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Remarks on Spectra of Modular Operators of von Neumann Algebras

Huzihiro Araki*

Department of Mathematics, Queen's University, Kingston, Ontario, Canada

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Abstract. It is shown that if ϱ is an invariant state of an asymptotically abelian C^* algebra $\mathfrak{A}$, then the spectrum of modular operator for ϱ is contained in the spectrum of any other modular operator for the von Neumann algebra $\pi_\varrho(\mathfrak{A})''$.

It is also shown that a modular operator can not have an isolated spectrum with a finite multiplicity at 1 unless the associated Hilbert space is of finite dimension. It is further shown that if a modular operator has an isolated spectrum with a finite multiplicity at $x \neq 1$, then the von Neumann algebra $\mathfrak{R}$ is a direct sum of $\mathfrak{R}_1$ and $\mathfrak{R}_2$ where $\mathfrak{R}_1$ is represented on a finite dimensional Hilbert space and the modular operator for $\mathfrak{R}_2$ does not have its spectrum at x.

Applications to Connes invariant are given.

§ 1. Preliminaries

A net of operators Q_α in a von Neumann algebra $\mathfrak{R}$ is called weakly (or strongly) central if there exists weakly total self adjoint subset $\mathfrak{R}_0$ of $\mathfrak{R}$ such that $[Q_\alpha, Q] \to 0$ weakly (or strongly) for every $Q \in \mathfrak{R}_0$. If Q_α is uniformly bounded and weakly central, then $\text{w-lim}\,[Q_\alpha, Q] = 0$ for all $Q \in \mathfrak{R}$ ([1]).

A subset $\mathfrak{A}$ of $\mathfrak{R}$ is called weakly (or strongly) τ_α central relative to a net of * automorphisms τ_α of $\mathfrak{R}$ if $\tau_\alpha Q$ is weakly (or strongly) central in $\mathfrak{R}$ for each $Q \in \mathfrak{A}$.

For any state ϱ of $\mathfrak{R}$, we denote by H_ϱ, π_ϱ and Ω_ϱ a Hilbert space, a representation of $\mathfrak{R}$ on H_ϱ and a cyclic vector in H_ϱ associated with ϱ through the relation

$$\varrho(Q) = (\Omega_\varrho, \pi_\varrho(Q)\,\Omega_\varrho)\,, \qquad Q \in \mathfrak{R}\,.$$

J_ϱ and Δ_ϱ denote modular conjugation operator and modular operator for Ω_ϱ when ϱ is faithful. $\bar\tau_\varrho(t)\,Q \equiv \Delta_\varrho^{it}\,Q\,\Delta_\varrho^{-it}$.

If ϱ is τ_α invariant, then there exists a unitary U_α such that $U_\alpha \pi_\varrho(Q)\,\Omega_\varrho = \pi_\varrho(\tau_\alpha Q)\,\Omega_\varrho$ for all $Q \in \mathfrak{R}$. We denote $U_\alpha Q U_\alpha^* = \bar\tau_\alpha Q$ for $Q \in \mathscr{B}(H_\varrho)$.

The following result has been obtained in [1]. (See also appendix.)

* On leave from Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan.

Lemma 1. *Let a weakly dense * subalgebra $\mathfrak{A}$ of $\mathfrak{R}$ be strongly τ_α central, ϱ be a faithful normal state on $\mathfrak{R}$, invariant under all τ_α and $\hat{\mathfrak{A}}$ be the C^* algebra generated by $\pi_\varrho(\mathfrak{A}) j_\varrho\{\pi_\varrho(\mathfrak{A})\}$ where $j_\varrho(Q) = J_\varrho Q J_\varrho$. Let $\hat{\varrho}$ denote the vector state on $\mathscr{B}(H_\varrho)$ by the vector Ω_ϱ and ϱ' be any normal state on $\mathscr{B}(H_\varrho)$, such that its restriction to the center $\mathfrak{Z} = \pi_\varrho(\mathfrak{R}) \cap \pi_\varrho(\mathfrak{R})'$ of $\mathfrak{R}$ is the same as that of $\hat{\varrho}: \hat{\varrho}(z) = \varrho'(z)$ for all $z \in \mathfrak{Z}$. Then*

$$\lim \varrho'(\bar{\tau}_\alpha Q) = \hat{\varrho}(Q), \qquad Q \in \hat{\mathfrak{A}}.$$

To achieve the situation $\varrho'|_{\mathfrak{Z}} = \hat{\varrho}|_{\mathfrak{Z}}$, we use the following commutative Radon-Nikodym Theorem. Here, $s(\varrho)$ denotes the support projection of ϱ.

Lemma 2. *Let ϱ_1 and ϱ_2 be normal states of a commutative von Neumann algebra $\mathfrak{Z}$ and $s(\varrho_1) \geqq s(\varrho_2)$. (The last condition is automatically fulfilled if ϱ_1 is faithful.) Then there exists a non-negative self adjoint operator $A^{\mathfrak{Z}}(\varrho_2/\varrho_1)$ affiliated with $\mathfrak{Z}$ such that Ω_{ϱ_1} is in the domain of $\pi_{\varrho_1}(A^{\mathfrak{Z}}(\varrho_2/\varrho_1)) (\equiv \int \lambda d\pi_{\varrho_1}(E_\lambda)$ if $A^{\mathfrak{Z}}(\varrho_2/\varrho_1) = \int \lambda dE_\lambda)$ and the vector state on $\mathfrak{Z}$ by the vector $\Omega' \equiv \pi_{\varrho_1}(A^{\mathfrak{Z}}(\varrho_2/\varrho_1))\,\Omega_{\varrho_1}$ is ϱ_2.*

$A(\varrho_2/\varrho_1)$ is the positive square root of Radon-Nikodym derivate in measure theoretical sense.

Lemma 3. *Let $\mathfrak{R}$ be a von Neumann algebra on H and Ω and Ω' be two cyclic and separating vectors related by $\Omega' = A\Omega$ where A is a positive self adjoint operator affiliated with center $\mathfrak{Z} = \mathfrak{R} \cap \mathfrak{R}'$. Then $\Delta_\Omega = \Delta_{\Omega'}$.*

Proof. Let $z \in \mathfrak{Z}$, $z = z^*$ and $S_\Omega = J_\Omega \Delta_\Omega^{1/2}$.

Then

$$S_\Omega Q z \Omega = z Q^* \Omega = Q^* z \Omega, \qquad Q \in \mathfrak{R}.$$

Let $A = \int \lambda dE_\lambda$, $A_L = AE_L$. Then $A_L \in \mathfrak{Z}$, $A_L^* = A_L$. Further,

$$\lim_{L \to +\infty} Q A_L \Omega = Q\Omega', \qquad \lim_{L \to +\infty} Q^* A_L \Omega = Q^* \Omega'$$

for $Q \in \mathfrak{R}$. Since S_Ω is closed, we have

$$S_\Omega Q \Omega' = Q^* \Omega' = S_{\Omega'} Q \Omega'.$$

Hence $S_\Omega \supset S_{\Omega'}$. Since $\Omega = A^{-1}\Omega'$, we have $S_{\Omega'} \supset S_\Omega$. Therefore $S_\Omega = S_{\Omega'}$, which implies $\Delta_\Omega = S_\Omega^* S_\Omega = S_{\Omega'}^* S_{\Omega'} = \Delta_{\Omega'}$. Q.E.D.

The following Lemma has been given by Connes [2].

Lemma 4. *$t \in [0, \infty)$ is in the spectrum of Δ_Ω if and only if there exist operators $x \in \mathfrak{R}$ and $y \in \mathfrak{R}'$ for each given $\varepsilon > 0$ such that $\|x\Omega\| = 1$, $\|t^{1/2} x\Omega - y\Omega\| < \varepsilon$ and $\|x^* \Omega - t^{1/2} y^* \Omega\| < \varepsilon$.*

§ 2. Invariant State of Asymptotically Abelian System

Theorem 1. *Let τ_α be a net of * automorphisms of $\mathfrak{R}$ such that a weakly dense sub * algebra $\mathfrak{A}$ of $\mathfrak{R}$ is strongly τ_α central and ϱ be a faithful normal state of $\mathfrak{R}$, invariant under all τ_α. Then the spectrum of Δ_ϱ is contained in the spectrum of $\Delta_{\varrho'}$ for any faithful normal state ϱ' on $\mathfrak{R}$.*

Remark. This theorem with an assumption of strong clustering has been given by Størmer [4].

Proof. Let $t \in [0, \infty)$ be in the spectrum of Δ_ϱ and $\varepsilon > 0$ be given. By Lemma 4, there exists $x \in \mathfrak{R}$ and $y \in \mathfrak{R}'$ satisfying

$$\|x\Omega_\varrho\| = 1\,,$$
$$\|t^{1/2} x\Omega_\varrho - y\Omega_\varrho\| < \varepsilon/4\,,$$
$$\|x^* \Omega_\varrho - t^{1/2} y^* \Omega_\varrho\| < \varepsilon/4\,.$$

Since $\mathfrak{A}$ is a self adjoint linear weakly dense subset of $\mathfrak{R}$, it is * strongly dense in $\mathfrak{R}$. Hence there exist $x_1 \in \pi_\varrho(\mathfrak{A})$ and $y_1 \in J_\varrho \pi_\varrho(\mathfrak{A}) J_\varrho$ such that

$$\|x_1 \Omega_\varrho\| = 1\,,$$
$$\|t^{1/2}(x - x_1)\,\Omega_\varrho - (y - y_1)\,\Omega_\varrho\| < \varepsilon/4\,,$$
$$\|(x^* - x_1{}^*)\,\Omega_\varrho - t^{1/2}(y^* - y_1^*)\,\Omega_\varrho\| < \varepsilon/4\,.$$

Since Ω_ϱ is cyclic and separating for $\pi_\varrho(\mathfrak{R}) \sim \mathfrak{R}$, there exists a vector $\Omega_{\varrho'} \in H_\varrho$ such that the vector state by $\Omega_{\varrho'}$ on $\mathfrak{R}$ is ϱ'. By Lemma 2, there exists a positive self adjoint operator z affiliated with $\mathfrak{Z} = \pi_\varrho(\mathfrak{R}) \cap \pi_\varrho(\mathfrak{R})'$ such that $z\Omega_{\varrho'} \equiv \Omega'$ gives the same vector state on $\mathfrak{Z}$ as Ω_ϱ. Let ϱ'' be the vector state on $\mathscr{B}(H_\varrho)$ by the vector Ω'.

By Lemma 1, there exists α such that

$$|(\varrho'' - \hat{\varrho})\,(\bar{\tau}_\alpha(x_1{}^* x_1))| < 1/2\,,$$
$$|(\varrho'' - \hat{\varrho})\,(\bar{\tau}_\alpha\{(t^{1/2} x_1 - y_1)^* \,(t^{1/2} x_1 - y_1)\})| < \varepsilon^2/4\,,$$
$$|(\varrho'' - \hat{\varrho})\,(\bar{\tau}_\alpha\{(x_1{}^* - t^{1/2} y_1{}^*)^* \,(x_1{}^* - t^{1/2} y_1{}^*)\})| < \varepsilon^2/4\,.$$

We define $\lambda = \varrho''(\bar{\tau}_\alpha(x_1{}^* x_1))^{1/2}$. Then $\lambda^2 > 1 - 1/2 = 1/2$, due to $\hat{\varrho}(\bar{\tau}_\alpha(x_1{}^* x_1)) = \hat{\varrho}(x_1{}^* x_1) = 1$. We define

$$x_2 = \lambda^{-1} \bar{\tau}_\alpha x_1\,, \qquad y_2 = \lambda^{-1} \bar{\tau}_\alpha y_1\,.$$

By previous estimates and $\bar{\tau}_\alpha$ invariance of $\hat{\varrho}$, we have

$$\varrho''(x_2{}^* x_2) = 1\,,$$
$$\varrho''((t^{1/2} x_2 - y_2)^* \,(t^{1/2} x_2 - y_2)) < \varepsilon^2\,,$$
$$\varrho''((x_2{}^* - t^{1/2} y_2{}^*)^* \,(x_2{}^* - t^{1/2} y_2{}^*)) < \varepsilon^2\,.$$

Since $x_2 \in \pi_\varrho(\mathfrak{R})$, $y_2 \in \pi_\varrho(\mathfrak{R})'$, t is in the spectrum of $\Delta_{\Omega'}$ by Lemma 4. By Lemma 3, $\Delta_{\Omega'} = \Delta_{\Omega_{\varrho'}}$. Hence t is in the spectrum of $\Delta_{\varrho'}$. Q.E.D.

§ 3. Isolated Spectrum with a Finite Multiplicity at 1

Theorem 2. *If* 1 *is an isolated spectrum of* Δ_ϱ *with a multiplicity* n, *then* $\dim H_\varrho \leqq n^2$.

We need a few preparations for the proof of this Theorem. Let $\mathfrak{A}$ be any weakly dense $\bar{\tau}_\varrho(t)$ invariant norm closed linear subset of $\pi_\varrho(\mathfrak{R})$. Let $\Delta_\varrho = \int_{-\infty}^{\infty} e^\lambda \mathrm{d}E_\lambda$. For any bounded open interval $I=(a,b)$, define $\mathfrak{A}_I$ as the set of all operators Q in $\mathfrak{A}$ such that

$$QH((\alpha,\beta)) \subset H((\alpha+a, \beta+b)),$$
$$H((\alpha,\beta)) = (E_{\beta-0} - E_{\alpha+0})\, H_\varrho .$$

From the definition

$$\mathfrak{A}_{I_1}\mathfrak{A}_{I_2} \subset \hat{\mathfrak{R}}_{I_1+I_2}, \qquad \hat{\mathfrak{R}} = \pi_\varrho(\mathfrak{R}). \tag{2.1}$$

Let $I \subset\subset J$ denotes $\bar{I} \subset J$ where $\bar{I}$ is the closure of I.

Lemma 5. $H(I)$ *is the closure of* $\bigcup_{J \subset\subset I} \mathfrak{A}_J \Omega_\varrho$.

Proof. Since $\Omega_\varrho \in H(I'')$ for any I'' containing 0, we have $\mathfrak{A}_J \Omega_\varrho \subset H(I)$ if $J \subset\subset I$.

Since $\bigcup_{J \subset\subset I} A_J \Omega_\varrho$ is a linear set, it is enough to show that for any unit vector $\Phi \in H(I)$, there exists $J \subset\subset I$ and $Q \in \mathfrak{A}_J$ such that $(Q\Omega_\varrho, \Phi) \neq 0$.

Let $I=(a,b)$. By definition, there exist $a<a'<b'<b$ such that $\|(E_{b'-0} - E_{a'+0})\,\Phi\| \neq 0$. ($\|\Phi\|=1$ by assumption.) Let $J=(a',b')$. Since Ω_ϱ is cyclic, there exists $Q_1 \in \mathfrak{A}$ such that $(Q_1\Omega_\varrho, (E_{b'-0} - E_{a'+0})\,\Phi) \neq 0$. Let

$$\mathrm{d}\mu(\lambda) = \mathrm{d}(Q_1\Omega_\varrho, E_\lambda \Phi).$$

It is a finite complex measure and its restriction to J is not identically 0. The set $C_0(J)$ of all continuous functions vanishing outside of J is separating for finite measures on J. Since C^∞ functions vanishing outside of J is norm dense in $C_0(J)$, there exists a C^∞ function $\tilde{f}(\lambda)$ whose support is in $\bar{J}$ and $\int \tilde{f}(\lambda)\, \mathrm{d}\mu(\lambda) \neq 0$.

Let $f(t) = (2\pi)^{-1} \int \tilde{f}(\lambda)^* e^{-it\lambda} \mathrm{d}\lambda$. f is in $\mathscr{S}$. Define

$$Q = \int_{-\infty}^{\infty} \bar{\tau}_\varrho(t)\, Q_1\, f(t)\, dt \in \mathfrak{A}.$$

Then

$$Q\Omega_\varrho = \int \tilde{f}(\lambda)^* \, \mathrm{d}E_\lambda Q_1 \Omega_\varrho ;$$

$$(Q\Omega_\varrho, \Phi) = \int \tilde{f}(\lambda) \, \mathrm{d}\mu(\lambda) \neq 0 .$$

Lemma 5 is proved if we show $Q \in \mathfrak{A}_J$. This follows from the next Lemma. Q.E.D.

Lemma 6. *Let $\tilde{f}(\lambda)$ be a C^∞ function with its support in a compact interval $\bar{J}$ and*

$$f(t) = (2\pi)^{-1} \int \tilde{f}(\lambda) \, e^{-it\lambda} \, \mathrm{d}\lambda .$$

Then, for any Q_1 in a $\bar{\tau}_\varrho$-invariant norm closed linear set $\mathfrak{A}$,

$$Q(f) \equiv \int_{-\infty}^{\infty} \bar{\tau}_\varrho(t) \, Q_1 \, f(t) \, \mathrm{d}t \in \mathfrak{A}_J .$$

Proof. Let I be a bounded open interval and I_1 be another open interval such that $I_1 \subset\subset I$. Since the union of $H(I_1)$ for all such I_1 is dense in $H(I)$, it is enough to prove that for $\Phi \in H(I_1)$ and any Ψ such that the measure $\mathrm{d}(\Psi, E_\lambda \Psi)$ has a compact support with empty intersection with $I + \bar{J} (= I + J)$, $Q(f)$ satisfies

$$(\Psi, Q(f) \, \Phi) = 0 .$$

Let

$$F(t, s) = (\Psi, \Delta_\varrho^{it} Q_1 \Delta_\varrho^{-is} \Phi) .$$

F is a uniformly bounded continuous function of (t, s), analytic in t and s. Its Fourier transform

$$\tilde{F}(p, q) = \int e^{i(-pt+qs)} F(t, s) \, \mathrm{d}t \, \mathrm{d}s / (2\pi)^2$$

is a tempered distribution with support in the direct product of the support of $\mathrm{d}(\Psi, E_\lambda \Psi)$ and $\bar{I}_1 \subset I$. This support has an empty intersection with the support of $\tilde{f}(p - q)$, which is a C^∞ function. Hence

$$\begin{aligned} 0 &= \int \tilde{F}(p, q) \, \tilde{f}(p - q) \, \mathrm{d}p \, \mathrm{d}q \\ &= \int F(t, s) \, f(t) \, \delta(t - s) \, \mathrm{d}t \, \mathrm{d}s \\ &= \int F(t, t) \, f(t) \, \mathrm{d}t = (\Psi, Q(f) \, \Phi) . \quad \text{Q.E.D.} \end{aligned}$$

Lemma 7. $\mathfrak{A}_J^* \subset \hat{\mathfrak{R}}_{-J}$.

Proof. Let I_1 and I_2 be open bounded intervals such that $I_1 + J$ and I_2 has an empty intersection. Then $\mathfrak{A}_J H(I_1) \perp H(I_2)$. Hence $\mathfrak{A}_J^* H(I_2) \perp H(I_1)$. Given an open bounded interval I. Let $I_2 \subset\subset I$. Then $I_2 - J \subset\subset I - J$ and $\mathfrak{A}_J^* H(I_2) \perp H(I_1)$ whenever $I_2 - J$ has an empty intersection with an open bounded interval I_1. Since $I_2 - J \subset\subset I - J$, this implies $\mathfrak{A}_J^* H(I_2) \subset H(I - J)$. Since the union of $H(I_2)$ is dense in $H(I)$, we have $\mathfrak{A}_J^* H(I) \subset H(I - J)$ and hence $\mathfrak{A}_J^* \subset \hat{\mathfrak{R}}_{-J}$. Q.E.D.

Proof of Theorem 2 when $n=1$. Assume that

$$\dim H((-\delta, \delta)) = 1$$

for some $\delta > 0$. Since the spectrum of $\log \Delta_\varrho$ is symmetric due to $J_\varrho(\log \Delta_\varrho) J_\varrho = -\log \Delta_\varrho$, there exists $t \geqq \delta$ in the spectrum of Δ_ϱ if $\dim H_\varrho > 1$. By Lemma 5, there exist $Q \in \hat{\mathfrak{R}}_I$, $I \subset\subset (t-\delta/4, t+\delta/4)$, such that $\|Q\Omega_\varrho\| = 1$, because $H((t-\delta/4, t+\delta/4)) \neq 0$. Let

$$\Phi = Q\Omega_\varrho \in H((t-\delta/4, t+\delta/4)).$$

By Lemma 7 and (2.1), we have $Q^* Q \in \hat{\mathfrak{R}}((-\delta/2, \delta/2))$ and hence

$$Q^* Q \Omega_\varrho = c \Omega_\varrho$$

for some complex number c, which is determined by

$$c = \|Q\Omega_\varrho\|^2 = 1.$$

Since Ω_ϱ is separating for $\mathfrak{R}$, we have $Q^* Q = 1$. Hence $\|Q^*\| = 1$.

We now have

$$\begin{aligned} 1 &\geqq \|Q^* \Omega_\varrho\| = \|J_\varrho Q^* \Omega_\varrho\| = \|\Delta_\varrho^{1/2} Q \Omega_\varrho\| \\ &= \|\Delta_\varrho^{1/2} \Phi\| \geqq \{\exp(1/2)(t-\delta/4)\} \|\Phi\| \\ &> 1, \end{aligned}$$

which is a contradiction. Q.E.D.

Proof of Theorem 2 for a general n. Let H_0 be the set of all Δ_ϱ invariant vectors in H_ϱ and $\hat{\mathfrak{R}}_0$ be the set of all $\bar{\tau}_\varrho(t)$ invariant elements of $\hat{\mathfrak{R}} = \pi_\varrho(\mathfrak{R})$. By assumption, there exists $\delta > 0$ such that $H(I) = H_0$ for $I = (-\delta, \delta)$. $\dim H_0 = n$.

For any $J \subset\subset I$, $Q \in \hat{\mathfrak{R}}_J$ satisfies $Q\Omega_\varrho \in H(I) = H_0$ because $\Omega_\varrho \in H(I_1)$ for small I_1 containing 0 such that $J + I_1 \subset I$. Hence $\{\bar{\tau}_\varrho(t) Q\} \Omega_\varrho = \Delta_\varrho^{it} Q \Omega_\varrho = Q\Omega_\varrho$. Since Ω_ϱ is separating, $\bar{\tau}_\varrho(t) Q = Q$ and hence $\hat{\mathfrak{R}}(I) \subset \hat{\mathfrak{R}}_0$. If $0 \in J$, then $\hat{\mathfrak{R}}(J) \supset \hat{\mathfrak{R}}_0$. Hence $\hat{\mathfrak{R}}(J) = \hat{\mathfrak{R}}_0$ for $J \subset I$. By Lemma 5, $\hat{\mathfrak{R}}_0 \Omega_\varrho$ is dense in $H(I) = H_0$ and hence $\hat{\mathfrak{R}}_0 \Omega_\varrho = H_0$. Since Ω_ϱ is separating for $\hat{\mathfrak{R}}$, it is cyclic and separating for $\hat{\mathfrak{R}}_0$ in H_0. By KMS condition, Ω_ϱ is a trace vector for $\hat{\mathfrak{R}}_0$.

There exists a set of mutually orthogonal minimal projections $s_i \in \hat{\mathfrak{R}}_0$ such that $\Sigma s_i = 1$. Let $\Omega_i = s_i \Omega_\varrho$. Since $J_\varrho s_i \Omega_\varrho = s_i \Omega_\varrho$ because Δ_ϱ is 1 on H_0, we have $s_i \Omega_\varrho = j_\varrho(s_i) \Omega_\varrho = s_i^2 \Omega_\varrho = s_i j_\varrho(s_i) \Omega_\varrho$. Let $s_i j_\varrho(s_i) H = H_i$. Then $\Omega_i = s_i \Omega_\varrho \in H_i$. Since $(s_i \hat{\mathfrak{R}} s_i \Omega_i)^- = (s_i \hat{\mathfrak{R}} j(s_i) \Omega_\varrho)^- = (s_i j(s_i) \hat{\mathfrak{R}} \Omega_\varrho)^- = H_i$, Ω_i is cyclic for $\mathfrak{R}_i \equiv s_i \hat{\mathfrak{R}} s_i$. Since $Q\Omega_i = Q\Omega_\varrho$ for $Q \in \mathfrak{R}_i$, Ω_i is separating for $\mathfrak{R}_i$. For $Q \in \mathfrak{R}_i$, we have

$$\begin{aligned} S_\varrho Q \Omega_i &= S_\varrho Q s_i \Omega_\varrho = S_\varrho Q \Omega_\varrho = Q^* \Omega_\varrho \\ &= Q^* \Omega_i, \end{aligned}$$

where $S_\varrho = J_\varrho \Delta_\varrho^{1/2}$. Hence the restriction of J_ϱ and Δ_ϱ are J_{Ω_i} and Δ_{Ω_i} in H_i.

Since s_i is minimal in $\hat{\mathfrak{R}}_0$ and Ω_ϱ is cyclic separating trace vector, $j_\varrho(s_i)$ is minimal in the commutant of $\hat{\mathfrak{R}}_0$ in H_0 and $\Omega_i = s_i j_\varrho(s_i)\,\Omega_\varrho$ spans $s_i j_\varrho(s_i)\,H_0$. Hence Δ_{Ω_i} has an isolated spectrum at 1 with multiplicity 1 and hence $\dim H_i = 1$. Hence $s_i \mathfrak{R} s_i = \mathfrak{R}_i \sim C$. Therefore s_i is also a minimal projection of $\mathfrak{R}$. Since the number of s_i can not exceed $\dim H_0 = n$, $\mathfrak{R}$ has at most n mutually orthogonal minimal projections with sum 1. This implies $\dim H_\varrho \leqq n^2$. Q.E.D.

§ 4. Isolated Spectrum with a Finite Multiplicity at $x \neq 1$

Theorem 3. *If x is an isolated spectrum of Δ_ϱ with a finite multiplicity, then there exists a direct sum decomposition*

$$\pi_\varrho(\mathfrak{R}) = \mathfrak{R}_a \oplus \mathfrak{R}_b, \qquad \Omega_\varrho = \Omega_a \oplus \Omega_b, \qquad \Delta_\varrho = \Delta_{\Omega_a} \oplus \Delta_{\Omega_b}$$

such that $\mathfrak{R}_a$ is of type I with a finite atomic center and Δ_{Ω_b} does not have its spectrum at x and x^{-1}.

Let H_t denote the set of all eigenvectors of Δ_ϱ belonging to an eigenvalue e^t and s_t be the projection to the subspace spanned by $\hat{\mathfrak{R}}' H_t + \hat{\mathfrak{R}}' H_{-t}$. As a preparation for our proof, we have the following:

Lemma 8. *Assume that $H((t-\delta, t+\delta)) = H_t$.*
Then
(a) $[s_t, \Delta_\varrho] = 0$.
(b) *1 is an isolated spectrum of $\Delta_\varrho | s_t H$.*
(c) *If $\dim H_t < \infty$, then $\dim s_t H_0 < \infty$.*
(d) *$(1 - s_t)\,\Delta_\varrho$ does not have its spectrum at $e^{\pm t}$.*

Proof. If $t = 0$, then $s_t = 1$ and all statements become trivial. Hence we assume $t \neq 0$.

(a) Since H_t and H_{-t} are invariant under Δ_ϱ^{it} and $\hat{\mathfrak{R}}'$ is invariant under $\bar{\tau}_\varrho(t)$, $s_t H_\varrho$ is invariant under Δ_ϱ^{it} and hence $[s_t, \Delta_\varrho] = 0$.

(b) For any $J \subset\subset (-\delta, \delta)$, there exists $I \ni t$ such that $J + I \subset (t-\delta, t+\delta)$. Then $H_t \subset H(I)$ and $\hat{\mathfrak{R}}_J H_t \subset H((t-\delta, t+\delta)) = H_t$. For $Q \in \hat{\mathfrak{R}}_J$ and $\Psi \in H_t$,

$$\bar{\tau}_\varrho(u)\, Q \Psi = e^{-iu} \Delta_\varrho^{iu} Q \Psi = Q \Psi .$$

Hence

$$\{\bar{\tau}_\varrho(u)\, Q - Q\}\, \Psi = 0 \tag{4.1}$$

for all $\Psi \in \hat{\mathfrak{R}}' H_t$.

Since $J_\varrho \Delta_\varrho J_\varrho = \Delta_\varrho^{-1}$, $\log \Delta_\varrho$ has a symmetric spectrum and hence $H((-t-\delta, -t+\delta)) = H_{-t}$. By the same argument as above, (4.1) holds for $\Psi \in \hat{\mathfrak{R}}' H_{-t}$ and hence for $\Psi \in s_t H_\varrho$. We have

$$\bar{\tau}_\varrho(u)\, \{Q s_t\} = \{\bar{\tau}_\varrho(u)\, Q\}\, s_t = Q s_t .$$

Hence $\hat{\mathfrak{R}}_J s_t \subset \hat{\mathfrak{R}}_0$ for any $J \subset\subset (-\delta, \delta)$. Clearly, $\hat{\mathfrak{R}}_0 s_t \subset \hat{\mathfrak{R}}_J s_t$. Hence $\hat{\mathfrak{R}}_J s_t = \hat{\mathfrak{R}}_0 s_t$. Taking adjoint, $s_t \hat{\mathfrak{R}}_J = s_t \hat{\mathfrak{R}}_0$.

By Lemma 5, $s_t H((-\delta, \delta))$ is generated by

$$s_t \hat{\mathfrak{R}}_J \Omega_\varrho = s_t \hat{\mathfrak{R}}_0 \Omega_\varrho \subset s_t H_0, \quad J \subset\subset (-\delta, \delta).$$

Hence 1 is an isolated spectrum of $\Delta_\varrho|_{s_t H}$. Moreover, $s_t H_0 \subset s_t \hat{\mathfrak{R}}_J \Omega_\varrho \subset s_t H_0$ and hence $s_t \hat{\mathfrak{R}}_0 \Omega_\varrho = s_t H_0$.

(c) $\dim H_t < \infty$ implies $\dim H_{-t} = \dim J_\varrho H_t < \infty$. Since $QH_t = 0$, $QH_{-t} = 0$, $Q \in \hat{\mathfrak{R}}$ imply $Qs_t = 0$, we have

$$\dim H_t + \dim H_{-t} \geqq \dim \hat{\mathfrak{R}}_J s_t = \dim \hat{\mathfrak{R}}_0 s_t = \dim s_t H_0 .$$

(d) This follows from (1) and the definition of s_t. Q.E.D.

Proof of Theorem 3. Let $x = e^t$. If $t = 0$, then Theorem 3 holds with $\mathfrak{R}_b = 0$ due to Theorem 2. Assume that $t \neq 0$. Let

$$K = s_t j_\varrho(s_t) H_\varrho ,$$
$$\mathfrak{M} = s_t \hat{\mathfrak{R}} s_t |_K ,$$
$$\Psi = s_t j_\varrho(s_t) \Omega_\varrho .$$

By (a) of Lemma 8, we have $s_t \Omega_\varrho = \Delta_\varrho^{1/2} s_t \Omega_\varrho = J_\varrho s_t \Omega_\varrho = j_\varrho(s_t) \Omega_\varrho = s_t^2 \Omega_\varrho = s_t j_\varrho(s_t) \Omega_\varrho = \Psi$. Hence $\mathfrak{M}\Psi = s_t \hat{\mathfrak{R}} j_\varrho(s_t) \Omega_\varrho = s_t j_\varrho(s_t) \hat{\mathfrak{R}} \Omega_\varrho$ is dense in K and $\mathfrak{M}' \Psi = j_\varrho(s_t) \hat{\mathfrak{R}}' j_\varrho(s_t) s_t \Omega_\varrho = j_\varrho(s_t) \hat{\mathfrak{R}}' s_t \Omega_\varrho = j_\varrho(s_t) s_t \hat{\mathfrak{R}}' \Omega_\varrho$ is also dense in K. Hence Ψ is cyclic and separating for $\mathfrak{M}$ in K. For $Q \in \mathfrak{M}$ and $S_\varrho = J_\varrho \Delta_\varrho^{1/2}$,

$$S_\varrho Q \, s_t \Omega_\varrho = s_t Q^* \Omega_\varrho = Q^* s_t \Omega_\varrho$$

and hence $S_\varrho|_K = S_\Psi$, $\Delta_\varrho|_K = \Delta_\Psi$ and $J_\varrho|_K = J_\Psi$.

By (c) of Lemma 8, Δ_Ψ has an isolated spectrum with a finite multiplicity. Hence $\mathfrak{M}$ is a finite matrix algebra by Theorem 2.

Since $s_t H_t = H_t$, $j_\varrho(s_t) H_t = J_\varrho s_t J_\varrho H_t = J_\varrho s_t H_{-t} = J_\varrho H_{-t} = H_t$. Similarly $j_\varrho(s_t) H_{-t} = H_{-t}$. Hence $H_t + H_{-t} \subset K$.

Let $c(s_t)$ be the central support of s_t. Since $j_\varrho(c(s_t)) = c(s_t)$ (for any central projection), $c(j_\varrho(s_t)) = c(s_t)$. $\mathfrak{M}$ is isomorphic to $s_t \hat{\mathfrak{R}} s_t$ restricted to $\hat{\mathfrak{R}}' K = s_t \hat{\mathfrak{R}}' \Omega_\varrho = s_t H$. Hence $c(s_t) \hat{\mathfrak{R}}$ must be of type I with a finite atomic center.

$$\mathfrak{R}_a = c(s_t) \hat{\mathfrak{R}}, \quad \mathfrak{R}_b = (1 - c(s_t)) \hat{\mathfrak{R}}, \ \Omega_a = c(s_t) \Omega_\varrho ,$$

$\Omega_b = (1 - c(s_t)) \Omega_\varrho$ satisfy required properties. Q.E.D.

§ 5. Applications

Connes has introduced the invariant

$$S(\mathfrak{R}) = \bigcap_\varrho \text{spectrum } \Delta_\varrho .$$

Our result gives the following application for $S(\mathfrak{R})$.

Theorem 4. *Let ϱ be a faithful normal state of $\mathfrak{R}$ invariant under a net of * automorphisms τ_α of $\mathfrak{R}$. Assume that $\mathfrak{R}$ has a weakly dense sub * algebra $\mathfrak{A}$ which is strongly τ_α central. Then*

$$S(\mathfrak{R}) = \text{Spectrum}\, \Delta_\varrho \,.$$

If ϱ is ergodic with respect to modular automorphisms in addition, then either $S(\mathfrak{R})$ is $[0, \infty)$ or H_ϱ is of one dimension.

Proof. The first half follows from Theorem 1. If ϱ is τ_ϱ ergodic, then ϱ is primary and hence Spectrum $\Delta_\varrho \setminus \{0\}$ is a multiplicative group. If 1 is not an isolated spectrum of Δ_ϱ, then Spectrum $\Delta_\varrho = [0, \infty)$. If 1 is an isolated spectrum of Δ_ϱ, then Theorem 2 is applicable where $n=1$ due to τ_ϱ ergodicity. Hence $\dim H_\varrho = 1$. Q.E.D.

Remark 1. Størmer [4] proved the first part under the assumption of strong clustering. The second part is stated in [4] with the assumption that τ_ϱ is asymptotically abelian.

Theorem 5. $S(\mathfrak{R}) = \bigcap_\varrho$ essential spectrum Δ_ϱ.

Proof. Obvious from Theorem 3. Q.E.D.

Remark 2. Connes invariant is additive under direct sum $S(\mathfrak{R}_1 \oplus \mathfrak{R}_2) = S(\mathfrak{R}_1) \cup S(\mathfrak{R}_2)$, whereas the asymptotic ratio set satisfies $r_\infty(\mathfrak{R}_1 \oplus \mathfrak{R}_2) = r_\infty(\mathfrak{R}_1) \cap r_\infty(\mathfrak{R}_2)$. $S(\mathfrak{R})$ is more closely related to the union of S_x over non-zero portion of partial central decomposition of $\mathfrak{R}$ according to asymptotic ratio set.

Remark 3. In the situation of Theorem 4, if $\mathfrak{R}$ is ITPFI, then $\mathfrak{R} = \mathfrak{R}_x$, $0 < x \leqq \infty$. If ϱ is τ_ϱ ergodic, then $\mathfrak{R} = \mathfrak{R}_\infty$. $\mathfrak{R}$ appearing in Gibbs states of a lattice system is hyperfinite but it is not known whether it is an ITPFI in general.

Appendix

The following result is a part of Theorem 4 in [1] and is a basis for Lemma 1 of § 1.

Lemma 9. *If Q_α is a uniformly bounded weakly central net in R and if ϱ and ϱ' are normal states of $\mathfrak{R}$ such that $\varrho(z) = \varrho'(z)$ for all $z \in \mathfrak{Z} = (\mathfrak{R} \cap \mathfrak{R}')$, then*

$$\lim \{\varrho(Q_\alpha) - \varrho'(Q_\alpha)\} = 0 \,. \tag{A.1}$$

The following direct proof is due to Elliott.

Proof. Let $Q_{\alpha(\beta)}$ be weakly converging subnet of Q_α. Since Q_α is weakly central,

$$z = \text{w} - \lim Q_{\alpha(\beta)} \in \mathfrak{Z} \,.$$

Hence $\varrho(z)=\varrho'(z)$, i.e.

$$\lim\{\varrho(Q_{\alpha(\beta)})-\varrho'(Q_{\alpha(\beta)})\}=0.$$

In view of weak compactness of the unit ball of $\mathfrak{R}$, this implies (A.1).
Q.E.D.

Somewhat stronger conclusion can be drawn if $Q_\alpha=\tau_\alpha Q$, and ϱ is a faithful invariant state. An example is seen in the following:

Lemma 10. *Let $\mathfrak{A}$ be a weakly dense * subalgebra of $\mathfrak{R}$, ϱ be a faithful normal state of $\mathfrak{R}$, τ_α be a net of * automorphisms of $\mathfrak{R}$ such that ϱ is invariant and $\mathfrak{A}$ is weakly τ_α central, and ϱ' be a normal state of $\mathfrak{R}$ such that $\varrho'(z)=\varrho(z)$ for every $z\in\mathfrak{R}\cap\mathfrak{R}'$. Then*

$$\lim\varrho'(\tau_\alpha Q)=\varrho(Q),\qquad Q\in\mathfrak{R}. \tag{A.2}$$

Proof. By Theorem 4 of [1],

$$\mathrm{w}-\lim\{\tau_\alpha Q_1-\tau_\alpha F_\varrho^{\mathfrak{Z}\mathfrak{R}}(Q_1)\}=0$$

for $Q_1\in\mathfrak{A}$, which implies

$$\mathrm{w}-\lim U_\alpha\pi_\varrho(Q_1-F_\varrho^{\mathfrak{Z}\mathfrak{R}}(Q_1))\,\Omega_\varrho=0.$$

Since $F_\varrho^{\mathfrak{Z}\mathfrak{R}}$ is strongly continuous on the unit ball, there exists $Q_1\in\mathfrak{A}$ for given $Q\in\mathfrak{R}$, $\Phi_j\in H_\varrho$, $j=1\,\ldots\,n$, and $\varepsilon>0$ such that

$$\|\{\pi_\varrho(Q_1-F_\varrho^{\mathfrak{Z}\mathfrak{R}}(Q_1))-\pi_\varrho(Q-F_\varrho^{\mathfrak{Z}\mathfrak{R}}(Q))\}\,\Omega_\varrho\|\;\|\Phi_j\|<\varepsilon/2.$$

For this Q_1, there exists α_0 such that for $\alpha>\alpha_0$,

$$|(\Phi_j,U_\alpha\pi_\varrho(Q_1-F_\varrho^{\mathfrak{Z}\mathfrak{R}}(Q_1))\,\Omega_\varrho)|<\varepsilon/2.$$

These two equations imply

$$|(\Phi_j,U_\alpha\pi_\varrho(Q-F_\varrho^{\mathfrak{Z}\mathfrak{R}}(Q))\,\Omega_\varrho)|<\varepsilon$$

and hence

$$\mathrm{w}-\lim\pi_\varrho(\tau_\alpha\{Q-F_\varrho^{\mathfrak{Z}\mathfrak{R}}(Q)\})\,\Omega_\varrho=0.$$

Multiplying $Q'\in\pi_\varrho(\mathfrak{R})'$ and using the cyclicity of Ω_ϱ for $\pi_\varrho(\mathfrak{R})'$, we obtain

$$\mathrm{w}-\lim\pi_\varrho(\tau_\alpha Q-\tau_\alpha F_\varrho^{\mathfrak{Z}\mathfrak{R}}(Q))=0,$$

which implies

$$\mathrm{w}-\lim(\tau_\alpha Q-\tau_\alpha F_\varrho^{\mathfrak{Z}\mathfrak{R}}(Q))=0,\qquad Q\in\mathfrak{R}. \tag{A.3}$$

Since $F_\varrho^{\mathfrak{Z}\mathfrak{R}}(Q)\in\mathfrak{Z}$, we obtain

$$\varrho'(\tau_\alpha F_\varrho^{\mathfrak{Z}\mathfrak{R}}(Q))=\varrho(\tau_\alpha F_\varrho^{\mathfrak{Z}\mathfrak{R}}(Q))=\varrho(F_\varrho^{\mathfrak{Z}\mathfrak{R}}(Q))=\varrho(Q).$$

Hence we obtain from (A.3)

$$\begin{aligned}\lim \varrho'(\tau_\alpha Q) &= \lim \varrho'(\tau_\alpha F_\varrho^{3\mathfrak{R}}(Q)) \\ &= \varrho(Q). \qquad \text{Q.E.D.}\end{aligned}$$

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H. Araki
Research Institute for Mathematical Sciences
Kyoto University
Kyoto, Japan