

Calculation of Superpropagators in Non-linear Quantum Field Theories

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Received April 16, 1971

Abstract. A new method of constructing the superpropagators, i.e. the Fourier transforms of the expressions of the form $\sum_{n=1}^{\infty} c_n \Delta_F^n(x)$ is suggested. The method makes it possible to derive by use of the same technique explicit analytic expressions for the superpropagators for a wide class of field theories – from strictly local up to essentially non-local. The essence of the method is the construction of a differential equation for the superpropagator which in general is of an infinite order. By use of the boundary condition at $p^2 = 0$ we find the solution of this equation depending on one arbitrary real parameter. Simple examples are given to illustrate the method.

I. Introduction

In the theories with the essentially nonlinear (non-polynomial) Lagrangians (for example, in the chiral field theories) one encounters the necessity of calculating the superpropagators, i.e. the Fourier transforms of the expressions of the form $\sum_{n=1}^{\infty} c_n [g^2 \Delta_F(x)]^n$. The same problem holds for the polynomial nonrenormalizable field theories treated by use of equivalence theorems. In the present paper a rather general method of constructing the superpropagators is suggested. This method makes it possible to derive explicit analytic expressions for the superpropagators for a wide class of field theories – from strictly local up to essentially non-local – by use of the same technique.

The idea of the method originates in the earlier investigations of the approximate linear integral equations for the Green's functions in nonrenormalizable theories [1–6]. In particular, the Edwards equations, the Bethe-Salpeter equations and the integral equation for the simplest superpropagator which corresponds to the expression

$$\Delta_F(x) [1 - g^2 \Delta_F(x)]^{-1}$$

¹ The notations are introduced below.

have been extensively studied. To construct the superpropagators of a more general form there have been proposed other methods than those connected with solving the equations for the Green's functions [7–11]. However, as was shown in papers [5, 6], the superpropagator corresponding to $S_F(x) e^{g^2 D_F(x)}$ obeys some linear integral equation which was solved in the Euclidean momentum space by reducing it to the differential equation.

At this point it is important to note the following. The integral equation has a solution only if its kernel is made quadratically integrable by suitable regularizations (e.g., a cut-off on high virtual momenta). This regularization can be removed only after that the exact solution is derived and renormalized. In the differential equation, to which we reduce the integral one, the regularization can be eliminated from the very beginning and in solving the differential equation there is no more necessity to use any divergent expressions and ill-defined limiting procedures. The solutions derived in such a way are exactly the same as those obtained in [8–11] and depend on one arbitrary constant².

A generalization of the method of Refs. [5, 6] applied earlier to the investigation of the simplest superpropagators makes it possible to derive the linear differential equation for the superpropagator of a general form. This method seems to us to be simpler and more general than that of [8–11] because it allows to find the analytic expressions for the superpropagators for a wider class of field theories.

2. Superpropagator and S-matrix

We will study the neutral scalar field with the Lagrangian³

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 = -\frac{1}{2} [\partial_\mu \varphi(x)]^2 + f : U[g\varphi(x)] : , \quad (2.1)$$

where

$$U(\varphi) = \sum_{n=1}^{\infty} \frac{u_n}{n!} \varphi^n, \quad \partial_\mu \equiv \frac{\partial}{\partial x_\mu}, \quad x^2 \equiv \mathbf{x}^2 - x_0^2. \quad (2.2)$$

We meet the problems of this kind in the theories with a partial symmetry. Consider for example the theory with the Lagrangian

$$\mathcal{L} = \mathcal{L}_S + \mathcal{L}_M, \quad \mathcal{L}_S = -\frac{1}{2} f^2(\varphi) (\partial_\mu \varphi)^2, \quad \mathcal{L}_M = -\frac{m^2}{2} \varphi^2. \quad (2.3)$$

² This ambiguity will be discussed in detail in what follows.

³ Our analysis can be applied to the fields with spin and isospin without difficulties of principle encountered.

Here $\varphi(x) = \varphi[\psi(x)]$ and $\psi(x)$ is chosen in such a way that

$$f^2(\varphi) \left(\frac{d\varphi}{d\psi} \right)^2 \equiv 1$$

(so, we may assume that $\psi = \int f(\varphi) d\varphi$). The Lagrangian $\mathcal{L}_S = -\frac{1}{2}(\partial_\mu \psi)^2$ is obviously invariant under the group of one-parameter transformations of the field φ generated by the replacement $\psi \rightarrow \psi + C$ but \mathcal{L}_M is non-invariant. This can be illustrated by a simple example

$$f(\varphi) = (1 - g^2 \varphi^2)^{-\frac{1}{2}}, \quad g\psi = \arcsin(g\varphi), \quad g\varphi = \sin(g\psi). \quad (2.4)$$

The replacement $\psi \rightarrow \psi + C$ defines the nonlinear transformation of φ :

$$g\varphi \rightarrow g\varphi \cos Cg + \sqrt{1 - g^2 \varphi^2} \sin Cg.$$

The Lagrangian \mathcal{L}_M apparently is not invariant under this transformation:

$$\mathcal{L}_M = -\frac{m^2}{2g^2} \sin^2(g\psi)$$

$$\xrightarrow{\psi \rightarrow \psi + C} -\frac{m^2}{2g^2} [\sin^2(g\psi) \cos 2Cg + \frac{1}{2} \sin 2(g\psi) \sin 2Cg + \sin^2 Cg].$$

Note, that the Lagrangian $U(\varphi)$ deduced above is not normal ordered and in general $:U(\varphi):$ is not simply connected with $U(\varphi)$ [10]. However, the normal-ordered Lagrangian is more convenient for practical calculations. As we do not consider the problems of equivalence of theories described in various field variables so we use this simplification.

Sometimes (see, for example, [8]) the term $-\frac{m^2}{2}\varphi^2$ is inserted in the Lagrangian \mathcal{L}_0 . As a result, the superpropagator is expressed in terms of $\Delta_F(x)$ (instead of $D_F(x)$). A generalization of our method to this case will be considered in the next paper. In what follows a general case with \mathcal{L}_0 containing the term $-\frac{m^2}{2}\varphi^2$ will be investigated up to Eq. (3.6), all the subsequent formulae being valid only for the case $m=0$.

The S-matrix in the interaction representation has the form

$$S = 1 + \sum_{n=1}^{\infty} f^n S_n, \quad (2.5)$$

$$S_n = \frac{i^n}{n!} \int d^4 x_1 \dots d^4 x_n T\{ :U[g\varphi(x_1)]: \dots :U[g\varphi(x_n)]: \}. \quad (2.6)$$

Let us consider the term of second order in f

$$S_2 = \frac{i^2}{2!} \int d^4 x_1 d^4 x_2 \sum_{n_1, n_2=0}^{\infty} g^{n_1+n_2} F_{n_1 n_2}^{(2)}(x_1 - x_2) : \frac{\varphi^{n_1}(x_1)}{n_1!} \frac{\varphi^{n_2}(x_2)}{n_2!} :. \quad (2.7)$$

It is not hard to check that

$$F_{n_1 n_2}^{(2)}(x) = \sum_{n=0}^{\infty} \frac{u_{n+n_1} u_{n+n_2}}{n!} [g^2 \Delta_F(x)]^n, \quad (2.8)$$

where $u_0 \equiv 0$ by definition and

$$\Delta_F(x) = \int \frac{d^4 p}{(2\pi)^4 i} \frac{e^{-i p x}}{m^2 + p^2 - i0}; \quad D_F(x) = \Delta_F(x)|_{m=0}. \quad (2.9)$$

It appears that the calculation of $F_{n_1 n_2}^{(2)}$ for arbitrary n_1 and n_2 is reduced to that of $F_{00}^{(2)}$

$$F_{00}^{(2)}(x) = \sum_{n=1}^{\infty} \frac{u_n^2}{n!} [g^2 \Delta_F(x)]^n. \quad (2.10)$$

To prove this we use the recipe which will be systematically applied hereafter. We put⁴

$$\frac{u_{n+1}}{u_n} \equiv v(n-1), \quad n = 1, 2, \dots \quad (2.11)$$

and continue the function $v(z)$ from the integer $z = n$ to the whole complex z -plane. This continuation is unique for the class of functions which satisfy the Carlson's conditions (see e.g. [12]): 1) $v(z)$ is a regular function in the halfplane $\text{Re } z \geq 0$; 2) $|v(z)| < M \exp(A|z|)$ for $\text{Re } z \geq 0$, $A > 0$; 3) $|v(iy)| < M \exp\{(\pi - \varepsilon)|y|\}$, $-\infty < y < +\infty$, $\varepsilon > 0$. These conditions on $v(z)$ (not on $u_n!$) are not very restrictive because the ratio u_{n+1}/u_n depends on n more smoothly than u_n . For example, if $u_n \sim (n!)^a$ then $\frac{u_{n+1}}{u_n} \sim (n+1)^a$ and continuation of the v -function, $v(z) = (z+2)^a$, satisfies the Carlson's conditions for all a ; but, generally speaking, it is not so for that of the function $(n!)^a$. It is clear that

$$\frac{u_{n+m}}{u_n} = \prod_{k=0}^{m-1} v(n-1+k) \equiv v_m(n-1), \quad m = 1, 2, \dots, \quad v_0 \equiv 1, \quad (2.12)$$

and the functions $v_m(z)$ are uniquely defined and regular at least for $\text{Re } z \geq 0$. Thus,

$$F_{n_1 n_2}^{(2)}(x) = u_{n_1} u_{n_2} + \sum_{n=1}^{\infty} v_{n_1}(n-1) v_{n_2}(n-1) \frac{u_n^2}{n!} [g^2 \Delta_F(x)]^n. \quad (2.13)$$

⁴ It is assumed here that $u_n \neq 0$ ($n = 1, 2 \dots$). However, if for example $u_{2n} = 0$, $u_{2n+1} \neq 0$ then it is possible to use the same recipe for $\tilde{u}_n \equiv u_{2n+1}$.

Let us now introduce the operator which will hereafter be systematically used

$$\delta_{g^2} \equiv g^2 \frac{\partial}{\partial g^2}, \quad \delta_{g^2} \cdot g^{2n} = n g^{2n}, \quad \delta_{g^2} \{g^{2n} f(g^2)\} = g^{2n} (\delta_{g^2} + n) f(g^2) \quad (2.14)$$

and construct the functions of this operator

$$v_m(\delta_{g^2} - 1) = \sum_{k=0}^{m-1} v(\delta_{g^2} - 1 + k) \quad (2.15)$$

which is possible due to the regularity of the function $v(z)$. Using the properties (2.14) of the operator δ_{g^2} it is easy to show that

$$v_m(\delta_{g^2} - 1) g^{2n} = v_m(n - 1) g^{2n}$$

i.e.,

$$F_{n_1 n_2}^{(2)}(x) = u_{n_1} u_{n_2} + v_{n_1}(\delta_{g^2} - 1) v_{n_2}(\delta_{g^2} - 1) F_{00}^{(2)}(x). \quad (2.16)$$

The latter relation is derived in x -representation, nevertheless it may be used in momentum representation in that domain of the parameter g^2 where $\tilde{F}_{00}^{(2)}(p^2)$ is an analytic function of g^2 ⁵.

To clarify the practical use of Eq. (2.16) we would like to mention the following. In the most general case the function $\tilde{F}_{00}^{(2)}(p^2)$ can be expanded in a series of powers of $(g^2)^{an} \ln^k g^2$ and it is necessary to calculate an action of $v_m(\delta_{g^2} - 1)$ on such terms. To do this it is more convenient to represent them as $\lim_{\varepsilon \rightarrow 0} \frac{d^k}{d\varepsilon^k} (g^2)^{an+\varepsilon}$. Then

$$\begin{aligned} v_m(\delta_{g^2} - 1) g^{2an} \ln^k g^2 &= \lim_{\varepsilon \rightarrow 0} \frac{d^k}{d\varepsilon^k} \{v_m(an + \varepsilon - 1) g^{2(an+\varepsilon)}\} \\ &= v_m^{(k)}(an - 1) g^{2an} + \dots + v_m(an - 1) g^{2an} \ln^k g^2. \end{aligned}$$

This trick makes it possible to find the expression of the function $\tilde{F}_{n_1 n_2}^{(2)}(p^2)$ if $\tilde{F}_{00}^{(2)}(p^2)$ is known⁶.

We would like to stress that if the singularity in g^2 at $g^2 = 0$ (non-integer a or $k \neq 0$) is taken into account then at once there arises the actual necessity to know the value of the interpolating function $v(z)$ in the noninteger points. The necessity of imposing the Carlson's conditions follows from the considerations of the simple case $u_n \equiv 1$. Indeed, it is obvious that $F_{n_1 n_2}^{(2)} \equiv F_{00}^{(2)}$, but a continuation of $v(n)$ from the integer points is, in general, not unique. In fact, we have $v(z) = 1 + f(z) \sin \pi z$

⁵ It is known [1, 6, 8] that momentum representations of the superpropagators have a singularity in g^2 at $g^2 = 0$. So the relations written above are not applicable to this point.

⁶ In the simplest cases it is possible to obtain an analytic form of $\tilde{F}_{n_1 n_2}^{(2)}(p^2)$ using (2.14) without expanding $\tilde{F}_{00}^{(2)}(p^2)$ in powers of g^2 and $\ln g^2$ (see examples considered below).

where $f(z)$ is an arbitrary function regular in the right halfplane. Since in this case the superpropagator $\tilde{F}_{00}^{(2)}$ contains the terms $\sim g^{2n} \ln g^2$ (see [1]) then, for example, the expansion of the function $\tilde{F}_{10}^{(2)}$ has the terms of the form

$$v(n-1)g^{2n} \ln g^2 + v'(n-1)g^{2n} = g^{2n} \ln g^2 + (-1)^n \pi f(n-1)g^{2n}$$

and we get additional (extra) terms $(-1)^n \pi f(n-1)g^{2n}$. The Carlson's conditions make it possible to eliminate such terms and thus it is natural to employ them in a more general case as well.

3. Equation for the Superpropagator

In this Section we will construct the equation for the superpropagator $F_{00}^{(2)}$ in the momentum representation. To simplify the notations we put $u_n^2 \equiv n! c_n$, i.e., we write down $F_{00}^{(2)}$ in the form

$$F_{00}^{(2)}(x) = \sum_{n=1}^{\infty} c_n [g^2 \Delta_F(x)]^n, \tag{3.1}$$

and pass to the Euclidean metric by substituting $x_0 \rightarrow ix_0$, $p_0 \rightarrow ip_0$ ⁷. Assuming the theory to be somehow regularized we find in the momentum representation

$$F_{00}^{(2)}(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-ipx} \tilde{F}(p^2); \quad \Delta_F(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-ipx} (m^2 + p^2)^{-1} \tag{3.2}$$

where $x^2 = \mathbf{x}^2 + x_0^2$, $p^2 = \mathbf{p}^2 + p_0^2$. It is convenient to substitute the dimensionless variable $g^2 p^2$ for the variable p^2 . The function $F_{00}^{(2)}(x)$ depends only on the dimensionless combinations g^2/x^2 and $g^2 m^2$. From this and Eq. (3.2) it follows that $\tilde{F}(p^2)$ can be represented in the form

$$\tilde{F}(p^2) = g^4 F(g^2 p^2, g^2 m^2). \tag{3.3}$$

To construct the equation for $\tilde{F}(p^2)$ we use the recipe considered in detail in the previous Section. Putting

$$c_{n+1}/c_n \equiv R(n-1), \quad n = 1, 2, \dots \tag{3.4}$$

and assuming the existence of the unique analytic continuation of⁸ $R(n)$ to the complex z -plane it is easy to show that $F_{00}^{(2)}$ obeys the equation

$$F_{00}^{(2)}(x) = c_1 g^2 \Delta_F(x) + g^2 \Delta_F(x) R(\delta_{g^2} - 1) F_{00}^{(2)}(x). \tag{3.5}$$

⁷ More detailed considerations how to use the Euclidean metric in solving linear equations in nonrenormalizable theories can be found in the papers [2-6].

⁸ We recall that $R(z)$ is assumed to satisfy the Carlson's conditions (see Section 2).

In the momentum representation this equation is of the form

$$\tilde{F}(p^2) = \frac{c_1 g^2}{m^2 + p^2} + g^2 R(\delta_{g^2} - 1) \int \frac{d^4 q}{(2\pi)^4} \frac{\tilde{F}(q^2)}{m^2 + (p - q)^2}.$$

Using the relation

$$g^2 \frac{\partial}{\partial g^2} F(g^2 p^2, g^2 m^2) \equiv \left(p^2 \frac{\partial}{\partial p^2} + m^2 \frac{\partial}{\partial m^2} \right) F(g^2 p^2, g^2 m^2)$$

and setting $\delta_{p^2} = p^2 \frac{\partial}{\partial p^2}$, $\delta_{m^2} = m^2 \frac{\partial}{\partial m^2}$ we derive the equation for F :

$$F(g^2 p^2, g^2 m^2) = \frac{c_1}{g^2(m^2 + p^2)} + R(\delta_{p^2} + \delta_{m^2}) \int \frac{d^4 q}{(2\pi)^4} \frac{g^2 F(g^2 q^2, g^2 m^2)}{m^2 + (p - q)^2}. \quad (3.6)$$

The integro-differential Eq. (3.6) is rather complicated, so hereafter we suppose $m = 0$. Then Eq. (3.6) reduces to the more simple integro-differential equation

$$F(g^2 p^2, 0) = \frac{c_1}{g^2 p^2} + R(p^2) \int \frac{d^4(q)}{(2\pi)^4} \frac{F(g^2 q^2, 0)}{g^2(p - q)^2}. \quad (3.7)$$

Using the relation

$$g^2 \int \frac{d^4 q}{(2\pi)^4} \frac{F(g^2 q^2, 0)}{(p - q)^2} = \frac{1}{\xi} \int_0^\xi d\eta \eta F(\eta) + \int_\xi^\infty d\eta F(\eta) \quad (3.8)$$

where $F(\xi) \equiv F(g^2 p^2, 0)$, $\xi = \frac{g^2 p^2}{(2\pi)^4}$ and acting on the both members of Eq. (3.7) by $\delta_\xi(\delta_\xi + 1)$ we derive the differential equation for the function $F(\xi)$ ⁹

$$\{\delta_\xi(\delta_\xi + 1) + \xi R(\delta_\xi + 1)\} F(\xi) = 0; \quad \delta_\xi \equiv \xi \frac{\partial}{\partial \xi}. \quad (3.9)$$

In general, this differential equation is of infinite order. Nevertheless, it can be easily solved by use of the Frobenius method [13]. To this end we note that the characteristic values of Eq. (3.9) which determine the behaviour of solutions for small ξ coincide with zeros of $\alpha(\alpha + 1)$ and poles of the function $R(\alpha + 1)$ ¹⁰. Thus, we can choose the linear independent solutions which behave for small ξ as follows: ξ^0 , ξ^{-1} and ξ^{α_i} ,

⁹ To derive this equation more rigorously it is necessary to introduce a regularization in (3.6)–(3.8). This does not influence the considerations and the results obtained, so we omit the discussion of this problem (for the details see [6]).

¹⁰ All the subsequent considerations are valid only if the solutions are determined by the convergent series of positive powers of ξ .

where α_i is defined by the condition $[R(\alpha_i + 1)]^{-1} = 0$. Since $R(z)$ is regular for $\text{Re } z \geq 0$, then $\alpha_i < -1$ and the solutions corresponding to the characteristic values α_i are more singular for small ξ than the free propagator ξ^{-1} . Choosing the behaviour which is not more singular for small ξ than that of the free propagator we conclude that the superpropagator is a linear combination of two solutions corresponding to $\alpha = 0$ and $\alpha = -1$. To construct these solutions we use the Frobenius method [13]. To the characteristic value $\alpha = 0$ there corresponds the power series solution

$$F_1(\xi) = \sum_{n=0}^{\infty} a_n \xi^n; \quad a_0 = 1, \quad a_n = \frac{(-1)^n \prod_{k=1}^n R(k)}{n!(n+1)!} = \frac{(-1)^n c_{n+2}}{c_2 \cdot n!(n+1)!}, \quad (3.10)$$

which is normalized by the condition $F_1(0) = 1$. For $\xi < 0$ this solution is proportional to the difference between the imaginary part of the superpropagator

$$\text{Im } \tilde{F}(p^2) = c_1 \pi g^2 \delta(p^2) + \frac{g^4}{16\pi} \theta(-p^2) \theta(p_0) c_2 F_1(\xi) \quad (3.11)$$

and the imaginary part of the free propagator $c_1 \pi g^2 \delta(p^2)$ (Eq. (3.11) can be directly calculated by the Cutkosky rule [14]). The solutions corresponding to other characteristic values are also searched in the form

$$F(\alpha, \xi) = \sum_{n=0}^{\infty} a_n(\alpha) \xi^{n+\alpha}; \quad (3.12)$$

$$a_n(\alpha) = \frac{(-1)^n R(n+\alpha) R(n+\alpha-1) \dots R(\alpha+1)}{(n+\alpha+1)(n+\alpha)^2 \dots (\alpha+2)^2 (\alpha+1)} a_0(\alpha).$$

For $\alpha = 0$ the solution is exactly the same as written above, but for $\alpha = -1$ the solution (3.12) does not make sense as the denominator in the recurrent formula (3.12) vanishes for $\alpha = -1$. To construct the solution corresponding to $\alpha = -1$ we employ the Frobenius method as developed for the finite order equations. To this end we put $a_0(\alpha) = (\alpha + 1)$ and differentiate (3.12) with respect to α :

$$\frac{\partial F(\alpha, \xi)}{\partial \alpha} = \sum_{n=0}^{\infty} \frac{da_n(\alpha)}{d\alpha} \xi^{n+\alpha} + \ln \xi \sum_{n=0}^{\infty} a_n(\alpha) \xi^{n+\alpha}. \quad (3.13)$$

As $\alpha \rightarrow -1$ this expression determines the solution which is linearly independent with (3.10):

$$F_2(\xi) = \left. \frac{\partial F(\alpha, \xi)}{\partial \alpha} \right|_{\alpha=-1} \quad (3.14)$$

$$= -R(0) \ln \xi \cdot F_1(\xi) - \sum_{n=0}^{\infty} b_n (-\xi)^{n-1}$$

where

$$b_0 = 1, \quad b_n = \left\{ \frac{d}{d\alpha} \left[(n + \alpha + 1) \prod_{k=1}^n \frac{R(k + \alpha)}{(k + \alpha + 1)^2} \right] \right\}_{\alpha=-1}, \quad n = 1, 2, \dots \quad (3.15)$$

The solutions corresponding to other α_i can be constructed by an analogous procedure. However, we neglect these solutions because they are more singular for $\xi \rightarrow 0$ than the free propagator. The solution (3.14) is real for $\xi > 0$ and is defined in the entire complex ξ -plane (i.e., p^2 -plane) with a cut $-\infty < \xi < 0$. A discontinuity across the cut is $-2\pi i R(0) F_1(\xi)$ and being properly normalized it coincides with the imaginary part of the superpropagator. So we arrive at the final expression for the superpropagator

$$F(\xi) = AF_2(\xi) + BF_1(\xi) \quad (3.16)$$

where $A = \frac{c_1}{(4\pi)^2}$ and B is an arbitrary real constant¹¹.

Now the problem of determining the asymptotic behaviour of the solution (3.16) as $\xi \rightarrow \infty$ arises. In general this problem is rather complicated. However, if $R(z)$ has a simple form the asymptotic behaviour can be derived by use of standard methods. For instance, for many theories of the physical importance $R(z)$ is a rational function

$$R(z) = P_p(z)/Q_{q-2}(z)$$

where P_p and Q_{q-2} are polynomials of degree p and $q-2$, respectively. Then Eq. (3.9) takes the form

$$[\delta_\xi(\delta_\xi + 1) Q_{q-2}(\delta_\xi) + \xi P_p(\delta_\xi + 1)] F(\xi) = 0. \quad (3.17)$$

Factorizing the polynomials P_p and Q_{q-2}

$$P_p(\delta_\xi + 1) = \prod_{j=1}^p (\delta_\xi - a_j + 1);$$

$$Q_{q-2}(\delta_\xi) = \prod_{j=1}^{q-2} (\delta_\xi - b_j); \quad b_{q-1} \equiv -1, \quad b_q \equiv 0;$$

we easily find that the solutions of Eq. (3.17) are linear combinations of the Meijer G -functions [16]

$$G_{pq}^{mn} \left((-1)^{p+1-m-n} \xi \begin{matrix} a_1 \dots a_p \\ b_1 \dots b_q \end{matrix} \right), \quad \text{where } 0 \leq m \leq q, \quad 0 \leq n \leq p.$$

¹¹ One can try to fix this constant by requiring the minimal singularity for $|\xi| \rightarrow \infty$ (or on the light cone) [6, 11]. This requirement is reduced to the condition $\text{Re} \tilde{F}(p^2)/\text{Im} \tilde{F}(p^2) \rightarrow 0$ as $p^2 \rightarrow -\infty$ (see [6]) which is very suitable for calculations. However, it is possible that, in calculating higher orders in the constant f, B in (3.16) will appear to be fixed by the unitarity condition (cf., e.g. [15]), so the final choice of B can be made only after the detailed investigation of higher orders.

The asymptotic behaviour of these functions has been completely investigated by Meijer [16] and by using his results we can solve the boundary value problem for Eq. (3.17). The Meijer G -function can have an essential singularity at $\xi \rightarrow \infty$ or be polynomially bounded. Thus, even within this simplest class of theories all types of asymptotics (localizable, non-localizable and essentially nonlocal) can be obtained [17].

So far in constructing the superpropagator we used only the boundary conditions at $\xi \rightarrow 0$ and derived the solution depending on one arbitrary constant. However, the differential Eq. (3.9) is equivalent to the integro-differential Eq. (3.6) only if $F(\xi)$ vanishes at infinity. As it can be seen from the examples considered in the next section, in localizable theories the superpropagator does not vanish at infinity (in fact, it grows exponentially) and therefore in this case we will treat the differential Eq. (3.9) with the boundary condition at $\xi \rightarrow 0$ as an extension of the integral Eq. (3.6) which has a solution even when the integral Eq. (3.6) has no solution (cf. the theory of extensions of the symmetric operators [18]). Any choice of the parameter B (see Eq. (3.16)) is equivalent to some boundary condition at infinity, the most natural one being given in footnote 11. The same results can be obtained by introducing some regularization removed at the end of calculations or by the analytic continuation in the coupling constant g^2 (see for the detailed discussion Ref. [6]). So we may say that the transition to the differential equation is the convenient method for regularizing nonrenormalizable theories in momentum space without any reference to co-ordinate space. This approach makes it possible to avoid the problems connected with the regularization of the nontempered distributions.

4. Examples and Discussion

1. Consider the superpropagator (3.1) with $c_n \equiv 1$. In this case $R(z) \equiv 1$ and Eq. (3.9) for $F(\xi)$ reduces to the Bessel equation. Normalizing the solution in such a way that $\tilde{F}(p^2) \simeq \frac{g^2}{p^2}$ as $p^2 \rightarrow 0$, we find

$$\tilde{F}(p^2) = - \frac{g^4}{16\pi \sqrt{\xi}} \{N_1(2\sqrt{\xi}) + C J_1(2\sqrt{\xi})\}, \quad (4.1)$$

where N_1 is the Neumann function, J_1 is the Bessel function and C is an arbitrary constant. The function \tilde{F} has a logarithmic branch point in g^2 at $g^2 = 0$ and an essential singularity at $p^2 \rightarrow \infty$. As $p^2 \rightarrow +\infty$ the superpropagator (4.1) vanishes, but when $p^2 \rightarrow -\infty + i0$ it grows for any choice of C . Nevertheless, the parameter C can be chosen so that

$\operatorname{Re} \tilde{F}(p^2 + i0)/\operatorname{Im} \tilde{F}(p^2 + i0) \xrightarrow{p^2 \rightarrow -\infty} 0$ (in accordance with the remarks of the previous section).

If g^2 is changed in sign the integro-differential Eq. (3.6) has the unique solution satisfying the boundary condition at $\xi = 0$ and vanishing for $\xi \rightarrow +\infty$:

$$\tilde{F}(p^2) = \frac{g^4}{8\pi^2 \sqrt{\xi}} K_1(2\sqrt{\xi}) \quad \text{where} \quad \xi = \frac{|g^2| p^2}{16\pi^2}. \quad (4.2)$$

This solution exponentially vanishes as $|\xi| \rightarrow \infty$ in any direction in the complex ξ -plane except for the cut $-\infty < \xi < 0$ where it falls off with oscillations (the imaginary part oscillates too). Note that the superpropagator (4.2) has been originally derived by Okubo [1], who however has not found the physically acceptable expression (4.1).

2. Consider now the exponential superpropagator: $c_n = 1/n!$, $R(z) = (z+2)^{-1}$. It is not difficult to show that in this case

$$\begin{aligned} \tilde{F}(p^2) = & -\frac{g^4}{32\pi^2} \{G_{03}^{20}(\xi e^{i\pi}|0, -1, -2) \\ & + G_{03}^{20}(\xi e^{-i\pi}|0, -1, -2) + CG_{03}^{10}(\xi|0, -1, -2)\}. \end{aligned} \quad (4.3)$$

The superpropagator (4.3) grows as $p^2 \rightarrow \infty$ in all directions in the complex p^2 -plane and its imaginary part is positive for $p^2 < 0$. (The condition $\operatorname{Re} \tilde{F}(p^2 + i0)/\operatorname{Im} \tilde{F}(p^2 + i0) \xrightarrow{p^2 \rightarrow -\infty} 0$ is fulfilled for $C=0$.) Reversing the sign of g^2 we get the following solution

$$\tilde{F}(p^2) = \frac{g^4}{16\pi^2} G_{03}^{20}(\xi|0, -1, -2) \quad \text{where} \quad \xi = \frac{|g^2| p^2}{16\pi^2} \quad (4.4)$$

vanishing as $\xi \rightarrow +\infty$. On the cut this solution exponentially grows and its asymptotics is of the localizable type. The same expression has been previously derived by the present authors and Arbutov (see [5, 6]). In a different way the superpropagator (4.3) has been obtained by Volkov [8].

3. The next one is the case of $C_n = n!$, $R(z) = z+2$. Here Eq. (3.9) reduces to the confluent hypergeometric equation. The solution is

$$\tilde{F}(p^2) = -\frac{g^4}{16\pi^2} \{\psi(3, 2; \xi e^{i\pi}) + \psi(3, 2; \xi e^{-i\pi}) + C\phi(3, 2; -\xi)\} \quad (4.5)$$

where

$$\phi(3, 2; z) = \sum_{n=0}^{\infty} \frac{n+2}{2} \frac{z^n}{n!} \quad (4.6)$$

is the confluent hypergeometric function, and

$$\psi(3, 2; z) = \frac{1}{2z} + \ln z \cdot \phi(3, 2; z) + \frac{e^z}{2} - \sum_{n=0}^{\infty} \frac{n+2}{2 \cdot n!} \psi(n+1) z^n. \quad (4.7)$$

The branch of logarithm is so chosen in Eq. (4.7) that $\ln z$ is real for $z > 0$. The asymptotic behaviour of $\phi(3, 2; z)$ and $\psi(3, 2; z)$ is well known [19]. The solution (4.5) for any C vanishes like ξ^{-3} as $\xi \rightarrow \infty$ in the right halfplane, but it grows exponentially in the left halfplane. For the reversed sign of g^2 we get the unique solution $\psi(3, 2; \xi)$ which vanishes like ξ^{-3} as $\xi \rightarrow \infty$ in the whole complex ξ -plane $\left(\xi = \frac{|g^2| p^2}{16\pi^2}\right)$. Thus, in the case of nonlocalizable theories for negative g^2 one can construct solutions vanishing as $p^2 \rightarrow \infty$ which makes it possible to define the superpropagator in x -space as well.

4. As a matter of fact, one can investigate as many examples as one wishes choosing growing or vanishing coefficients c_n . For $c_n \xrightarrow{n \rightarrow \infty} \infty$ the series (3.1) is obviously divergent, however our approach requires no preliminary summation of this series which produces the well-known ambiguity (one can add any arbitrary function with zero asymptotic expansion). If c_n grows more slowly than $(n!)^2$ it is possible to obtain the unique (up to the parameter B) solution determined by the series (3.14). If c_n grows faster than $(n!)^2$ the series (3.14) diverge, the differential equation having an essential singularity at $\xi = 0$ and a regular singularity at $\xi = \infty$. In this case the solution cannot be represented as a power series (3.14) and expansions of another type are required to solve the equation. The corresponding solutions are essentially non-local [17]. If $c_n \sim (n!)^2$ then both $\xi = 0$ and $\xi = \infty$ are regular singularities but, in addition, there arises a singularity on the circle $|\xi| = \text{const}$ (i.e., on the "unitary limit"). The method described above seems to be useful tool in investigating all such superpropagators and we hope to do this later.

We are indebted to N. Bogoliubov, D. Blokhintsev, A. Tavkhelidze, A. Efremov, V. Ogievetsky and M. Volkov for helpful discussions and the interest in the work.

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