \infty$.*

(c) *For $2 \leqq p < \infty$ there exist $t_0(p) \geqq 0$ such that for $t \geqq t_0(p)$, e^{-tH_0} is a bounded map from $L_2(Q, dq)$ to $L_p(Q, dq)$.*

Proposition 2.2. (a) *$H(g) = H_0 + H_I(g)$ is essentially self-adjoint on $D(H_0) \cap D(H_I(g))$ and bounded below.*

(b) *$e^{-tH(g)}$ is bounded in $L_p(Q, dq)$ for all $t \geqq 0$ and $1 < p < \infty$.*

(c) *For $2 \leqq p < \infty$ there exists $T_0(p) \geqq 0$ such that for $t \geqq T_0(p)$ $e^{-tH(g)}$ is bounded map from $L_2(Q, dq)$ to $L_p(Q, dq)$.*

Also we shall need:

Lemma 2.3. (a) *$C^\infty(H_0) = \bigcap_{n=1}^{\infty} D(H_0^n)$ is a core for $H(g)$.*

(b) *For $T \geqq T_0(p)$ for any $p > 2$, $e^{-TH(g)} L_2(Q, dq) \subset D(H_0)$.*

Remark. Lemma 2.3 (a) was proved by Simon [10]. However, we give here a slightly different proof based on the techniques we will be using later.

Proof. (a) Let $D_t = e^{-tH(g)} L_\infty(Q, dq)$. Then $D_t \subset L_p(Q, dq)$ for all $p < \infty$ and $D_t \subset D(H_0) \cap D(H_I(g))$. (See proof of Theorem II.16, Ref. [5].) Also D_t is a core for $H(g)$. For any $\varphi \in D_t$, let $\varphi_\varepsilon = e^{-\varepsilon H_0}\varphi$. Then $\varphi_\varepsilon \xrightarrow{s} \varphi$ in $L_p(Q, dq)$ for $p < \infty$ by Lemma 2.1 (b), and

$$\|H(g)(\varphi_\varepsilon - \varphi)\| \leqq \|H_0(\varphi_\varepsilon - \varphi)\| + \|H_I(g)(\varphi_\varepsilon - \varphi)\| \leqq \|(1 - e^{-\varepsilon H_0}) H_0 \varphi\| + \|H_I(g)\|_4 \|\varphi_\varepsilon - \varphi\|_4 \to 0 \quad \text{as} \quad \varepsilon \to 0 .$$

Since $e^{-\varepsilon H_0} D_t \subset C^\infty(H_0)$, this proves (a).

(b) For any $\psi \in L_2(Q, dq)$, choose a sequence of vectors $\{\psi_i\}$, $\psi_i \in L_\infty(Q, dq)$, such that $\psi_i \xrightarrow{s} \psi$ in $L_2(Q, dq)$. Then $e^{-TH(g)}\psi_i \xrightarrow{s} e^{-TH(g)}\psi$ in $L_2(Q, dq)$ and for $T \geqq T_0(p)$, $p > 2$, we have that for $p^{-1} + q^{-1} = 1$

$$\|H_I(g)\, e^{-TH(g)}(\psi_i - \psi)\| \leqq \|H_I(g)\|_q \, \|e^{-TH(g)}(\psi_i - \psi)\|_p \leqq \text{const} \, \|\psi_i - \psi\|_2 \tag{2.1}$$

by Theorem 2.2(c). Since $D_T \subset D(H_0) \cap D(H_I(g))$, we have that

$$\|H_0 e^{-TH(g)}(\psi_i - \psi_j)\| \leqq \|H(g)\, e^{-TH(g)}(\psi_i - \psi_j)\| + \|H_I(g)\, e^{-TH(g)}(\psi_i - \psi_j)\|$$

$$\leqq \|H(g)\, e^{-TH(g)}\| \, \|\psi_i - \psi_j\|_2 + \text{const} \, \|\psi_i - \psi_j\|_2 \to 0 \quad \text{as} \quad i, j \to \infty \, .$$

Thus $\{e^{-TH(g)}\psi_i\}$ is an H_0-convergent sequence. Since H_0 is closed, $e^{-TH(g)}\psi \in D(H_0)$ and $H_0 e^{-TH(g)}\psi_i \xrightarrow{s} H_0 e^{-TH(g)}\psi$ in $L_2(Q, dq)$. ▮

The locally correct Lorentzian

$$M(g_0, g_1) = M_0(g_0) + H_I(g_1) \tag{2.2}$$

where

$$M_0(g_0) = \alpha H_0 + H_0(g_0) \, , \tag{2.3}$$

$$H_0(g_0) = \tfrac{1}{2} \int : [\pi(x)^2 + (\nabla\varphi(x))^2 + m^2 \varphi(x)^2] : g_0(x)\, dx \, , \tag{2.4}$$

$\alpha > 0$, $g_0, g_1 \in \mathscr{S}(R)$ and $g_0, g_1, \geqq 0$, was introduced and studied by Cannon and Jaffe [1] for the $(\varphi^4)_2$ theory, and their results have been extended to the $(\varphi^{2n})_2$ by Rosen [9]. Recently Klein [7] has proved the existence of a probability measure $d\mu$ on Q-space such that the Fock space $\mathscr{F}$ can be represented by $L_2(Q, d\mu)$ and $e^{-tM_0(g_0)}$ has the sane properties on $L_p(Q, d\mu)$ as e^{-tH_0} on $L_p(Q, dq)$. We renormalize $M_0(g_0)$ such that $M_0(g_0) \geqq 0$.

Lemma 2.4. (a) *$e^{-tM_0(g_0)}$ is a contraction in $L_p(Q, d\mu)$ for all $t \geqq 0$ and all $p \geqq 1$.*

(b) *$e^{-tM_0(g_0)}$ is a strongly continuous semi-group on $L_p(Q, d\mu)$ for $1 \leqq p < \infty$.*

(c) *For $2 \leqq p < \infty$ there exists $t_0(p) \geqq 0$ such that $e^{-tM_0(g_0)}$ is a contraction from $L_2(Q, d\mu)$ to $L_p(Q, d\mu)$ for $t \geqq t_0(p)$.*

Proposition 2.5. (a) *$M(g_0, g_1) = M_0(g_0) + H_I(g_1)$ is essentially self-adjoint on $D(M_0(g_0)) \cap D(H_I(g_1))$ and bounded below.*

(b) *$e^{-tM(g_0, g_1)}$ is bounded on $L_p(Q, d\mu)$ for all $t \geqq 0$ and for all $1 < p < \infty$.*

(c) *For $2 \leqq p < \infty$ there exists $T_0(p) \geqq 0$ such that for $T \geqq T_0(p)$ $\cdot\, e^{-TM(g_0, g_1)}$ is a bounded map from $L_2(Q, d\mu)$ to $L_p(Q, d\mu)$.*

Lemma 2.6. (a) *$C^\infty(M_0(g_0))$ is a core for $M(g_0, g_1)$.*

(b) *For $T \geqq T_0(p)$ for any $p > 2$, $e^{-TM(g_0, g_1)} L_2(Q, d\mu) \subset D(M_0(g_0))$.*

Proof of Lemma 2.4 – Lemma 2.6. Lemma 2.4(a), (c) and Proposition 2.5(a) are Klein's results [7, Theorem I and Corollary of Theorem II].

The rest of the results can be proved by the techniques used in proving Lemma 2.1 – Lemma 2.3 after replacing H_0, $H(g)$ and $L_p(Q, dq)$ by $M_0(g_0)$, $M(g_0, g_1)$ and $L_p(Q, d\mu)$ respectively. ∎

The remainder of this section is devoted to investigating some useful properties of $M_0(g_0)$ and $M(g_0, g_1)$. We note that $H_0(g_0)$ is symmetric operator defined on $D(H_0)$ [1] and can be written as

$$\begin{aligned} H_0(g_0) = H_{0,1}(g_0) + H_{0,2}(g_0) = \int W_1(k,k')\, a^*(k)\, a(k')\, dk dk' \\ + \int W_2(k,k')\, [a^*(k)\, a^*(-k') + a(-k)\, a(k')]\, dk dk' , \end{aligned} \tag{2.5}$$

where the kernel $W_2(k, k')$ of $H_{0,2}(g_0)$ belongs to $L_2(R)$. We introduce the following operators

$$P(g) = \tfrac{1}{2} \int : [\pi(x)\, \nabla\varphi(x) + (\nabla\varphi(x))\, \pi(x)] : g(x)\, dx , \tag{2.6}$$

$$\dot{P}(g) = H_0(g) - m^2 \int : \varphi(x)^2 : g(x)\, dx \tag{2.7}$$

for $g = \bar{g} \in \mathscr{S}(R)$. $P(g)$ and $\dot{P}(g)$ are also symmetric operators on $D(H_0)$ [1]. Furthermore we have

Lemma 2.7. (a) $M_0(g_0)$ *is a self-adjoint operator and it is bounded below.*

(b) *The following operators are all bounded:*

$$M_0(g_0)\,(H_0+1)^{-1} , \quad P(g)\,(H_0+1)^{-1} , \quad \dot{P}(g)\,(H_0+1)^{-1} ,$$
and
$$H_0(M_0(g_0)+b)^{-1} \tag{2.8}$$

for some positive constant b.

Proof. (a) This lemma can be obtained from Theorem 5.3, Ref. [1] by setting $g_1(x) = 0$.

(b) The boundedness of the first three operators in (2.8) is proven in Theorem 3.2.1, Ref. [1]. We consider the last operator. We have that on $D(H_0) \times D(H_0)$

$$\begin{aligned} M_0(g_0)^2 = (\alpha H_0 + H_0(g_0))^2 = \alpha^2 H_0^2 + \alpha[H_0(g_0)\, H_0 + H_0 H_0(g_0)] + H_0(g_0)^2 \\ \geqq (\alpha^2 - \varepsilon)\, H_0^2 - d(\varepsilon) \end{aligned} \tag{2.9}$$

from Lemma 4.2, Ref. [1]. Choose ε sufficiently small so that $\alpha^2 - \varepsilon > 0$. Then boundedness of the last operator in (2.8) follows from the above inequality and the self-adjointness of $M_0(g_0)$ on $D(H_0)$.

Let D_o be the dense domain of vectors in $\mathscr{F}$ with finite number of particles and wave functions in $C_0^\infty(R^n)$. Notice that the wave functions have compact support in momentum space and consequently D_0 is invariant under H_0. Then D_o is a core for H_o^n for any n, since any vector $\varphi \in D_o$ is an analytic vector for H_0^n and moreover $D_o \subset C^\infty(H_0)$. We denote that

$$(ad\, A)^n B = [A, (ad\, A)^{n-1} B], (ad\, A)^0 B = B . \qquad R = [M_0(g_0) + b]^{-1} , \tag{2.10}$$

and

$$H_{0,2}{}^{(j)} = 2^j \int W_2(k,k') [a^*(k)\, a^*(-k') + (-1)^j a(-k) a(k')]\, dk dk'$$

We note that $H^{(j)}_{0,2} R$ and $R H^{(j)}_{0,2}$ are bounded operators for any j by the boundedness of (2.8).

Lemma 2.8. *As an operator relation on* $D(H_0^n)$

$$(\mathrm{ad}\, N)^n R = M_n , \tag{2.11}$$

where M_n *is some bounded operator for any* n.

Proof. First we consider (2.11) for the case for $n = 1$. By direct computation on $D_o \times D_o$, we obtain

$$[N, M_0(g_0) + b] = H^{(1)}_{0,2}(g_0) . \tag{2.12}$$

Each term in (2.12) is bounded on $D(H_0) \times D(H_0)$ by Lemma 2.7. Therefore (2.12) holds on $D(H_0) \times D(H_0)$, since H_0 is essentially self-adjoint on D_o. Thus we have that as bounded operators

$$[N, R] = (-1)\, R H^{(1)}_{0,2}(g_0)\, R . \tag{2.13}$$

Hence the Lemma holds for the case of $n = 1$.

From (2.13) we obtain that on $D(H_0^2) \times D(H_0^2)$

$$(\mathrm{ad}\, N)^2 R = (-1)^2 R[H^{(1)}_{0,2}(g_0)\, R]^2\, 2! + (-1)\, R H^{(2)}_{0,2}(g_0)\, R \equiv M_2 . \tag{2.14}$$

Let $\chi \in D(H_0^2)$ and $\varphi \in D(H_0^2)$. Then from (2.14) we have

$$(N\chi, [N, R]\, \varphi) \leqq \text{const}\, \|\chi\|\, \{\|[N, R]\, N\varphi\| + \|M_2 \varphi\|\} \leqq \text{const}\, \|\chi\| ,$$

since M_2 is a bounded operator. Hence (2.14) holds on $D(H_0^2)$, since $D(H_0^n)$ is a core for N, $n > 1$, and so $[N, R]\, D(H_0^2) \subset D(N)$.

By repeating above arguments n times and by noting that on $D(M_0(g_0)) \times D(M_0(g_0))$

$$[N, H^{(j)}_{0,2}(g_0)] = H^{(j+1)}_{0,2}(g_0) , \tag{2.15}$$

we prove the lemma. ▮

Proposition 2.9. *There exist constant* a *and* b *such that for any* $n > 0$

$$N^n \leqq a(M_0(g_0) + b)^n . \tag{2.16}$$

Proof. For the cases $n = 1, 2$, the proposition follows from the boundedness of (2.8). We assume that for given $n > 1$, $N^{n-1} R^{n-1}$ is bounded. Let $\chi \in C^\infty(H_0)$ and $\psi \in R^n \varphi$. Then

$$(N\chi, N^{n-1}\psi) = (RN^n\chi, (M_0(g_0)+b)\psi). \tag{2.17}$$

Since

$$[N^n, R] \supset \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} N^{n-i}[(\mathrm{ad}\,N)^i R] \tag{2.18}$$

and since each term in (2.18) is defined on $C^\infty(H_0)$ as a corollary of Lemma 2.8, we have that

$$\begin{aligned} |(RN^n\chi, (M_0(g_0)+b)\psi)| &= |(NR\chi, N^{n-1}(M_0(g_0)+b)\psi)| \\ &\quad + \left| \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} ((\mathrm{ad}\,N)^i R\chi, N^{n-i}(M_0(g_0)+b)\psi) \right| \\ &\leqq \mathrm{const}\, \|\chi\|\, \|\varphi\|. \end{aligned} \tag{2.19}$$

Here we have used Lemma 2.8 and the induction hypothesis. Since N^n is essentially self-adjoint on $C^\infty(H_0)$, $N^n R^n$ is bounded. This proves the proposition. ∎

Corollary 2.10. (a) $C^\infty(M_0(g_0)) \subset C^\infty(N)$.
(b) *$D(H_0) \cap C^\infty(N)$ is a core for $M(g_0, g_1)$.*

Proof. (a) This follows from Proposition 9.

(b) From the boundedness of (2.8) and part (a) it follows that $C^\infty(M_0(g_0)) \subset D(H_0) \cap C^\infty(N)$. Using this and Lemma 2.6(a) part (b) follows. ∎

Proposition 2.11. *Assume that there exist constants* c *and* d *such that $cg \leqq g_1 \leqq dg$. Then the following operators are bounded:*

$$H(g)^{\frac{1}{2}} (M(g_0, g_1)+1)^{-\frac{1}{2}} \quad \textit{and} \quad M(g_0, g_1)^{\frac{1}{2}} (H(g)+1)^{-\frac{1}{2}}, \tag{2.20}$$

where we have renormalized $H(g)$ and $M(g_0, g_1)$ such that these are positive.

Proof. First we prove that on $D(H_0) \cap D(N^n)$

$$H(g) \leqq \mathrm{const}(M(g_0, g_1)+1). \tag{2.21}$$

Then (2.21) gives the boundedness of $H(g)^{\frac{1}{2}} (M(g_0, g_1)+1)^{-\frac{1}{2}}$, since $D(H_0) \cap D(N^n) \supset D(H_0) \cap C^\infty(N)$ is a core for $M(g_0, g_1)^{\frac{1}{2}}$.

On $D(H_0) \cap D(N^n)$ we have that

$$\begin{aligned} a(M(g_0, g_1)+1) - H(g) &= (\alpha a - 1) H_0 + a H_0(g_0) + H_I(a g_1 - g) \\ &\quad + \mathrm{const}. \end{aligned} \tag{2.22}$$

Choose a large enough so that $\alpha a - 1 > 0$ and $a g_1 - g \geqq 0$, then the right hand side of (2.22) has a form similar to $M(g_0, g_1)$. Thus it is bounded below on $D(H_0) \cap D(N^n)$. This proves the inequality (2.21).

Next we prove that on $D(H_0)\cap D(N^n)$

$$M(g_0, g_1) \leqq \text{const}\,(H(g)+1)\,. \tag{2.23}$$

But we have that on $D(H_0)\cap D(N^n)$

$$a(H(g)+1) - M(g_0, g_1) = (a-\alpha)H_0 - H_0(g_0) + H_I(ag-g_1) + \text{const}\,. \tag{2.23}$$

From the boundedness of (2.8) there exists a constant e such that on $D(H_0)$

$$eH_0 - H_0(g_0) \geqq 0\,.$$

Again we choose the constant a sufficiently large enough so that $a-\alpha > e > 0$ and $ag - g_1 \geqq 0$. Then $(a-\alpha-e)H_0 + H_I(ag-g_1)$ has a form similar to $H(g)$. Thus the right hand side of (2.23) is bounded below and so we have the inequality (2.23). The proof is complete. ∎

III. Local Lorentz Transformation of Field Operators

In this section we shall study the transformation of the Heisenberg picture field operators $\varphi(f)$, $f=\bar{f}\in C_o^\infty(B')$ under the unitary group generated by the local Lorentzian $M(g_0, g_1)$ introduced in the previous section. We note that the field operators $\varphi(f)$ are essentially self-adjoint on any core for $H(g)^{\frac{1}{2}}$ and independent of the space cutoff g provided that the support of $g(x)$ is large enough [2, 3]. All these properties may be shown without using the higher order estimates (for instance, see Theorem 8.7, Ref. [3]).

The Lorentzian on bounded regions B' with $B'\cup\Lambda B'\subset B_I$ has the form

$$M = \alpha H_0 + H_0(xg_0) + H_I(xg_1)\,. \tag{3.1}$$

We impose certain conditions on α, g_0, and g_1 [1], and show that M is an infinitesimal generator for the locally correct Lorentz transformation of the field operators without making use of higher order estimates [1, 9]. The assumptions are

$$\text{(a)}\ \ \alpha>0,\ xg_i(x)=h_i(x)^2,\ h_i(x)>0,\ h_i\in C_o^\infty(R)\,, \tag{3.2}$$

$$\text{(b)}\ \ \alpha + xg_0(x) = x = xg_1(x) \quad \text{on} \quad I=[a,b]\subset R+\,, \tag{3.3}$$

$$\text{(c)}\ \ xg_1(x) = (\alpha + xg_0(x))\,g_1(x) \quad \text{for all} \quad x\in R\,. \tag{3.4}$$

The above conditions are satisfied by choosing the suitable α, g_0 and g_1 (see Ref. [1], p. 299). We denote B_I as

$$B_I = \{(x,t) : a+|t| < x < b-|t|\}\,. \tag{3.5}$$

The Hamiltonian (assume the coupling constant $\lambda = 1$)

$$H = H_0 + H_I(g_1) \tag{3.6}$$

is correct in the region B_I [2, 3].

We shall work with this choice of Hamiltonian H and Lorentzian M. All condition on H and M in the previous section are satisfied, and again we renormalize M and H such that these are positive. According to the discussion at the end of Section I, the following result is sufficient for the proof of Theorem 1.1.

Theorem 3.1. *Let* $f = \bar{f} \in C_o^\infty(B_I)$ *and* $\operatorname{supp} f_{A_\beta} \subset B_I$. *Then*

$$e^{i\beta M} \varphi(f)\, e^{-i\beta M} = \varphi(f_{A\beta}) \tag{3.7}$$

as an equality for the self-adjoint operators.

This section is devoted to proving Theorem 3.1. Although the overall structure of the proof is the same as that of Cannon and Jaffe [1, Section 6 and 9, Section 6], we carry out the proof by using the hypercontractive properties of e^{-tH} and e^{-tM} stated in the previous section. We shall give a sketch of the proof in the last part of this section.

The most difficult part in proving Theorem 3.1 is in controlling the domain for various commutators of H and M. For this reason we introduce the domains

$$D_T = e^{-(T+1)H} L_2(Q, dq) \tag{3.8}$$

and

$$F_T = e^{-(T+1)M} L_2(Q, d\mu)\,, \tag{3.9}$$

where $T \geqq T_0(4)$. Notice that D_T and F_T are cores for H^m and M^m respectively for any $m > 0$. We shall need some technical lemmas.

Lemma 3.2. *Let A be one of* $H_0, M_0(g), P(g), \dot{P}(g), H$ *and* M, *where* $g = \bar{g} \in \mathscr{S}(R)$. *For any* $m \geqq 0$ *we have*

(a) $H^m D_T \subset D(A)$ *and* $M^m F_T \subset D(A)$, (3.9)

(b) *As* $\varepsilon \to 0$, $\varepsilon > 0$,

$$A e^{-\varepsilon H_0} \Psi \xrightarrow{s} A\Psi\,, \quad \Psi \in H^m D_T\,, \tag{3.10}$$

$$A e^{-\varepsilon M_0(xg_0)} \Psi \xrightarrow{s} A\Psi\,, \quad \Psi \in M^m D_T\,.$$

Proof. (a) Since $H^m D_T \subset L_p(Q, dq)$ for $p \leqq 4$ from Proposition 2.2(c) and since $H_I(g) \in L_p(Q, dq)$ for $p < \infty$, it follows that $H^m D_T \subset D(H_I(g))$. Thus $H^m D_T \subset D(H_0) \cap D(H_I(g))$ by Lemma 2.3(b). The first part of (a) follows from Lemma 2.7(b). The second part follows from similar arguments in $L_p(Q, d\mu)$.

(b) We first consider the case for $A = H_0$, $M_0(g)$, $P(g)$ or $\dot{P}(g)$. We note that for $\Psi \in H^m D_T$

$$\|A e^{-\varepsilon H_0} \Psi - A\Psi\| \leqq \|A(H_0+1)^{-1}\| \, \|(e^{-\varepsilon H_0} - 1)(H_0+1)\Psi\| \to 0 \quad \text{as} \quad \varepsilon \to 0 .$$

Here we have used Lemma 2.1(b) and Lemma 2.3(a). For $A = H_I(g)$ we also have that for $\Psi \in H^m D_T$

$$\|H_I(g) e^{-\varepsilon H_0} \Psi - H_I(g)\Psi\| \leqq \|H_I(g)\|_4 \, \|(e^{-\varepsilon H_0} - 1)\Psi\|_4 \to 0$$

by Lemma 2.1(b). The case for $A = H$ or M is obvious. This proves the first part of (b). Similar arguments in $L_p(Q, d\mu)$ prove the second part. ∎

Theorem 3.3. *For any* $m \geqq 0$ *we have*

(a) *As operators on* $H^m D_T$ *and on* $M^m F_T$

$$[iH, M] = P\left(\frac{d}{dx}(xg_0)\right). \tag{3.11}$$

(b) *As operators on* $H^m D_T$

$$[iH, [iH, M]] = S, \tag{3.12}$$

where

$$S = \dot{P}\left(\frac{d^2}{dx^2}(xg_0)\right) - H_I\left(\frac{d}{dx} g_1\right). \tag{3.13}$$

(c) $M H^m D_T \subset D(H^2)$, (3.14)

$$H^m D_T \subset D(M^{\frac{3}{2}}) \quad \textit{and} \quad M^m F_T \subset D(H^{\frac{3}{2}}). \tag{3.15}$$

Proof. For simplification of the proof we only consider the case for $m = 0$. The case for arbitrary m will be obvious.

(a) We first prove (3.11) on D_T. As bilinear forms on $D_0 \times D_0$ (3.11) holds [1.9]. Since each term in (3.11) is bounded on $D(H_0 + N^n) \times D(H_0 + N^n)$ and since D_0 is a core for $H_0 + N^n$, (3.11) holds on $D(H_0 + N^n) \times D(H_0 + N^n)$. Let $\Psi \in D_T$ and $\Psi_\varepsilon = e^{-\varepsilon H_0}\Psi$. Using the identity

$$(A\Psi, B\Psi) - (A\Psi_\varepsilon, B\Psi_\varepsilon) = (A(\Psi - \Psi_\varepsilon), B\Psi) - (A\Psi_\varepsilon, B(\Psi_\varepsilon - \Psi))$$

and

$$\|B\Psi_\varepsilon\| \leqq \|B\Psi\| + \|B(\Psi - \Psi_\varepsilon)\|$$

for $\Psi, \Psi_\varepsilon \in D(A) \cap D(B)$, we conclude that

$$\begin{aligned}(\Psi, [iH, M]\Psi) - (\Psi, P\Psi) &= [(\Psi, [iH, M]\Psi) - (\Psi_\varepsilon [iH, M]\Psi_\varepsilon) \\ &\quad - [(\Psi, P\Psi) - (\Psi_\varepsilon, P\Psi_\varepsilon)] \\ &\to 0 \quad \text{as} \quad \varepsilon \to 0\end{aligned}$$

by Lemma 3.2, where $P = P\left(\frac{d}{dx}(xg_0)\right)$. By passing to the limit we show the relation (3.11) on $D_T \times D_T$. Let $\chi, \Psi \in D_T$. Then from (3.11) on $D_T \times D_T$ we find that

$$\begin{aligned}|(H\chi, M\Psi)| &\leqq \|\chi\| \, \{\|MH\Psi\| + \|P\Psi\|\} \\ &\leqq \text{const} \, \|\chi\|\end{aligned} \tag{3.16}$$

by Lemma 3.2(a). Since D_T is a core for $H, MD_T \subset D(H)$ and (3.11) holds on D_T.

By replacing $e^{-\varepsilon H_0}$ by $e^{-\varepsilon M_0(xg_0)}$ and D_T by F_T, and by noting $e^{-\varepsilon M_0(xg_0)} F_T \subset D(H_0 + N^n)$ from Corollary 2.10(a) and Lemma 2.4(b) we have proved (3.11) on F_T.

(b) As bilinear forms on $D_T \times D_T$

$$[iH, [iH, M]] = \left[iH, P\left(\frac{d}{dx}(xg_0)\right)\right].$$

Here we have used part (a). But on $D_0 \times D_0$[1]

$$\left[iH, P\left(\frac{d}{dx}(xg_0)\right)\right] = S. \tag{3.17}$$

(3.17) also holds on D_T by arguments similar to those in proving part (a). Thus (3.12) holds on D_T by repeating the similar arguments used in (16).

(c) This follows as a corollary from the theorem (a) and (b), and Proposition 2.11. ▌

For $f = \bar{f} \in C_o^\infty(B_I)$ we write

$$A(f, t) = \int \varphi(x) \, f(x, t) \, dx \tag{3.18}$$

and

$$B(f, t) = \int \pi(x) \, f(x, t) \, dx. \tag{3.19}$$

Then $A(f, t)$ and $B(f, t)$ are essentially self-adjoint on any core of $H^{\frac{1}{2}}$ [3]. We restrict ourselves to functions with support contained in

$$B_\varepsilon = \{(x, t) : a + \varepsilon + |t| < x < b - \varepsilon - |t| \text{ and } |t| < \varepsilon\}, \tag{3.20}$$

where $\varepsilon > 0$ is some small number. Any $f \in C_o^\infty(B_I)$ can be written as a sum of such function.

By following the main steps in Section 6, Ref. [1] we summarize the proof of Theorem 3.1 with no use of higher order estimates.

Sketch of the Proof of Theorem 3.1. The main steps are as follows:

Step 1. For $\Psi \in D_T$ and $\text{supp} f \subset B_\varepsilon$ we consider the function

$$F(t) = i[(M(t)\,\Psi, \varphi(f)\,\Psi) - (\varphi(f)\,\Psi, M(t)\,\Psi)], \tag{3.21}$$

where $M(t)=e^{itH}Me^{itH}$. $F(t)$ is well defined by Lemma 3.2(a). In fact $F(t)$ is n times continuously differentiable (via Theorem 3.3(a), Proposition 2.11, Lemma 2.7(b) and Proposition 2.2(c)). Obviously

$$F'(t) = -([H, M(t)]\,\Psi, \varphi(f)\,\Psi) - (\varphi(f)\,\Psi, [H, M(t)]\,\Psi) \quad (3.22)$$

$$F''(t) = -i([H,[H,M(t)]]\,\Psi, \varphi(f)\,\Psi) + i(\varphi(f)\,\Psi, [H,[H,M(t)]]\,\Psi). \quad (3.23)$$

Note that each term in (3.22) and (3.23) is well defined by Theorem 3.3.

Step 2. We wish to show that for $|s| \leqq \varepsilon$ and $\operatorname{supp} f \subset B_\varepsilon$

$$[S, e^{isH}\varphi(f)\,e^{isH}] = 0 \quad \text{on} \quad D_T \times D_T\,. \quad (3.24)$$

Let $W(I)$ be von Neumann algebra generated by the spectral projections of the time zero fields $\int \varphi(x)\,h_1(x)\,dx$ and $\int \pi(x)\,h_2(x)\,dx$, $h_i = \bar{h}_i \in C_o^\infty(I)$. Then on $D_o \times D_o$[1]

$$[S, W(I)] = 0\,, \quad (3.25)$$

and so also on $D(H_0^n) \times D(H_0^n)$ by the boundedness of $S(H_0+1)^{-n}$. Using Lemma 3.2(b) one finds that (3.25) holds on $D_T \times D_T$. Since

$$e^{isH}\varphi(f)\,e^{-isH}$$

is affiliated with $W(I)$ for $|s| \leqq \varepsilon$, this gives (3.24).

Step 3. Together with the expansion of $F(t)$ by Taylor's Theorem

$$F(t) = F(o) + tF'(o) + \frac{t^2}{2}F''(s)$$

for $|s| < |t|$, (3.22) – (3.24) and Theorem 3.3 we find that

$$[iM(t), \varphi(f)] = [iM, \varphi(f)] - t\left[iP\left(\frac{d}{dx}(xg_0), \varphi(f)\right)\right] \quad \text{on} \quad D_T \times D_T\,. \quad (3.26)$$

Step 4. With the technique used in proving Theorem 3.3(a), one expects that

$$\begin{aligned} &[iM, A(f,t)] = B(xf,t) \quad \text{on} \quad D_T\,, \\ &\left[iP\left(\frac{d}{dx}(xg_0)\right), A(f,t)\right] = A\left(\frac{\partial}{\partial x}f, t\right) \quad \text{on} \quad D_T\,. \end{aligned} \quad (3.27)$$

By passing to the sharp field $(\varphi(f) \to A(f,t))$ via Theorem 3.3(c) and by using (3.27), (3.26) become

$$[iM(t), A(f,t)] = B(xf,t) - tA\left(\frac{\partial}{\partial x}f, t\right) \quad \text{on} \quad D_T \times D_T\,. \quad (3.28)$$

Multiplying (3.28) by e^{itH} on left and by e^{-itH} on right, and integrating with respect to t we obtain that

$$\begin{aligned}[iM, \varphi(f)] &= \pi(xf) - \varphi\left(t\frac{\partial f}{\partial x}\right) \\ &= -\varphi\left(x\frac{\partial f}{\partial t} + t\frac{\partial f}{\partial x}\right) \quad \text{on} \quad D_T \times D_T\,.\end{aligned} \tag{3.28}$$

Step 5. In order to deduce Theorem 3.1 from (3.28) we must show that (3.28) holds on $D(M^{\frac{3}{2}}) \times D(M^{\frac{3}{2}})$. Each term in (3.28) is bounded on $D(H^{\frac{3}{2}}) \times D(H^{\frac{3}{2}})$ by Proposition 2.11 and the relation $[iH, \varphi(f)] = \pi(f)$ on $D(H^{\frac{3}{2}})$. Hence (3.28) holds on $D(H^{\frac{3}{2}}) \times D(H^{\frac{3}{2}})$ and so on $F_T \times F_T$ by Theorem 3.3(c). Thus (3.28) holds on $D(M^{\frac{3}{2}}) \times D(M^{\frac{3}{2}})$ by Proposition 2.11. In fact, for supp$f \subset B_\varepsilon$,

$$[iM, \varphi(f)] = -\varphi\left(x\frac{\partial f}{\partial t} + t\frac{\partial}{\partial x}f\right) \quad \text{on} \quad D(M^{\frac{3}{2}}) \tag{3.29}$$

by the method used in (3.16).

Step 6. The relation (3.29) is a differential form of (3.7). We note that

$$\varphi(x, t) = e^{itH}\varphi(x)\,e^{-itH}$$

is a bilinear form on $D(M^{\frac{1}{2}}) \times D(M^{\frac{1}{2}})$ and also $D(M^{\frac{1}{2}})$ is a core for $\varphi(f)$ by Proposition 2.11. Therefore the relation (3.29) implies Theorem 3.1 by the arguments similar to those used to prove Theorem 6.1, Ref. [1] from Lemma 6.14, Ref. [1]. ∎

Acknowledgement. The author wishes to thank Professor J. Challifour for many enlightening discussions, criticisms and helpful suggestions. I also would like to express my appreciation for the extremely helpful comments made by the referee. These have helped me rewrite the paper in much better form.

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