Commun. math. Phys. 33, 97—117 (1973)

Absence of Strong Interaction Corrections to the Axial Anomaly in the σ Model

C. Becchi
CERN – Geneva
and
Istituto di Fisica dell'Università, Genova, Sezione INFN di Genova

Received January 29, 1973

Abstract. The absence of strong interaction corrections to the axial anomaly in the σ model is proved in a cut-off independent way using Zimmermann's normal product algorithm.

1. Introduction

In 1969, Adler [1] suggested that there are no higher order corrections to the axial anomaly [1–3]. This suggestion was supported later by Adler and Bardeen [4] with convincing cut-off dependent arguments in the framework of spinor electrodynamics and in a simple version of the Gell-Mann and Lévy σ model [5] coupled to the electromagnetic field.

In the case of the σ model the arguments proposed by Adler and Bardeen are, however, much weaker than in the case of spinor electrodynamics. In fact, Adler and Bardeen do not prove the renormalizability of the model and use Ward identities without being sure that they are not affected by the renormalization procedure. Unfortunately the more relevant case is actually the former because, using the Adler-Bardeen result in the framework of the model, it is possible to compute the low energy value of the $\pi^0 \to 2\gamma$ amplitude.

Recently, Zee [6] and, independently, Lowenstein and Schroer [7] have proved the absence of radiative corrections to the axial anomaly using the Callan-Symanzik equation [8]. In particular the proof given by Lowenstein and Schroer using the Zimmermann's normal product algorithm (NPA) [9] does not involve any cut-off procedure. Using the method of Lowenstein and Schroer we prove in this paper the Adler-Bardeen theorem in the simplified version of the σ model in which the π is an isoscalar meson and only one fermion field (say, the proton field) exists.

The paper is organized as follows. First we state the renormalization rules for the σ model using the NPA (Section 2). Then we derive an equation analogous to the Callan-Symanzik equation for our model using the method developed by Lowenstein [10] (Section 3). In Section 4 we discuss the consequences of the coupling to an external electromagnetic field and we prove the theorem.

2. Renormalization of the σ Model

The renormalization rules for the σ model are widely discussed in the existing literature [11–13]. However, the σ model with spinors has never been treated before in the framework of the NPA, therefore we have to study it in some detail.

We consider a truncated version of the σ model which contains only a proton field (ψ) a neutral pseudoscalar (π) and a scalar meson (σ). By definition of the σ model, an axial current $j_5^\mu(x)$ exists that satisfies the Ward identity:

$$\partial_\mu \langle j_5^\mu(x) X\rangle_+ = -c\langle \pi(x) X\rangle_+ + i\sum_1^m{}_i\, \delta(x-x_i)\,\langle \pi(x) X_{\widehat{\sigma(x_i)}}\rangle_+$$

$$-i\sum_1^m{}_j\, \delta(x-y_j)\,\langle (\sigma(x)+F) X_{\widehat{\pi(y_j)}}\rangle_+ - \tfrac{1}{2}\left\{\sum_1^p{}_k\, \delta(x-z_k')\,\langle (\overline{\psi}(x)\gamma_5)^{\beta_k} X_{\widehat{\psi(z_k')^{\beta_k}}}\rangle_+\right.$$

$$\left.+\sum_1^p{}_l\, \delta(x-z_l)\,\langle (\gamma_5\,\psi(x))_{\alpha_l} X_{\widehat{\psi(z_l)_{\alpha_l}}}\rangle_+\right\} \tag{1}$$

where $\langle\ \rangle_+$ means the vacuum expectation value of the covariant time ordered product, X is any product of fields:

$$X = \prod_1^n{}_i\, \sigma(x_i) \prod_1^m{}_j\, \pi(y_j) \prod_1^p{}_l\, \psi(z_l)_{\alpha_l} \prod_1^p{}_k\, \overline{\psi}(z_k')^{\beta_k} \tag{2}$$

and the symbols $X_{\widehat{\sigma(x_i)}}$, etc., mean suppression of the corresponding field from the product.

We shall first show that the existence of the identity (1) is sufficient to determine the parameters of the effective Lagrangian of the model in terms of the physical parameters [12]. Afterwards we will show that in the model corresponding to this effective Lagrangian, a current j_5^μ actually exists that satisfies Eq. (1).

Let us begin to show how Eq. (1) determines the parameters of the effective Lagrangian in terms of the physical parameters. Since we are interested in deriving from Eq. (1) conditions on the proper vertices that can be directly translated into relations among the parameters of the Lagrangian it is convenient to recast Eq. (1) into an equation for the generating functional of the Green's functions.

Let us consider the vacuum functional of the model $S_0[J_\pi, J_\sigma, \eta, \overline{\eta}]$. Here $J_\pi(x)$, $J_\sigma(x)$, $\eta(x)$, $\overline{\eta}(x)$ are the external sources of the π, σ, ψ, $\overline{\psi}$ fields respectively [14]. The functional generating the connected parts of the time-ordered Green's functions is:

$$Z[J_\pi, J_\sigma, \eta, \bar{\eta}] = -i \log S_0[J_\pi, J_\sigma, \eta, \bar{\eta}] .$$

Integrating (1) with respect to the variable x we get the equation for the vacuum functional:

$$\begin{aligned}\int dx \Big\{ & i(c + J_\sigma(x)) \frac{\delta}{\delta J_\pi(x)} S_0 - i\, J_\pi(x) \left(\frac{\delta}{\delta J_\sigma(x)} + F \right) S_0 \\ & - \tfrac{1}{2} \left(\bar{\eta}(x)\, \gamma_5 \frac{\delta}{\delta \bar{\eta}(x)} S_0 + S_0 \frac{\overleftarrow{\delta}}{\delta \eta(x)} \gamma_5\, \eta(x) \right) \Big\} = 0\end{aligned} \tag{3}$$

and the equation for Z:

$$\begin{aligned}\int dx \Big\{ & J_\pi(x) \left(\frac{\delta}{\delta J_\sigma(x)} Z + F \right) - (c + J_\sigma(x)) \frac{\delta}{\delta J_\pi(x)} Z \\ & - \frac{i}{2} \left(\bar{\eta}(x) \gamma_5 \frac{\delta}{\delta \bar{\eta}(x)} Z + Z \frac{\overleftarrow{\delta}}{\delta \eta(x)} \gamma_5 \eta(x) \right) \Big\} = 0 .\end{aligned} \tag{4}$$

It is now convenient to define:

$$\begin{aligned} \Pi(x) &= \frac{\delta}{\delta J_\pi(x)} Z ; \qquad \Sigma(x) = \frac{\delta}{\delta J_\sigma(x)} Z ; \\ \Psi(x) &= \frac{\delta}{\delta \bar{\eta}(x)} Z ; \qquad \bar{\Psi}(x) = Z \frac{\overleftarrow{\delta}}{\delta \eta(x)} \end{aligned} \tag{5}$$

and to perform the Legendre transformation:

$$\begin{aligned} & W[\Pi, \Sigma, \Psi, \bar{\Psi}] \\ & \quad = Z - \int dx \big(\Pi(x)\, J_\pi(x) + \Sigma(x)\, (J_\sigma(x) + c) + \bar{\Psi}(x)\, \eta(x) + \bar{\eta}(x)\, \Psi(x) \big) . \end{aligned} \tag{6}$$

$W + \int dx c\, \Sigma(x)$ is the functional generating the proper vertices [15]. Since

$$\begin{aligned} \frac{\delta}{\delta \Pi(x)} W &= -J_\pi(x) ; \qquad \frac{\delta}{\delta \Sigma(x)} W = -J_\sigma(x) ; \\ \frac{\delta}{\delta \bar{\Psi}(x)} W &= -\eta(x) ; \qquad W \frac{\overleftarrow{\delta}}{\delta \Psi(x)} = -\bar{\eta}(x) \end{aligned} \tag{7}$$

we get, from Eq. (4):

$$\begin{aligned}\int dx \Big\{ & \Pi(x) \frac{\delta}{\delta \Sigma(x)} W - (\Sigma(x) + F) \frac{\delta}{\delta \Pi(x)} W \\ & + \frac{i}{2} \left(\bar{\Psi}(x) \gamma_5 \frac{\delta}{\delta \bar{\Psi}(x)} W + W \frac{\overleftarrow{\delta}}{\delta \Psi(x)} \gamma_5\, \Psi(x) \right) \Big\} = 0 .\end{aligned} \tag{8}$$

If we assume that all the terms of the effective Lagrangian are Zimmermann's normal products N_4, then their coefficients are proportional to functional derivatives of W.

Indeed let us consider the most general effective Lagrangian for the π, σ and ψ fields:

$$\begin{aligned}\mathscr{L}_{\text{eff}} = N_4\Big[& i(1+a)\bar{\psi}\,\partial\!\!\!/\psi - A\bar{\psi}\psi - ig_1\bar{\psi}\gamma_5\psi\pi - g_2\bar{\psi}\psi\sigma + \frac{1+b_1}{2}(\partial_\mu\pi)^2 \\ &+ \frac{1+b_2}{2}(\partial_\mu\sigma)^2 - \frac{B_1}{2}\pi^2 - \frac{B_2}{2}\sigma^2 - \lambda_1\pi^2\sigma - \lambda_2\sigma^3 - \lambda_3\pi^4 \\ &- \lambda_4\sigma^4 - \lambda_5\pi^2\sigma^2\Big]. \end{aligned} \tag{9}$$

The values at zero external momenta of the superficially divergent proper vertices are given by the coefficients of the Lagrangian. If W is the generator of the proper vertices, we have, for example[1]:

$$\begin{aligned} &\tfrac{1}{4}\operatorname{Tr}\left\{\gamma_\mu\left[\partial p_\mu \frac{\delta}{\delta\tilde{\bar{\Psi}}(-p)}\,W\,\frac{\overleftarrow{\delta}}{\delta\tilde{\Psi}(p)}\right]_{\varphi=p=0}\right\} = 1+a\,, \\ &\tfrac{1}{4}\operatorname{Tr}\left\{\left[\frac{\delta}{\delta\tilde{\bar{\Psi}}(-p)}\,W\,\frac{\overleftarrow{\delta}}{\delta\tilde{\Psi}(p)}\right]_{\varphi=p=0}\right\} = -A\,. \end{aligned} \tag{10}$$

We can solve globally Eq. (8) by using the method of Symanzik [12] and we obtain for the σ model:

$$\begin{aligned}\mathscr{L}^{(\sigma)}_{\text{eff}} = N_4\Big[& i(1+a)\bar{\psi}\,\partial\!\!\!/\psi - (m+A)\bar{\psi}\left(1+\frac{g}{m}(i\pi\gamma_5+(1+d)\sigma)\right)\psi \\ &+ \frac{1+b}{2}((\partial_\mu\pi)^2+(\partial_\mu\sigma)^2) + \frac{f}{2}(\partial_\mu\sigma)^2 - \frac{\mu^2+B}{2}(\pi^2+\sigma^2) \\ &- \frac{\delta^2+C}{2}\sigma^2 - \frac{\delta^2+C}{2}\,\frac{g}{m}\,\sigma(\pi^2+\sigma^2) - \frac{\delta^2+C}{8}\,\frac{g^2}{m^2}(\sigma^2+\pi^2)^2 \\ &- 2D\frac{g}{m}\sigma^3 - 3D\frac{g^2}{m^2}\sigma^2(\pi^2+\sigma^2) - E\frac{g^2}{m^2}\sigma^4\Big] \end{aligned} \tag{11}$$

where m, μ and $\sqrt{\mu^2+\delta^2}$ are the masses of the proton, π and σ, $F=m/g$, a, b, A, B and C are given by fixing the position and the residue of the

[1] We put

$$\int\prod_1^n{}_i\, dx_i\, e^{-i\sum_1^n{}_i q_i x_i}\prod_1^n{}_i\frac{\delta}{\delta\varphi(x_i)}W = (2\pi)^4\,\delta\left(\sum_1^n{}_i q_i\right)\prod_1^n{}_i\frac{\delta}{\delta\tilde{\varphi}(q_i)}W$$

where φ is any field.

poles of the proton and π propagators and the position of the pole of the σ propagator, and we have:

$$\frac{m}{g}\left[\frac{\delta}{\delta\tilde{\Sigma}(0)}\left(\frac{\delta}{\delta\tilde{\Pi}(0)}\right)^4 W\right]_{\varphi=0}$$

$$=\left[\left(3\left(\frac{\delta}{\delta\tilde{\Pi}(0)}\right)^2\left(\frac{\delta}{\delta\tilde{\Sigma}(0)}\right)^2-\left(\frac{\delta}{\delta\tilde{\Pi}(0)}\right)^4\right)W\right]_{\varphi=0}=-36D\frac{g^2}{m^2},$$

$$\frac{m}{g}\left[\left(\frac{\delta}{\delta\tilde{\Sigma}(0)}\right)^3\left(\frac{\delta}{\delta\tilde{\Pi}(0)}\right)^2 W\right]_{\varphi=0}$$

$$=\left[\left(\left(\frac{\delta}{\delta\tilde{\Sigma}(0)}\right)^4-3\left(\frac{\delta}{\delta\tilde{\Sigma}(0)}\right)^2\left(\frac{\delta}{\delta\tilde{\Pi}(0)}\right)^2\right)W\right]_{\varphi=0}=-(36D+24E)\frac{g^2}{m^2},$$

$$\frac{m}{g}\,\frac{1}{8}\left[\partial p_\mu\,\partial p^\mu\frac{\delta}{\delta\tilde{\Pi}(0)}\,\frac{\delta}{\delta\tilde{\Pi}(p)}\,\frac{\delta}{\delta\tilde{\Sigma}(-p)}\,W\right]_{p=\varphi=0}$$

$$=\frac{1}{8}\left[\partial p_\mu\,\partial p^\mu\left(\frac{\delta}{\delta\tilde{\Sigma}(p)}\,\frac{\delta}{\delta\tilde{\Sigma}(-p)}-\frac{\delta}{\delta\tilde{\Pi}(p)}\,\frac{\delta}{\delta\tilde{\Pi}(-p)}\right)W\right]_{p=\varphi=0}=f,$$

$$\frac{m}{4g}\left[\left(\frac{\delta}{\delta\tilde{\Pi}(0)}\right)^2\mathrm{Tr}\left\{\frac{\delta}{\delta\tilde{\bar{\Psi}}(0)}\,W\,\frac{\overleftarrow{\delta}}{\delta\tilde{\Psi}(0)}\right\}\right]_{\varphi=0}$$

$$=\frac{im}{4g}\left[\frac{\delta}{\delta\tilde{\Pi}(0)}\,\frac{\delta}{\delta\tilde{\Sigma}(0)}\mathrm{Tr}\left\{\frac{\delta}{\delta\tilde{\bar{\Psi}}(0)}\,W\,\frac{\overleftarrow{\delta}}{\delta\tilde{\Psi}(0)}\gamma_5\right\}\right]_{\varphi=0}$$

$$=\frac{1}{4}\left[\mathrm{Tr}\left\{\frac{\delta}{\delta\tilde{\Sigma}(0)}\,\frac{\delta}{\delta\tilde{\bar{\Psi}}(0)}\,W\,\frac{\overleftarrow{\delta}}{\delta\tilde{\Psi}(0)}+i\frac{\delta}{\delta\tilde{\Pi}(0)}\,\frac{\delta}{\delta\tilde{\bar{\Psi}}(0)}\,W\,\frac{\overleftarrow{\delta}}{\delta\tilde{\Psi}(0)}\gamma_5\right\}\right]_{\varphi=0}$$

$$=-\frac{g}{m}(m+A)\,d\,. \tag{12}$$

The parameters D, E, d, f are proportional to the value at zero external momenta of superficially convergent vertices, therefore they are known functions of the physical parameters m, μ, δ, and g.

We will now show that in the theory defined by the effective Lagrangian given in Eq. (11) an axial current $j_5^\mu(x)$ that satisfies Eq. (1) actually exists.

We shall start by showing that the current

$$j_{\mu 5}^{(0)}(x)=N_3[\sigma\,\partial_\mu\pi-\pi\,\partial_\mu\sigma+\tfrac{1}{2}\,\bar{\psi}\gamma_\mu\gamma_5\psi]\,(x)+\frac{m}{g}\,\partial_\mu\pi(x)$$

satisfies the corresponding integrated Ward identity. Indeed, using the method developed by Lowenstein [15], we have:

$$
\begin{aligned}
0 = & \int dx\, \partial_\mu \langle j_5^{(0)\mu}(x) X\rangle_+ = i \sum_{i=1}^{n} \langle \pi(x_i) X_{\widehat{\sigma(x_i)}}\rangle_+ \\
& - i \sum_{j=1}^{m} \left\langle \left(\sigma(y_j) + \frac{m}{g}\right) X_{\widehat{\pi(y_j)}} \right\rangle_+ \\
& - \tfrac{1}{2} \left\{ \sum_{k=1}^{p} \langle (\overline{\psi}(z_k')\gamma_5)^{\beta_k} X_{\widehat{\psi(z_k')^{\beta_k}}}\rangle_+ + \sum_{l=1}^{p} \langle (\gamma_5 \psi(z_l))_{\alpha_l} X_{\widehat{\psi(z_l)_{\alpha_l}}}\rangle_+ \right\} \\
& + \int dx \left\langle \left\{ N_4 \left[-f\sigma \Box \pi + (\delta^2 + C)\pi\sigma + \frac{\delta^2 + C}{2} \frac{g}{m} \pi(\pi^2 + \sigma^2) \right.\right.\right. \\
& + 6D \frac{g}{m} \pi\sigma^2 + 6D \frac{g^2}{m^2} \pi\sigma(\pi^2 + \sigma^2) + 4E \frac{g^2}{m^2} \sigma^3 \pi + i(m + A)\overline{\psi}\gamma_5\psi \\
& \left. + \frac{m + A}{m} g\, d\overline{\psi}(\pi - i\gamma_5\sigma)\psi \right](x) - N_3 \left[(\delta^2 + C)\pi\sigma \right. \\
& \left. + \frac{\delta^2 + C}{2} \frac{g}{m} \pi(\pi^2 + \sigma^2) + 6D \frac{g}{m} \pi\sigma^2 + i(m + A)\overline{\psi}\gamma_5\psi \right](x) \\
& \left.\left. - \frac{m}{g} (\mu^2 + B)\, \pi(x) \right\} X \right\rangle_+ .
\end{aligned}
\tag{13}
$$

In the right-hand side of Eq. (13), terms of the kind $\int dx\, \partial_\mu \langle O^\mu(x) X\rangle$ have been forgotten. Denoting by $S_\pi(x)$ the polynomial in the fields which is N_3 in Eq. (13) (that is the proper source of the π field), we have by Zimmermann's identity [9] relating normal products of different degree:

$$
\begin{aligned}
\int dx \langle N_3[S_\pi](x) X\rangle_+ = \int dx \langle N_4[S_\pi & + c_1 \pi \Box \sigma + c_2 \pi^3 \sigma \\
& + c_3 \sigma^3 \pi + c_4 \overline{\psi}\psi\pi + i c_5 \overline{\psi}\gamma_5\psi\sigma](x) X\rangle_+ ,
\end{aligned}
\tag{14}
$$

where:

$$
\begin{aligned}
c_1 &= -\tfrac{1}{8} \frac{m}{g} \left[\partial p_\mu \partial p^\mu \frac{\delta}{\delta \tilde{\Pi}(0)} \frac{\delta}{\delta \tilde{\Pi}(p)} \frac{\delta}{\delta \tilde{\Sigma}(-p)} W \right]_{p=\varphi=0} , \\
c_2 &= -\frac{m}{6g} \left[\frac{\delta}{\delta \tilde{\Sigma}(0)} \left(\frac{\delta}{\delta \tilde{\Pi}(0)} \right)^4 W \right]_{\varphi=0} , \\
c_3 &= -\frac{m}{6g} \left[\left(\frac{\delta}{\delta \tilde{\Sigma}(0)} \right)^3 \left(\frac{\delta}{\delta \tilde{\Pi}(0)} \right)^2 W \right]_{\varphi=0} , \\
c_4 &= -\frac{m}{4g} \left[\left(\frac{\delta}{\delta \tilde{\Pi}(0)} \right)^2 \mathrm{Tr} \left\{ \frac{\delta}{\delta \tilde{\overline{\Psi}}(0)} W \frac{\overleftarrow{\delta}}{\delta \tilde{\Psi}(0)} \right\} \right]_{\varphi=0} , \\
c_5 &= \frac{im}{4g} \left[\frac{\delta}{\delta \tilde{\Pi}(0)} \frac{\delta}{\delta \tilde{\Sigma}(0)} \mathrm{Tr} \left\{ \frac{\delta}{\delta \tilde{\overline{\Psi}}(0)} W \frac{\overleftarrow{\delta}}{\delta \tilde{\Psi}(0)} \gamma_5 \right\} \right]_{\varphi=0} .
\end{aligned}
\tag{15}
$$

Let us assume for a moment that $c_4 = c_5$ then, choosing for d, f, D, E the values given in Eq. (12) and taking into account Eq. (14) and Eq. (15), in the last term of the right-hand side of Eq. (13) everything cancels except $-\left(\frac{m}{g}\right)(\mu^2 + B) \int dx \langle \pi(x) X \rangle_+$ and we obtain:

$$
\begin{aligned}
0 = \int dx\, \partial_\mu \langle j_5^{(0)\mu}(x) X \rangle_+ &= i \sum_1^n{}_i \langle \pi(x_i) X_{\widehat{\sigma(x_i)}} \rangle_+ \\
&- i \sum_1^m{}_j \left\langle \left(\sigma(y_j) + \frac{m}{g} \right) X_{\widehat{\pi(y_j)}} \right\rangle_+ \\
&- \tfrac{1}{2} \left\{ \sum_1^p{}_k \langle (\bar{\psi}(z_k') \gamma_5)^{\beta_k} X_{\widehat{\psi(z_k')^{\beta_k}}} \rangle_+ + \sum_1^p{}_l \langle (\gamma_5 \psi(z_l))_{\alpha_l} X_{\widehat{\psi(z_l)_{\alpha_l}}} \rangle_+ \right\} \\
&- \frac{m}{g} (\mu^2 + B) \int dx \langle \pi(x) X \rangle_+ .
\end{aligned}
\tag{16}
$$

Thus the integrated Ward identity is proved.

Suppressing the integration and bringing to the left-hand side all the new terms appearing at the right-hand side which have the form $-\partial_\mu \langle j_5^{(1)\mu}(x) X \rangle_+$ we obtain Eq. (1) with $j_5^\mu = j_5^{(0)\mu} + j_5^{(1)\mu}$.

It remains to prove the $c_4 = c_5$.

If the integrated Ward identity is verified this happens order by order in $\hbar$ (i.e. in the loop number). The Feynman diagrams corresponding to c_4 and c_5 are superficially convergent, thus to a given order in $\hbar$ they contain renormalization corrections of lower orders. Now it is easy to convince oneself that if the integrated Ward identity is true to n^{th} order in $\hbar$ then $c_4 = c_5$ to $(n+1)^{\text{th}}$ order, this implies that the Ward identity is true to $(n+1)^{\text{th}}$ order. Since the integrated Ward identity is valid to 0^{th} order ($a = A = d = b = f = B = C = D = E = 0$) we can conclude that $c_4 = c_5$ to any order in $\hbar$.

3. The Callan-Symanzik Equation

We now study the Callan-Symanzik equations for the model defined in Section 2. Lowenstein [10] has shown that in the framework of the NPA a class of generalized Callan-Symanzik equations for the φ^4 model and for a massive vector meson model [7] can be obtained in a straightforward manner. Indeed the differentiation of a Green's function with respect to the parameters of the theory is equivalent to the insertion of new vertices in the corresponding Feynman diagrams. The independent vertex insertions (DVO's) correspond to the different terms in the Lagrangian. If there are more differential operations than DVO's, one gets directly linear relations among the differential operations (these are the Callan-Symanzik equations). Since our Lagrangian contains 13 terms, we have to exploit the content of the Ward identities in order to use this method.

We will show that six "differential operators" exist that leave unchanged the integrated Ward identities. They will thus be expressible in terms of symmetric linear combinations of the 13 DVO's.

Then we will show that the symmetric DVO's are only five and furthermore that only one symmetric vertex insertion of degree smaller than four exists (the actual degree being two). From these results the existence of two independent generalized Callan-Symanzik equations immediately follows.

We now study how the differential operators change the Ward identity (8). Taking into account explicitly the parameters of the model, we can write Eq. (8) in the symbolic form

$$R(F)\, W(g, m, \mu, \delta) = 0 \qquad \left(\text{where } F = \frac{m}{g}\right) \tag{17}$$

if we multiply g by $1+\eta$ we have to first order in η:

$$R(F(1-\eta))\, W(g(1+\eta), m, \mu, \delta) = 0\,. \tag{18}$$

Comparing Eq. (17) and Eq. (18) we get:

$$-F(\partial_F R)\, W + R(g\, \partial_g W) = 0\,. \tag{19}$$

From:

$$F(\partial_F R)\, W = F \int dx \frac{\delta}{\delta \Pi(x)} W = -R\left(F \int dx \frac{\delta}{\delta \Sigma(x)} W\right) = -R(F \Delta_\sigma W) \tag{20}$$

we have:

$$R((g\, \partial_g + F \Delta_\sigma)\, W) = 0 \tag{21.a}$$

and, by the same method, we get:

$$R((m\, \partial_m - F \Delta_\sigma)\, W) = 0\,, \tag{21.b}$$

$$R(\mu\, \partial_\mu W) = 0\,, \tag{21.c}$$

$$R(\delta\, \partial_\delta W) = 0\,. \tag{21.d}$$

The operator

$$N_B W = \int dx \left(\Pi(x) \frac{\delta}{\delta \Pi(x)} W + \Sigma(x) \frac{\delta}{\delta \Sigma(x)} W\right)$$

multiplies each vertex by the number of the corresponding boson legs. Since:

$$[R N_B - N_B R]\, W = R N_B W = F \int dx \frac{\delta}{\delta \Pi(x)} W = -R F \Delta_\sigma W\,. \tag{22}$$

We have:

$$R((N_B + F\Delta_\sigma)W) = 0\,. \tag{23}$$

In much the same way we can show that the operator N_P which multiplies each vertex by the number of proton legs satisfies the equation:

$$R(N_P W) = 0\,. \tag{24}$$

For $s = m, \mu, \delta, g$ Lowenstein has shown that:

$$s\,\partial_s W = \sum_{1}^{13} {}_j (s\,\partial_s c_j)\,\Delta_j W \tag{25}$$

where the c_j ($j = 1 \ldots 13$) are the coefficients of the 13 terms of the Lagrangian and the DVO's Δ_j represent the insertions of the corresponding vertices.

It is also easy to see that [10]

$$N_B W = \sum_{1}^{13} {}_j\, b_j \Delta_j W\,, \tag{26.a}$$

$$N_P W = \sum_{1}^{13} {}_j\, p_j \Delta_j W \tag{26.b}$$

and by Zimmermann's identity:

$$F\Delta_\sigma W = \sum_{1}^{13} {}_j\, f_j \Delta_j W\,. \tag{26.c}$$

By Eqs. (21.a–d), (23), (24), (25), (26), we know that the operators $m\,\partial_m - F\Delta_\sigma$ $\mu\,\partial_\mu$, $\delta\,\partial_\delta$, $g\,\partial_g + F\Delta_\sigma$, $N_B + F\Delta_\sigma$ and N_P correspond to linear combinations $\Delta^{(S)}$ of the Δ_i's such that:

$$R\,\Delta^{(S)}\,W = 0\,. \tag{27}$$

We now determine the number of independent $\Delta^{(S)}$. Suppose we change the coefficients of the effective Lagrangian in such a way that the generators of the proper vertices $W(\beta_1, \ldots, \beta_v)$ corresponding to the new Lagrangian

$$\mathscr{L}_{\text{eff}}(\beta_1, \ldots, \beta_v) = \mathscr{L}_{\text{eff}}^{(\sigma)} + \sum_{1}^{v} {}_i\, \beta_i N_4[O_i^{(S)}] \tag{28}$$

satisfies the equation $R(F)\,W(\beta_1, \ldots, \beta_v) = 0$ to first order in the $\beta_i S$'. Then to each $O_i^{(S)}$ there corresponds a $\Delta_i^{(S)}$ since

$$[\partial_{\beta_j} R\,W(\beta_1, \ldots, \beta_v)]_{\beta=0} = [R(\partial_{\beta_j} W(\beta_1, \ldots, \beta_v))]_{\beta=0} = R(\Delta_j^{(S)}W) = 0\,. \tag{29}$$

We know that, at fixed F, all the coefficients of the Lagrangian are known functions of five parameters (those which are fixed by the normalization conditions) hence we can infer that the number of independent $\Delta_i^{(S)}$'s is five. An alternative procedure to find the $\Delta_i^{(S)}$'s is to use the equation:

$$\begin{aligned}0 &= \int dx\, \partial_\mu \langle j_5^{(0)\mu}(x) \Delta_i^{(S)} X\rangle_+ = -c \int dx \langle \pi(x) \Delta_i^{(S)} X\rangle_+ - c_i' \int dx \langle \pi(x) X\rangle_+ \\ &+ i \sum_1^n {}_{i'} \langle \pi(x_{i'}) \Delta_i^{(S)} X_{\widehat{\sigma(x_{i'})}}\rangle_+ - i \sum_1^m {}_j \langle (\sigma(y_j) + F) \Delta_i^{(S)} X_{\widehat{\pi(y_j)}}\rangle_+ \\ &- \tfrac{1}{2} \left\{ \sum_1^p {}_k \langle (\overline{\psi}(z_k') \gamma_5)^{\beta_k} \Delta_i^{(S)} X_{\widehat{\overline{\psi}(z_k')^{\beta_k}}}\rangle_+ + \sum_1^p {}_l \langle (\gamma_5 \psi(z_l))_{\alpha_l} \Delta_i^{(S)} X_{\widehat{\psi(z_l)_{\alpha_l}}}\rangle_+ \right\},\end{aligned} \tag{30}$$

where $\Delta_i^{(S)} = i \int dx\, N_4[O_i^{(S)}](x)$, which is equivalent to Eq. (29). Indeed if $Z(\beta_1, \ldots, \beta_v)$ is the generator of the connected Green's functions for the Lagrangian $\mathscr{L}_{\text{eff}}(\beta_1, \ldots, \beta_v)$ it turns out from Eqs. (4) and (30) that $Z(\beta_1, \ldots, \beta_v)$ satisfies the integrated Ward identity (4) where c is replaced by $c(\beta_1, \ldots, \beta_v) = c + \sum_1^v {}_i c_i' \beta_i$. Then we can perform the Legendre transformation given by Eqs. (5) and (6) [replacing again c by $c(\beta_1, \ldots, \beta_v)$] and we obtain a new functional $W(\beta_1, \ldots, \beta_v)$ satisfying Eq. (8) and consequently Eq. (29). [It is easy to show that

$$W(\beta_1, \ldots, \beta_v) + \int dx\, c(\beta_1, \ldots, \beta_v)\, \Sigma(x)$$

generates the proper vertices.]

There remains to be studied how many independent vertex insertions $\Delta^{(S)} = i \int dx N[O^{(S)}](x)$ with $\delta < 4$ exist in the model which satisfy Eq. (30). For $\delta = 2$ we consider:

$$\Delta_0^{(S)} = \frac{i}{2} \int dx\, N_2[\xi(\pi^2 + \sigma^2) + \zeta \sigma^2](x). \tag{31}$$

By Zimmermann's formula, we obtain:

$$\begin{aligned}\int dx \langle (N_3[S_\pi](x) - N_4[S_\pi](x)) \Delta_0^{(S)} X\rangle_+ = \int dx \{ &\langle N_4[c_1 \pi \Box \sigma + c_2 \pi^3 \sigma \\ &+ c_3 \sigma^3 \pi + c_4 \overline{\psi} \psi \pi + i c_5 \overline{\psi} \gamma_5 \psi \sigma](x) \Delta_0^{(S)} X\rangle_+ \\ &+ (d_1 \xi + d_2 \zeta) \langle N_2[\sigma\pi](x) X\rangle_+ \}\end{aligned} \tag{32}$$

[compare with Eq. (14)]. Then we get in the same way as Eq. (13):

$$\begin{aligned}\int dx\, \partial_\mu \langle j_5^{(0)\mu}(x) \Delta_0^{(S)} X\rangle_+ &= i \sum_1^n {}_{i'} \langle \pi(x_{i'}) \Delta_0^{(S)} X_{\widehat{\sigma(x_{i'})}}\rangle_+ \\ &- \sum_1^m {}_j \left\langle \left(\sigma(y_j) + \frac{m}{g} \right) \Delta_0^{(S)} X_{\widehat{\pi(y_j)}} \right\rangle_+ - \tfrac{1}{2} \left\{ \sum_1^p {}_k \langle (\overline{\psi}(z_k') \gamma_5)^{\beta_k} \Delta_0^{(S)} X_{\widehat{\overline{\psi}(z_k')^{\beta_k}}}\rangle_+ \right. \\ &\left. + \sum_1^p {}_l \langle (\gamma_5 \psi(z_l))_{\alpha_l} \Delta_0^{(S)} X_{\widehat{\psi(z_l)_{\alpha_l}}}\rangle_+ \right\} - \frac{m}{g} (\mu^2 + B) \int dx \langle \pi(x) \Delta_0^{(S)} X\rangle_+ \\ &- \frac{m}{g} \xi \int dx \langle \pi(x) X\rangle_+ + (\zeta(1 - d_2) - d_1 \xi) \int dx \langle N_2[\sigma\pi](x) X\rangle_+ .\end{aligned} \tag{33}$$

In order to obtain Eq. (3), we have to put $\xi(1-d_2)-d_1\xi=0$. There is thus only one $\Delta^{(S)}$ for $\delta=2$. In much the same way it can be shown that there is only one $\Delta^{(S)}$ for $\delta=3$. It then follows from Zimmermann's reduction formula that the two symmetric insertion with $\delta=2$ and $\delta=3$ are proportional.

We can conclude that only one symmetric insertion ($\Delta_0^{(S)}$) exists of degree $\delta<4$.

Now it is easy to obtain the generalized Callan-Symanzik equations. Indeed, using Zimmermann's identity, we can write:

$$\Delta_0^{(S)}=\sum_1^5{}_j\, r_j \Delta_j^{(S)} \tag{34.a}$$

and by Eqs. (21.a–d), (23) and (24)

$$\begin{aligned} m\,\partial_m - F\,\Delta_\sigma &= \sum_1^5{}_j\, s_j \Delta_j^{(S)}\,,\\ \mu\,\partial_\mu &= \sum_1^5{}_j\, t_j \Delta_j^{(S)}\,,\\ \delta\,\partial_\delta &= \sum_1^5{}_j\, u_j \Delta_j^{(S)}\,,\\ g\,\partial_g + F\,\Delta_\sigma &= \sum_1^5{}_j\, v_j \Delta_j^{(S)}\,,\\ N_B + F\,\Delta_\sigma &= \sum_1^5{}_j\, w_j \Delta_j^{(S)}\,,\\ N_P &= \sum_1^5{}_j\, z_j \Delta_j^{(S)}\,. \end{aligned} \tag{34.b}$$

There are consequently two independent linear relations among the quantities (34.a–b) which we may take to be:

$$(D+t\,N_B+u\,N_P)\,W=(1-l-h-t)F\,\Delta_\sigma\,W+v\,\Delta_0^{(S)}\,W\,, \tag{35}$$

$$(D'+t'\,N_B+u'\,N_P)\,W=(1-l'-h'-t')F\,\Delta_\sigma\,W+v'\,\Delta_0^{(S)}\,W \tag{36}$$

where

$$\begin{aligned} D&=\lambda\,\partial_\lambda+hg\,\partial_g+l(\delta\,\partial_\delta-m\,\partial_m)\,,\\ D'&=\lambda\,\partial_\lambda+h'g\,\partial_g+l'(\mu\,\partial_\mu-m\,\partial_m) \end{aligned} \tag{37}$$

and

$$\lambda\,\partial_\lambda=m\,\partial_m+\mu\,\partial_\mu+\delta\,\partial_\delta\,. \tag{38}$$

The values of the coefficients h, l, t, u up to second order in g are computed in the Appendix.

Before concluding this section, it is convenient to see how Eq. (30) transforms if we suppress the integration. The new terms which appear in the left-hand side have the form

$$-\partial_\mu\{\langle j_5^{(1)\mu}(x)\Delta_i^{(S)}X\rangle_+ + \langle j_{5i}'^{\mu}(x)X\rangle_+\}$$

where $i=0, \ldots, 5$; bringing them to the right-hand side we obtain:

$$\begin{aligned}
&\partial_\mu\{\langle j_5^\mu(x)\Delta X\rangle_+ + \langle j_5'^\mu(x)X\rangle_+\} = -(c\langle\pi(x)\Delta X\rangle_+ + c'\langle\pi(x)X\rangle_+)\\
&+i\sum_{1}^{n}{}_i\,\delta(x-x_i)\,\langle\pi(x)\Delta X_{\widehat{\sigma(x_i)}}\rangle_+ - i\sum_{1}^{m}{}_j\,\delta(x-y_j)\,\langle(\sigma(x)+F)\Delta X_{\widehat{\pi(y_j)}}\rangle_+ \qquad (39)\\
&-\tfrac{1}{2}\sum_{1}^{p}{}_l\,\{\delta(x-z_l')\,\langle(\overline{\psi}(x)\gamma_5)^{\beta_l}\Delta X_{\widehat{\overline{\psi}(z_l')^{\beta_l}}}\rangle_+ + \delta(x-z_l)\,\langle(\gamma_5\psi(x))_{\alpha_l}\Delta X_{\widehat{\psi(z_l)_{\alpha_l}}}\rangle_+\}
\end{aligned}$$

where

$$j_5^\mu = j_5^{(0)\mu} + j_5^{(1)\mu}\,;\quad \Delta = \sum_{0}^{5}{}_i\,\beta_i\Delta_i^{(S)}\,;\quad c' = \sum_{0}^{5}{}_i\,\beta_i c_i'$$

and

$$j_5'^\mu = \sum_{0}^{5}{}_i\,\beta_i j_{5i}'^\mu\,.$$

We now write Eq. (1) and Eq. (39) more compactly in functional form. Let us consider the Lagrangian:

$$\mathscr{L}_{\text{eff}}^{(\sigma,\alpha)}(\beta_0, \ldots, \beta_5) = \mathscr{L}_{\text{eff}}^{(\sigma)} + \sum_{1}^{5}{}_i\,\beta_i N_4[O_i^{(S)}] + \beta_0 N_2[O_0^{(S)}] + \alpha_\mu j_5^\mu(\beta_0, \ldots, \beta_5) \qquad (40)$$

where $j_5^\mu(\beta_0, \ldots, \beta_5) = j_5^\mu + j_5'^\mu$ and α_μ is an external axial field. If the corresponding generator of the connected Green's functions is $Z[\alpha_\mu, \beta]$ we have by Eq. (1) and Eq. (39) up to first order in the β's:

$$\begin{aligned}
\partial_\mu\frac{\delta}{\delta\alpha_\mu(x)}Z[\alpha_\mu,\beta] &= J_\pi(x)\left(\frac{\delta}{\delta J_\sigma(x)}Z[\alpha_\mu,\beta]+F\right)\\
&-(J_\sigma(x)+c(\beta_0,\ldots,\beta_5))\frac{\delta}{\delta J_\pi(x)}Z[\alpha_\mu,\beta] \qquad (41)\\
&-\frac{i}{2}\left(\overline{\eta}(x)\gamma_5\frac{\delta}{\delta\overline{\eta}(x)}Z[\alpha_\mu,\beta]+Z[\alpha_\mu,\beta]\frac{\overleftarrow{\delta}}{\delta\eta(x)}\gamma_5\eta(x)\right).
\end{aligned}$$

4. Proof of the Theorem

We shall now discuss the coupling of the σ model to an external electromagnetic field. We will first study the changes of the Ward identities due to the electromagnetic field. Then using the modified Ward identities we shall show that the proper vertices containing photon legs satisfy the Callan-Symanzik equations (35) and (36). Finally, using Eq. (35), we shall prove the theorem.

Taking into account the vector current conservation we add to the effective Lagrangian the coupling term:

$$e(1+a)\,N_3[\overline{\psi}\gamma_\mu\psi]\,A^\mu = j_\mu A^\mu \tag{42}$$

where A^μ is the vector potential.

Let us consider how the Ward identities [Eqs. (1) and (39)] change in the presence of the electromagnetic field. We develop in the usual way the divergence $\partial_\mu\langle j_5^\mu X\,Y\rangle_+$ when $Y=\prod_1^r{}_i\, j_{\mu_i}(w_i)$. Since j_μ is formally chiral invariant each term of Eqs. (1) and (39) can be multiplied by Y (within the vacuum expectation value). In addition new terms coming from the reduction of $N_3[S_\pi]$ to $N_4[S_\pi]$ in the presence of the vertices (42) must be introduced. [Denoting

$$\langle(N_3[S_\pi]-N_4[S_\pi])X\rangle_+ = \langle A X\rangle_+\,,$$
$$\langle(N_3[S_\pi]-N_4[S_\pi])X\,Y\rangle_+ = \langle A X\,Y\rangle_+ + \langle B X\rangle_+\,,$$

we are just considering the terms $\langle B X\rangle_+$.] Because of vector current conservation and charge conjugation we see that:

(i) no proper superficially divergent diagram exists in the model that contains $N_4[S_\pi]$, one, three or four currents j_μ and any product of the insertions $\Delta_i^{(S)}$ $(i=0,\ldots,5)$;

(ii) the only proper superficially divergent vertices with one $N_4[S_\pi]$ vertex and two currents have no external leg except the two photons and contain no $\Delta_0^{(S)}$ insertion.

Comparing with the discussion at the end of the preceding section we can immediately write the Ward identity:

$$\begin{aligned}
\partial_\mu \frac{\delta}{\delta\alpha_\mu(x)}\,Z[\alpha_\mu,A_\mu,\beta] &= J_\pi(x)\left(\frac{\delta}{\delta J_\sigma(x)}\,Z[\alpha_\mu,A_\mu,\beta]+F\right)\\
&\quad-(J_\sigma(x)+c(\beta_0,\ldots,\beta_5))\frac{\delta}{\delta J_\pi(x)}\,Z[\alpha_\mu,A_\mu,\beta]\\
&\quad-\frac{i}{2}\left(\overline{\eta}(x)\gamma_5\frac{\delta}{\delta\overline{\eta}(x)}\,Z[\alpha_\mu,A_\mu,\beta]+Z[\alpha_\mu,A_\mu,\beta]\frac{\overleftarrow{\delta}}{\delta\eta(x)}\gamma_5\eta(x)\right)\\
&\quad+r(\beta_1,\ldots,\beta_5)\,\varepsilon^{\mu\nu\varrho\sigma}F_{\mu\nu}(x)F_{\varrho\sigma}(x)
\end{aligned}\tag{43}$$

where $Z[\alpha_\mu,A_\mu,\beta]$ is the generator of the connected Green's functions corresponding to the Lagrangian

$$\mathscr{L}_{\text{eff}}^{(\sigma,A,\alpha)}=\mathscr{L}_{\text{eff}}^{(\sigma,\alpha)}+A_\mu j^\mu\,,\qquad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\,,$$

and

$$r(\beta_1,\ldots,\beta_5)=r+\sum_1^5{}_i\, r_i\beta_i$$

with

$$r = \frac{1}{8 \cdot 4!} \varepsilon^{\mu\nu\varrho\sigma} [\partial_{q_\varrho} \partial_{k_\sigma} \langle N_3 [S_\pi] (0) \tilde{j}_\mu(q) \tilde{j}_\nu(k) \rangle_+^{PROP}]_{p=q=0}, \tag{44}$$

$$r_i = \frac{1}{8 \cdot 4!} \varepsilon^{\mu\nu\varrho\sigma} [\partial_{q_\varrho} \partial_{k_\sigma} \langle N_3 [S_\pi] (0) \Delta_i^{(S)} \tilde{j}_\mu(q) \tilde{j}_\nu(k) \rangle_+^{PROP}]_{p=q=0}. \tag{45}$$

The last term in Eq. (43) is the axial anomaly. Performing again the Legendre transformation (6) we obtain the new generator of the proper vertices $W[\alpha_\mu, A_\mu, \beta]$ which satisfies the equation:

$$\begin{aligned} -\partial_\mu \frac{\delta}{\delta \alpha_\mu(x)} W[\alpha_\mu, A_\mu, \beta] &= (\Sigma(x) + F) \frac{\delta}{\delta \Pi(x)} W[\alpha_\mu, A_\mu, \beta] \\ - \Pi(x) \frac{\delta}{\delta \Sigma(x)} W[\alpha_\mu, A_\mu, \beta] &- \frac{i}{2} \left(\overline{\Psi}(x) \gamma_5 \frac{\delta}{\delta \overline{\Psi}(x)} W[\alpha_\mu, A_\mu, \beta] \right. \\ \left. + W[\alpha_\mu, A_\mu, \beta] \frac{\overleftarrow{\delta}}{\delta \Psi(x)} \gamma_5 \Psi(x) \right) &- r(\beta_1, \ldots, \beta_5) \varepsilon^{\mu\nu\varrho\sigma} F_{\mu\nu}(x) F_{\varrho\sigma}(x) \end{aligned} \tag{46}$$

up to first order in the β_i's.

We now come to the Callan-Symanzik equations for the proper vertices with ν photon legs which are generated by:

$$\left[\prod_1^\nu {}_i \frac{\delta}{\delta A^{\mu_i}(x_i)} W[0, A_\mu, 0] \right]_{A_\mu = 0} = W^{\cdot}_{\mu_1 \ldots \mu_\nu}. \tag{47}$$

In this discussion we never consider the vacuum polarization vertex. Following the procedure used in order to obtain Eq. (25) and Eqs. (26.a–b), we get for $s = m, \mu, \delta, g$:

$$s\, \partial_s W_{\mu_1 \ldots \mu_\nu} - \nu (s\, \partial_s \log(1 + a)) W_{\mu_1 \ldots \mu_\nu} = \sum_1^{13} {}_j (s\, \partial_s c_j) \Delta_j W_{\mu_1 \ldots \mu_\nu}, \tag{48.a}$$

$$N_B W_{\mu_1 \ldots \mu_\nu} = \sum_1^{13} {}_j b_j \Delta_j W_{\mu_1 \ldots \mu_\nu}, \tag{48.b}$$

$$(N_P - 2\nu) W_{\mu_1 \ldots \mu_\nu} = \sum_1^{13} {}_j p_j \Delta_j W_{\mu_1 \ldots \mu_\nu}. \tag{48.c}$$

Zimmermann's identities (26.c) and (34.a) are modified in the following way:

$$F \Delta_\sigma W_{\mu_1 \ldots \mu_\nu} - \nu \gamma_\sigma (1 + a)^{-1} W_{\mu_1 \ldots \mu_\nu} = \sum_1^{13} {}_j f_j \Delta_j W_{\mu_1 \ldots \mu_\nu}, \tag{49}$$

$$\Delta_0^{(S)} W_{\mu_1 \ldots \mu_\nu} - \nu \gamma_S (1 + a)^{-1} W_{\mu_1 \ldots \mu_\nu} = \sum_1^{5} {}_j r_j \Delta_j^{(S)} W_{\mu_1 \ldots \mu_\nu} \tag{50}$$

where:

$$\gamma_\sigma = \tfrac{1}{4} F \Delta_\sigma \left[\mathrm{Tr} \left\{ \gamma_\mu \frac{\delta}{\delta \tilde{A}_\mu(0)} \frac{\delta}{\delta \tilde{\bar{\Psi}}(0)} \mathrm{W}\,[0, \mathrm{A}_\mu, 0] \frac{\overleftarrow{\delta}}{\delta \tilde{\Psi}(0)} \right\} \right]_{A=\varphi=0}, \tag{51}$$

$$\gamma_S = \tfrac{1}{4}\, \Delta_0^{(S)} \left[\mathrm{Tr} \left\{ \gamma_\mu \frac{\delta}{\delta \tilde{A}_\mu(0)} \frac{\delta}{\delta \tilde{\bar{\Psi}}(0)} W[0, A_\mu, 0] \frac{\overleftarrow{\delta}}{\delta \tilde{\Psi}(0)} \right\} \right]_{A=\varphi=0}. \tag{52}$$

Since the vector current is conserved, we also have:

$$\gamma_\sigma = \tfrac{1}{4} F \Delta_\sigma \left[\mathrm{Tr} \left\{ \partial\!\!\!/_p \frac{\delta}{\delta \tilde{\bar{\Psi}}(-p)} W \frac{\overleftarrow{\delta}}{\delta \tilde{\Psi}(p)} \right\} \right]_{p=\varphi=0}, \tag{53}$$

$$\gamma_S = \tfrac{1}{4}\, \Delta_0^{(S)} \left[\mathrm{Tr} \left\{ \partial\!\!\!/_p \frac{\delta}{\delta \tilde{\bar{\Psi}}(-p)} W \frac{\overleftarrow{\delta}}{\delta \tilde{\Psi}(p)} \right\} \right]_{p=\varphi=0}. \tag{54}$$

By applying Eq. (35) to the vertex

$$\left[\mathrm{Tr} \left\{ \partial\!\!\!/_p \frac{\delta}{\delta \tilde{\bar{\Psi}}(-p)} W \frac{\overleftarrow{\delta}}{\delta \tilde{\Psi}(p)} \right\} \right]_{p=\varphi=0} = 1 + a$$

we obtain:

$$D(1+a) + 2u(1+a) = (1-l-h-t)\gamma_\sigma + v\gamma_S\,. \tag{55}$$

Comparing Eqs. (48.a–c), (49), (50) with Eqs. (25), (26.a–c), (34.a), we get the Callan-Symanzik equation for $W_{\mu_1 \dots \mu_\nu}$ corresponding to Eq. (35):

$$\begin{aligned} (D + t N_B + u N_P) W_{\mu_1 \dots \mu_\nu} &- v(D \log(1+a) + 2u) W_{\mu_1 \dots \mu_\nu} \\ &= (1-l-h-t) F \Delta_\sigma W_{\mu_1 \dots \mu_\nu} + v \Delta_0^{(S)} W_{\mu_1 \dots \mu_\nu} \\ &\quad - v((1-l-h-t)\gamma_\sigma + v\gamma_S)(1+a)^{-1} W_{\mu_1 \dots \mu_\nu} \end{aligned} \tag{56}$$

which, by Eq. (55), becomes:

$$(D + t N_B + u N_P) W_{\mu_1 \dots \mu_\nu} = (1-l-h-t) F \Delta_\sigma W_{\mu_1 \dots \mu_\nu} + v \Delta_0^{(S)} W_{\mu_1 \dots \mu_\nu}\,. \tag{57}$$

Recalling that r is proportional to

$$F \varepsilon^{\mu\nu\varrho\sigma} \left[\partial_{q_\varrho} \partial_{k_\sigma} \frac{\delta}{\delta \tilde{A}_\mu(q)} \frac{\delta}{\delta \tilde{A}_\nu(k)} \frac{\delta}{\delta \tilde{\Pi}(-q-k)} W[0, A_\mu, 0] \right]_{A=\varphi=k=q=0}$$

and applying Eq. (57) to r, we obtain:

$$Dr = (D \log F) r + F D \frac{r}{F} = (1-l-h) r - t r + (1-l-h-t)\, F \Delta_\sigma r + v \Delta_0^{(S)} r\,. \tag{58}$$

To recast Eq. (58) in a simpler form, we remark that, from Eq. (46), we have:

$$\begin{aligned}
&-\left[\int dw \frac{\delta}{\delta\Sigma(w)}\partial_\varrho \frac{\delta}{\delta\alpha_\varrho(x)}\frac{\delta}{\delta A_\mu(y)}\frac{\delta}{\delta A_\nu(z)} W[\alpha_\mu, A_\mu, 0]\right]_{\alpha=A=\varphi=0}\\
&=F\left[\int dw \frac{\delta}{\delta\Sigma(w)}\frac{\delta}{\delta\Pi(x)}\frac{\delta}{\delta A_\mu(y)}\frac{\delta}{\delta A_\nu(z)} W[0, A_\mu, 0]\right]_{A=\varphi=0}\\
&+\left[\frac{\delta}{\delta\Pi(x)}\frac{\delta}{\delta A_\mu(y)}\frac{\delta}{\delta A_\nu(z)} W[0, A_\mu, 0]\right]_{A=\varphi=0}
\end{aligned} \tag{59}$$

$$\begin{aligned}
&-\left[\partial_{\beta_0}\partial_\varrho \frac{\delta}{\delta\alpha_\varrho(x)}\frac{\delta}{\delta A_\mu(y)}\frac{\delta}{\delta A_\nu(z)} W[\alpha_\mu, A_\mu, \beta]\right]_{\beta=\alpha=A=\varphi=0}\\
&=F\left[\partial_{\beta_0}\frac{\delta}{\delta\Pi(x)}\frac{\delta}{\delta A_\mu(y)}\frac{\delta}{\delta A_\nu(z)} W[0, A_\mu, \beta]\right]_{\beta=A=\varphi=0}.
\end{aligned} \tag{60}$$

Applying to the left-hand side of Eqs. (59) and (60) the well-known low energy theorem for the vacuum expectation value of the time ordered product of the divergence of the axial current and of two electromagnetic currents, we obtain:

$$\begin{aligned}
&\lim_{\substack{k\to 0\\ q\to 0}} \varepsilon^{\mu\nu\varrho\sigma}\partial_{q_\varrho}\partial_{k_\sigma}\left[F\frac{\delta}{\delta\tilde\Sigma(0)}\frac{\delta}{\delta\tilde\Pi(-k-q)}\frac{\delta}{\delta\tilde A_\mu(q)}\frac{\delta}{\delta\tilde A_\nu(k)} W[0, A_\mu, 0]\right.\\
&\qquad\left.+\frac{\delta}{\delta\tilde\Pi(-k-q)}\frac{\delta}{\delta\tilde A_\mu(q)}\frac{\delta}{\delta\tilde A_\nu(k)} W[0, A_\mu, 0]\right]_{A=\varphi=0} = 0\,,
\end{aligned} \tag{61}$$

$$\lim_{\substack{k\to 0\\ q\to 0}} \varepsilon^{\mu\nu\varrho\sigma}\partial_{q_\varrho}\partial_{k_\sigma}\left[\partial_{\beta_0}\frac{\delta}{\delta\tilde\Pi(-k-q)}\frac{\delta}{\delta\tilde A_\mu(q)}\frac{\delta}{\delta\tilde A_\nu(k)} W[0, A_\mu, \beta]\right]_{A=\beta=\varphi=0} = 0\,. \tag{62}$$

From Eqs. (61), (62), we obtain $F\Delta_\sigma r = -r$ and $\Delta_0^{(S)} r = 0$. Then Eq. (58) becomes:

$$Dr = 0\,. \tag{63}$$

For reasons of dimensionality r is a function of g and of the mass ratios. In terms of the variables g, $x = (\delta^2/m^2) - \sqrt{\tfrac{8}{5}}$, $y = (\delta m/\mu^2)$, Eq. (63) becomes:

$$(H(g,x)g\,\partial_g + L(g,x)\,\partial_x)\, r(g,x) = 0 \tag{64}$$

where the fixed parameter y is omitted. From the Appendix, we obtain:

$$H(g,x) = \frac{h(g,x)}{4\left(\dfrac{g}{4\pi}\right)^2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} H_{m,n} g^m x^n, \quad \text{with} \quad H_{0,0} = 1\,, \tag{65}$$

$$\begin{aligned}
L(g,x) &= \frac{l(g,x)}{4\left(\dfrac{g}{4\pi}\right)^2}\left(x + \sqrt{\tfrac{8}{5}}\right) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} L_{m,n} g^m x^n\,,\\
&\text{with} \quad L_{0,0} = 0 \quad \text{and} \quad L_{0,1} = \sqrt{40}
\end{aligned} \tag{66}$$

since H, L, and r are formal power series in g whose coefficients are analytic functions of x around $x=0$[2], we put:

$$r = r_0 + \sum_{1}^{\infty} {}_m \sum_{0}^{\infty} {}_n \, r_{m,n} g^m x^n$$

and we obtain, from Eq. (64):

$$\sum_{0}^{\infty} {}_{m'} \sum_{1}^{\infty} {}_m \sum_{0}^{\infty} {}_{n'} \sum_{0}^{\infty} {}_n (H_{m',n'} m g^{m+m'} x^{n+n'} + L_{m',n'} n g^{m+m'} x^{n+n'-1}) r_{m,n} = 0 \qquad (67)$$

which implies that for any M and N:

$$\sum_{1}^{M} {}_m \sum_{0}^{N} {}_n (m r_{m,n} H_{M-m,N-n} + (n+1) r_{m,n+1} L_{M-m,N-n}) = 0 \,. \qquad (68)$$

For $M=1$, $N=0$, Eq. (68) gives $r_{1,0} H_{0,0} = 0$, for $M=1$, $N=1$, we have $r_{1,1}(L_{0,1} + H_{0,0}) = 0$ and taking into account the relations obtained from $M=1$ up to $N = \bar{N} - 1$ we get for $M=1$, $N = \bar{N}: r_{1,\bar{N}}(H_{0,0} + \bar{N} L_{0,1}) = 0$. If we now increase M, we obtain for arbitrary values of M and N:

$$(M H_{0,0} + N L_{0,1}) r_{M,N} = 0 \qquad (69)$$

(since of course $M H_{0,0} + N L_{0,1}$ never vanishes). Equation (69) implies that $r = r_0$ which does not depend on y. Thus the Adler-Bardeen theorem is proved.

5. A Comment

The proof of the Adler-Bardeen theorem for the σ model is analogous to the one given by Zee and by Lowenstein and Schroer in the case of spinor electrodynamics with some differences which are due to the structures of the models.

Indeed a "true" Callan-Symanzik equation does not exist in our case. By "true", we mean an equation which does not contain derivatives with respect to mass ratios. It is interesting to point out that in the case of the σ model without fermions a "true" Callan-Symanzik equation does exist. The basic difference between the two models is that in the symmetric limit the proton is massless.

Acknowledgements. I am deeply indebted to Professor R. Stora for constant encouragement and help. I wish also to thank Professors J. S. Bell and V. Glaser for interest and discussions.

[2] For $x=0$ and $y \simeq 1$, all the particles of the theory are stable.

Appendix

To compute the coefficients of Eq. (35) we apply it to some simple vertices. To zeroth order in g we immediately obtain:

$$l = h = t = u = 0\,,$$
$$v = \frac{\delta^2}{2} - \mu^2\,. \tag{A.1}$$

To second order we start considering the vertex:

$$\frac{1}{8}\left[\partial_{p_\mu}\partial_{p^\mu}\frac{\delta}{\delta\tilde{\Pi}(p)}\,\frac{\delta}{\delta\tilde{\Pi}(-p)}\,W\right]_{p=\varphi=0} = 1 + b \tag{A.2}$$

where $b \sim O_{(g^2)}$. Since $\lambda\,\partial_\lambda(1+b) = 0$ we have, from Eq. (35)

$$\begin{aligned} hg\,\partial_g b + l(\delta\,\partial_\delta - m\,\partial_m)b + 2t(1+b) &\\ = (1-h-l-t)\frac{F\Delta_\sigma}{8}\left[\partial_{p_\mu}\partial_{p^\mu}\frac{\delta}{\delta\tilde{\Pi}(\mathrm{p})}\,\frac{\delta}{\delta\tilde{\Pi}(-\mathrm{p})}\,W\right]_{p=\varphi=0} &\\ + \frac{v}{8}\left[\partial_{p_\mu}\partial_{p^\mu}\frac{\delta}{\delta\tilde{\Pi}(p)}\,\frac{\delta}{\delta\tilde{\Pi}(-p)}\,\Delta_0^{(S)}\,W\right]_{p=\varphi=0} & \end{aligned} \tag{A.3}$$

selecting the terms which are $O_{(g^2)}$ we obtain:

$$2t = \frac{1}{8}\left[\partial_{p^\mu}\partial_{p_\mu}\frac{\delta}{\delta\tilde{\Pi}(p)}\,\frac{\delta}{\delta\tilde{\Pi}(-p)}\left(F\Delta_\sigma + \left(\frac{\delta^2}{2} - \mu^2\right)\Delta_0^{(S)}\right)W\right]_{p=\varphi=0} \tag{A.4}$$

which can be written in the form:

$$2t = -i\left[\int\frac{dq}{(2\pi)^4}\,\frac{\partial_{p_\mu}\partial_{p^\mu}}{8}\left(F\Delta_\sigma + \left(\frac{\delta^2}{2} - \mu^2\right)\Delta_0^{(S)}\right)I(q,p)\right]_{p=0} \tag{A.5}$$

where $I(q, p)$ is the integrand corresponding to the sum of Feynman diagrams in Fig. 1:

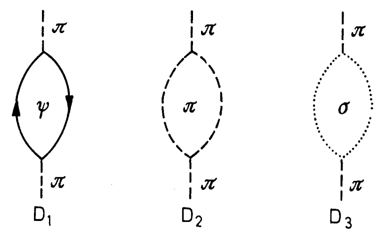

Fig. 1

Δ_σ represents the addition of a σ leg and $\Delta_0^{(S)}$ the insertion of a $(\pi^2+\sigma^2)/2$ vertex. Up to one loop the Feynman integrands satisfy the equation:

$$\left(F\,\Delta_\sigma+\left(\frac{\delta^2}{2}-\mu^2\right)\Delta_0^{(S)}\right)I(q,p)=\lambda\,\partial_\lambda I(q,p)\,. \tag{A.6}$$

Thus

$$2t=\frac{-i}{8}\left[\int\frac{dq}{(2\pi)^4}\,\lambda\,\partial_\lambda\,\partial_{p_\mu}\partial_{p^\mu}I(q,p)\right]_{p=0} \tag{A.7}$$

t is completely determined by the non-integrable part of $\partial_{p_\mu}\partial_{p^\mu}I(q,p)$ (for the integrable part we can extract ∂_λ from the integral and obtain zero). Thus the only contribution to t comes from D_1:

$$\begin{aligned} t&=-\frac{i}{16}g^2\int\frac{dq}{(2\pi)^4}\,m\,\partial_m\,\partial_{p_\mu}\partial_{p^\mu}\,\mathrm{Tr}\left\{\gamma_5\frac{1}{\not{p}+\not{q}-m}\gamma_5\frac{1}{\not{q}-m}\right\}\\ &=-ig^2\int\frac{dq}{(2\pi)^4}\,m\,\partial_m\frac{1}{(q^2-m^2)^2}=-2\left(\frac{g}{4\pi}\right)^2. \end{aligned} \tag{A.8}$$

In much the same way, applying Eq. (35) to

$$\frac{1}{4}\left[\mathrm{Tr}\left\{\bar{\partial}_p\frac{\delta}{\delta\bar{\tilde{\Psi}}(-p)}W\frac{\overleftarrow{\delta}}{\delta\tilde{\Psi}(p)}\right\}\right]_{p=\varphi=0}=1+a$$

we obtain to second order in g:

$$u=-\frac{i}{8}\int\frac{dq}{(2\pi)^4}\,\lambda\,\partial_\lambda\,\mathrm{Tr}\{\bar{\partial}_p I'(q,p)\} \tag{A.9}$$

where $I'(q,p)$ is the integrand corresponding to the sum of diagrams in Fig. 2

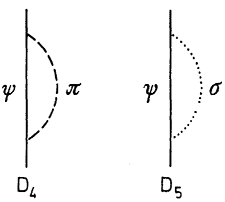

Fig. 2

Thus

$$\begin{aligned} u=&-\frac{i}{2}g^2\int\frac{dq}{(2\pi)^4}\,\lambda\,\partial_\lambda\left[\mathrm{Tr}\left\{\bar{\partial}_p\left(\frac{1}{\not{p}+\not{q}-m}\,\frac{1}{q^2-\mu^2-\delta^2}\right.\right.\right.\\ &\left.\left.\left.-\gamma_5\frac{1}{\not{p}+\not{q}-m}\,\frac{1}{q^2-\mu^2}\gamma_5\right)\right\}\right]_{p=0}\\ =&-\frac{i}{2}g^2\int\frac{dq}{(2\pi)^4}\,m\,\partial_m\frac{1}{(q^2-m^2)^2}=-\left(\frac{g}{4\pi}\right)^2. \end{aligned} \tag{A.10}$$

Then considering

$$\tfrac{1}{4}\left[\mathrm{Tr}\left\{\gamma_5 \frac{\delta}{\delta \tilde{\Pi}(0)} \frac{\delta}{\delta \tilde{\bar{\Psi}}(0)} W \frac{\overleftarrow{\delta}}{\delta \tilde{\Psi}(0)}\right\}\right]_{\varphi=0}$$

we obtain:

$$h+t+2u=0 \quad \text{and} \quad h=4\left(\frac{g}{4\pi}\right)^2 \tag{A.11}$$

since the divergent parts of the two diagrams in Fig. 3 cancel.

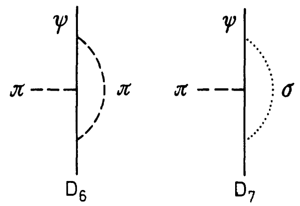

Fig. 3

Finally we consider the vertex

$$\left[\left(\frac{\delta}{\delta \tilde{\Pi}(0)}\right)^4 W\right]_{\varphi=0} = -3(\delta^2+C)\frac{g^2}{m^2}.$$

Taking into account the diagrams in Fig. 4:

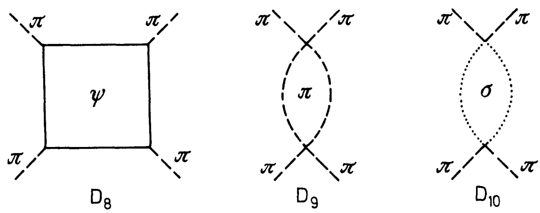

Fig. 4

we obtain:

$$(h+2l+2t)\frac{\delta^2 g^2}{m^2} = 2l\frac{\delta^2 g^2}{m^2} = -ig^4 \int \frac{dq}{(2\pi)^4} \lambda \partial_\lambda \left[\mathrm{Tr}\left\{\left(\gamma_5 \frac{1}{\not q - m}\right)^4\right\}\right.$$

$$\left.-\frac{\delta^4}{m^4}\left(\frac{9}{4}\,\frac{1}{(q^2-\mu^2)^2} + \frac{1}{4}\,\frac{1}{(q^2-\mu^2-\delta^2)^2}\right)\right] = \left(5\frac{\delta^4}{m^4} - 8\right)\frac{g^4}{(4\pi)^2}. \tag{A.12}$$

It then follows:

$$l = \left(\frac{g}{4\pi}\right)^2\left(\frac{5}{2}\,\frac{\delta^2}{m^2} - 4\frac{m^2}{\delta^2}\right). \tag{A.13}$$

For $(\delta^2/m^2)=\sqrt{\frac{8}{5}}\, l=0$ because the divergent parts of D_8, D_9, and D_{10} cancel.

Thus we have up to second order in g

$$D=\lambda\,\partial_\lambda+4\left(\frac{g}{4\pi}\right)^2\left(g\,\partial_g+\left(\frac{5}{2}\,\xi^2-4\right)\partial_\xi\right)$$

where $\xi=(\delta^2/m^2)$.

References

1. Adler, S. L.: Phys. Rev. **177**, 2426 (1969)
2. Bell, J. S., Jackiw, R.: Nuovo Cimento **60**, 47 (1969)
3. Adler, S. L.: In: 1970 Brandeis University Summer Institute Lectures. Cambridge; Mass.: M.I.T. Press
4. Adler, S. L., Bardeen, W.: Phys. Rev. **182**, 1517 (1969)
5. Gell-Mann, M., Lêvy, M.: Nuovo Cimento **16**, 705 (1960)
6. Zee, A.: Phys. Rev. Letters **29**, 1198 (1972)
7. Lowenstein, J. H., Schroer, B.: New York University Report No 18/72 (October 1972)
8. Callan, C. G., Jr.: Phys. Rev. D **2**, 1541 (1970). — Symanzik, K.: Commun. math. Phys. **18**, 227 (1970)
9. Zimmermann, W.: In: 1970 Brandeis University Summer Institute Lectures. Cambridge; Mass.: M.I.T. Press
10. Lowenstein, J. H.: Commun. math. Phys. **24**, 1 (1971)
11. Lee, B. W.: Nuclear Phys. B **9**, 649 (1969). — Gervais, J. L., Lee, B. W.: Nuclear Phys. B **12**, 627 (1969)
12. Symanzik, K.: Commun. math. Phys. **16**, 48 (1970). — Symanzik, K.: Lectures given at the Summer School of Theoretical Physics, Cargèse (July 1970), DESY Preprint No 70/62 (November 1970)
13. Schroer, B.: Lecture Notes in Physics, Vol. **17**. Berlin-Heidelberg-New York: Springer 1972
14. Jona-Lasinio, G.: Nuovo Cimento **34**, 1790 (1964)
15. Lowenstein, J. H.: Phys. Rev. D **4**, 2281 (1971)
16. Sutherland, D. G.: Nuclear Phys. B **2**, 433 (1967)

C. Becchi
Università di Genova
Istituto di Scienze Fisiche
Viale Benedetto XV, 5
Genova, Italia