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Gentle Perturbations

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Abstract. By introducing a specific type of perturbation, A, in the Hamiltonian, we define a class of gently perturbed states, $\varrho_{\beta,A}$, of a canonical ensemble, ϱ_β. The perturbations are chosen so as to preserve a relationship of the form $\varrho_{\beta,A} \leqq \text{constant} \times \varrho_\beta$. Applications in ergodic theory and phase transitions are described.

1. Introduction

It is not difficult to give examples wherein the state of a dynamical system is radically altered by the introduction of a perturbation. It is our purpose however, to investigate the effects on a canonical ensemble of a specific class of very weak perturbations, the choice being made so as to preserve a certain relationship with the unperturbed state. The relationship is known to be of use in studying problems of ergodic theory and of phase transitions, and we explicitly mention important, "non-sovable" models to which our results apply.

We are primarily concerned with infinite volume, quantum mechanical systems, and the C^*-algebra formalism is used. The system is thus assumed to be describable by the C^*-inductive limit [1] $\mathfrak{A}$ of an increasing sequence of finite volume subsystems $\mathfrak{A}_n$, i.e. sub-C^*-algebras, which are C^*-isomorphic to[1] $B(\mathscr{H}_n)$ for some sequence $\mathscr{H}_n$ of Hilbert spaces. To simplify the notation, we will identify $\mathfrak{A}_n$ with $B(\mathscr{H}_n)$ at will.

If H is a self-adjoint operator on a Hilbert space $\mathscr{H}$, with generalized resolution of the identity $\{E_\lambda \mid -\infty < \lambda < \infty\}$, we shall mean by $e^{\beta H}$ the self-adjoint operator with domain:

$$D(e^{\beta H}) = \left\{ \psi \in \mathscr{H} \mid \int_{-\infty}^{\infty} e^{2\beta\lambda} d\|E_\lambda \psi\|^2 < \infty \right\}$$

and definition:

$$e^{\beta H} : \psi \in D(e^{\beta H}) \to \int_{-\infty}^{\infty} e^{\beta\lambda} d(E_\lambda \psi) .$$

The meaning of the integral is that of [§ 29.2; 2].

2. Proposition I and Applications to Ergodic Problems

Our results will follow from the following proposition.

Proposition I. *Let $\mathfrak{A}$, $\mathfrak{A}_n$ and $\mathscr{H}_n$ be as described in* § 1, *and let H_n (resp. A_n) be positive (resp. bounded) self-adjoint operators on $\mathscr{H}_n$ for $n=1, 2, \cdots$. Let TR_n be a faithful trace*[1] *on $\mathfrak{A}_n^+$.*

Assuming

I. *the product $e^{\beta H_n} A_n e^{-\beta H_n}$ is defined and bounded on $\mathscr{H}_n$ and there exist positive constants β_C and C such that*

$$\|e^{\beta H_n} A_n e^{-\beta H_n}\| \leqq C$$

for all $\beta \in [0, \beta_C)$ and all $n=1, 2, \cdots$, and

II. *$TR_n(e^{-\beta_0 H_n}) < \infty$ and the states $\varrho_{\beta_0, n}$ defined on $\mathfrak{A}_n$ by*

$$\varrho_{\beta_0, n}(\cdot) = \frac{TR_n(e^{-\beta_0 H_n} \cdot)}{TR_n(e^{-\beta_0 H_n})}$$

have a w-limit*[2]*, ϱ_{β_0}, on $\mathfrak{A}$ for some $\beta_0 \in (0, \beta_C)$, and*

III. *the states $\varrho_{\beta_0, A, n}$ defined*[3] *on $\mathfrak{A}_n$ by*

$$\varrho_{\beta_0, A, n}(\cdot) = \frac{TR_n(\exp[-\beta_0(H_n + A_n)] \cdot)}{Tr_n(\exp[-\beta_0(H_n + A_n)])}$$

have a w-limit*[2]*, $\varrho_{\beta_0, A}$, on $\mathfrak{A}$ for the same $\beta_0 \in (0, \beta_C)$, then the following condition is satisfied:*

IV. $$\varrho_{\beta_0, A} \leqq e^{2\beta_0 C} \varrho_{\beta_0}.$$

The proof is given in Appendix I. To make the meaning of I clearer, consider the situation where TR_n is the usual (normal) trace on separable $\mathscr{H}_n$. Then since $e^{-\beta H_n}$ is positive, it follows from II that H_n only has isolated discrete spectrum. Therefore let $\{\phi_m | m=1, 2, \cdots\}$ be a basis in $\mathscr{H}_n$ with $H_n \phi_n = h_{n,m} \phi_m$. Condition I then reduces [Thrm., pg. 53; 3] to:

I′ $$\left| \sum_{m=1}^{p} \sum_{r=1}^{q} \exp[\beta(h_{n,m} - h_{n,r})] \langle \phi_n, A_n \phi_r \rangle x_m y_r \right| \leqq C \left(\sum_{m=1}^{p} |x_m|^2 \right)^{\frac{1}{2}} \left(\sum_{r=1}^{q} |y_m|^2 \right)^{\frac{1}{2}}$$

for all positive integers p and q, all $x_m, y_r \in \mathbb{C}$, and all $\beta \in [0, \beta_C)$.

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[1] This condition could easily be weakened.

[2] Strictly speaking we should extend the states to $\mathfrak{A}$ to use this terminology.

[3] The existence of these states will be shown in the proof.

Thus I becomes in this situation a restriction on the matrix elements of the perturbation, namely that the matrix elements between unperturbed energy levels vanish in a certain (strong) sense as the difference in the energies becomes infinite. In particular,

$$|\langle\phi_m, A_n\phi_r\rangle| \leqq C\exp[-\beta_C|h_{n,m}-h_{n,r}|]\,.$$

In order to describe our results we need some background material which we present at this point.

The usefulness of the (mean) ergodic theorem on L_1 is that it proves that those states[4] of the system represented by functions in L_1, maintain this characteristic not only for finite times, which is obvious, but also in the infinite time average. In [4] we have proven the following result which covers[5] quantum systems, and which reduces precisely to the mean ergodic theorem in the classical "limit" of commuting observables.

Proposition II[6]. *Let $\mathfrak{A}$ be a C^*-algebra with unit, and let $\{\alpha_t\,|\,t\in\mathbb{R}\}$ be a group representation of $\mathbb{R}$ as automorphisms of $\mathfrak{A}$. If $\bar{\varrho}$ is a state on $\mathfrak{A}$ invariant under the dual maps α_t^* for all $t\in\mathbb{R}$, then for each*

$$\varrho\in\overline{\bigcup_{\text{real}\,a}\{\phi\,|\,\phi \text{ is a state} \leqq a\bar{\varrho}\}}^{\text{norm}}\equiv\overline{\mathscr{S}(\bar{\varrho})}^{\text{norm}},$$

there exists an invariant state $\tilde{\varrho}$ in $\overline{\mathscr{S}(\bar{\varrho})}^{\text{norm}}$, given by the norm limit, under refinement[7], *of the net of "ergodic" averages:*

$$\left\{\sum_i a_i\alpha_{t_i}^*(\varrho)\,|\,a_i\geqq 0, \sum_i a_i=1\right\}.$$

It is clear that Proposition I is useful in elucidating the nature of the domain $\overline{\mathscr{S}(\bar{\varrho})}^{\text{norm}}$ of this ergodic theorem. To emphasize that the information is not vacuous, we mention the following examples. For the ν-dimensional spin-lattice models of Streater [6] and Robinson [7] (which includes all finite range, lattice invariant interactions), we take

[4] These states are to be distinguished from the *continuous* phase-space functions, which are observables. See for example [4].

[5] This is not meant to imply that important infinite systems with nontrivial time development will not be constructed in the future which do not satisfy all the hypotheses of this result.

[6] This is actually a restatement of [Prop. III; 4] which makes use of [Thrm. 2; 5].

[7] We define this partial ordering as follows.

$$\sum_i a_i\alpha_{t_i}^*(\varrho)\supseteqq\sum_j b_j\alpha_{t_j}^*(\varrho)$$

means that there exists an ergodic average $\sum_k c_k\alpha_{t_k}^*(\varrho)$ such that

$$\sum_i a_i\alpha_{t_i}^*(\varrho)=\sum_{j,k} b_jc_k\alpha_{t_j+t_k}^*(\varrho).$$

$\varrho = \bigotimes_i \mathrm{Trace}_i$ and $A_n = A =$ any local hermitian element, for instance $A = \sum_{i \in V} b_i \sigma_z^i$ where V is a finite volume and the field strengths b_i are real. Then condition I is satisfied with a known, model-dependent lower bound for β_C [6, 7]. Assuming condition II, $\bar{\varrho} \equiv \varrho_{\beta_0}$ is time invariant and the hypotheses of Proposition II are satisfied with[8] $\varrho \equiv \varrho_{\beta_0, A}$. Moreover, for the finite range, one-dimensional systems [8] and the ν-dimensional generalized Ising models [9], β_C may be taken arbitrarily large, and condition II can be derived. It is evident therefore that Propositions I and II together should be useful in proving the relaxation back to canonical equilibrium of a system initially in equilibrium with a local perturbation, which is turned off at $t = 0$.

In particular, in Appendix II we prove the following result[9]. Let ϱ_{β_0} be the infinite volume canonical ensemble, and α_t^* the Schrödinger time development, for the 1-dimensional $X - Y$ model (a-cyclic or free-ends and with arbitrary anisotropy parameter)[10]. Let $\varrho_{\beta_0, A}$ be the corresponding state[8] with a perturbation

$$A_n = A = \sum_{i \in V} b_i \sigma_z^i .$$

Then for any $B \in \mathfrak{A}$ we have

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \alpha_t^* \varrho_{\beta_0, A}(B) dt = \varrho_{\beta_0}(B) .$$

3. Proposition III and Application to Phase Transitions

We begin this section by strengthening Proposition I for those systems which, like the spin-lattice systems mentioned above, have the added feature that the $\mathscr{H}_n$ are finite dimensional.

Proposition III. In the notation and hypotheses of Proposition I we replace III with the conditions that the $\mathscr{H}_n$ be finite dimensional and that V $\exp(iH_n t) A_n \exp(-iH_n t)$ has a norm limit in $\mathfrak{A}$ for all $t \in [0, \beta_C)$. Then condition III can be derived (and therefore also IV).

The proof is given in Appendix I.

In the work of Yang [12], Ginibre [13] and others, it is proven that one can excite a phase transition from the canonical ensemble by means

8 The existence of these states is proven by Proposition III below.

9 We mention this result only as an example of a general technique. Stronger results for the $X - Y$ model have been published [10].

10 For this notation see [11].

of the following procedure. Using

$$\frac{\varrho(e^{-\beta H_n}\cdot)}{\varrho(e^{-\beta H_n})}$$

as a volume cutoff approximation to the canonical ensemble of an infinite spin system, one introduces a cutoff perturbation of the form

$$A_n = b_n \sum_{i \in V_n} \sigma_z^i \,.$$

Ginibre takes $b_n = b$ and Yang essentially takes $b_n = b/|V_n|^{\frac{1}{2}}$, where $|V_n|$ is the number of sites in the volume V_n. After removing the volume cutoff, the limit $b \to 0$ is taken and quantities are computed which show the existence of a phase transition.

If it can be assumed that the above procedure can be carried out in the sense of obtaining (infinite volume) states over the system, i.e. that the expectation values of *all* relevant observables have well defined limits, not just those computed in the work cited, then it is of interest to investigate any relation which may exist between these states. It seems reasonable to assume that the excited state obtained after taking the limit $b \to 0$ is what is usually called a pure phase, that is, an equilibrium state at a given temperature which cannot be decomposed into other equlibrium states of that temperature.

There has been much work concerned with the decomposition of impure phases [14–17]; they all fall into the following pattern. The set $\mathfrak{S}_\beta$ of equilibrium states at inverse temperature β is defined to consist of those states which are solutions of certain linear equations (variational principle, time or space invariance, Gallavotti-Miracle equations, KMS equation, etc.). $\mathfrak{S}_\beta$ is then convex, and the pure phases are the extreme points of $\mathfrak{S}_\beta$. It is easy to see that they can equivalently be characterized as those elements ϱ of $\mathfrak{S}_\beta$ for which the inequality $\varrho' \leqq a\varrho$, with $a \geqq 1$ and $\varrho' \in \mathfrak{S}_\beta$, has the unique solution $\varrho' = \varrho$.

Noticing the different strengths in the perturbation used by Yang and Ginibre, we now ask the natural question: are the same results obtainable with the perturbation

$$A_n = \frac{b}{|V_n|} \sum_{i \in V_n} \sigma_z^i$$

with (or without) a subsequent limit $b \to 0$? As these A_n satisfy $\|A_n\| \leqq |b|$, we first generalize the question to the following: is it possible to excite a pure phase (as defined above) from the canonical ensemble with a finite amount of energy in the (total) perturbation?

We cannot answer this in general, even for the Robinson spin systems. If however, $[A_n, H_n]=0$ as is the case for ν-dimensional Heisenberg, Ising and $X-Y$ models, then condition I precisely reduces to the restriction of finite energy. Moreover, in the proof of Proposition I the roles of ϱ_{β_0} and $\varrho_{\beta_0,A}$ are now interchangeable (since $T_{\beta,n}=e^{-\beta A_n}$ is now invertible) and we have that $\varrho_{\beta_0} \leqq e^{2\beta_0 C} \varrho_{\beta_0,A}$, which gives us a definite negative answer.

4. Summary and Remarks

We have introduced a certain class of very weak perturbations of a canonical ensemble, with the following results.

1. The perturbations are naturally suited for proving relaxation back to equilibrium, and the first step in this program is proven to go through for an important class of "non-solvable" spin models.

2. The question whether it is possible to excite a phase transition from the canonical ensemble by a finite energy perturbation is answered negatively when the perturbation cannot exchange energy with the unperturbed interaction. To these results we add the following corresponding remarks.

1. Unfortunately, the only case where we can complete the program is in the $X-Y$ model, and results stronger than ours have been obtained, making full use of the sovability of the model [10]. The pragmatist is therefore justified in remaining unconvinced of the usefulness of this result until time asymptotic abelianess or some similar ergodic property can be proven for some interesting "non-solvable" model.

2. The restriction $[A_n, H_n]=0$ severely limits the applicability of the result, although it still contains important models. It would be of interest to answer the unrestricted question.

Appendix I

Proof of Proposition I. It is easy to show that the bounded operators $T_{\beta,n}$ defined for $\beta \in (0, \beta_C)$ by:

$$T_{\beta,n}=I+\sum_{k=1}^{\infty}\int_0^{-\beta} d\beta_1 \int_0^{\beta_1} d\beta_2 \cdots \int_0^{\beta_{k-1}} d\beta_k \times \prod_{j=1}^{k} \exp[-\beta_j H_n] A_n \exp[\beta_j H_n] \tag{1}$$

satisfy

$$\|T_{\beta,n}\| \leqq e^{\beta C}. \tag{2}$$

Now we note that with our definitions, if H is a positive self-adjoint operator on a Hilbert space, $e^{-\beta H}$ is bounded and (strongly) differentiable for $\beta > 0$, with

$$\frac{d}{d\beta}\exp(-\beta H) = (-H)\exp(-\beta H).$$

It is clear that the $T_{\beta,n}$ are (strongly) differentiable for $\beta \in (0, \beta_C)$. In fact

$$\frac{d}{d\beta} T_{\beta,n} = -[\exp(\beta H_n) A_n \exp(-\beta H_n)]\, T_{\beta,n}.$$

Therefore, for $\beta \in (0, \beta_C)$, $\exp(-\beta H_n)\, T_{\beta,n}$ is (strongly) differentiable with

$$\frac{d}{d\beta}[\exp(-\beta H_n)\, T_{\beta,n}] = -(H_n + A_n)\exp(-\beta H_n)\, T_{\beta,n},$$

where $H_n + A_n$ is self-adjoint and bounded below [Thrm. V–4.3; 18]. Since $\exp(-\beta H_n)\, T_{\beta,n}$ and $\exp[-\beta(H_n + A_n)]$ are both solutions of:

$$\frac{d}{d\beta} K(\beta) = -(H_n + A_n) K(\beta) \qquad \beta \in (0, \beta_C)$$

with boundary condition $K(0) = I$, we know [pgs. 481, 2; 18] that

$$\exp(-\beta H_n)\, T_{\beta,n} = \exp[-\beta(H_n + A_n)] \qquad \beta \in [0, \beta_C).$$

It follows that $0 < TR_n\{\exp[-\beta(H_n + A_n)]\} < \infty$, and that the states $\varrho_{\beta_0, A, n}$ of III are well defined. It also follows that:

$$\varrho_{\beta_0, A, n}(\cdot) = \frac{TR_n[\exp(-\beta_0 H_n)\, T_{\beta_0, n}\cdot]}{TR_n[\exp(-\beta_0 H_n)]}\;\frac{TR_n[\exp(-\beta_0 H_n)]}{TR_n[\exp(-\beta_0 H_n)\, T_{\beta_0, n}]}. \tag{3}$$

Using (2) and applying [Prop. I; 19] to the second factor, F_n, on the *RHS* of (3) we find

$$0 < F_n \leqq e^{\beta_0 C}.$$

Absorbing F_n into the $T_{\beta_0, n}$ of the first factor, and applying [Prop. I; 19] again, we get

$$\varrho_{\beta_0, A, n} \leqq e^{2\beta_0 C} \varrho_{\beta_0, n},$$

from which IV is immediate. q.e.d.

Proof of Proposition III. By a simple generalization of its proof in the complex variable setting[11], Vitali's convergence theorem implies, given I (which by taking adjoints automatically holds for all $\beta \in (-\beta_C, \beta_C)$) and V, that

$$\exp(\beta H_n) A_n \exp(-\beta H_n) \quad \text{converges in norm for all} \quad \beta \in [0, \beta_C). \tag{4}$$

[11] See for example the proof of [5.21; 20].

Now by a norm estimate on the tail of the summation in (1) it is easy to see that given $\varepsilon > 0$, there exists K such that

$$\begin{aligned} \| T_{\beta,n} - T_{\beta,m} \| < \varepsilon + \int_{k=1}^{K} \int_{0}^{\beta} d\beta_1 \int_{0}^{\beta_1} d\beta_2 \cdots \int_{0}^{\beta_{n-1}} d\beta_n \\ \times \left\| \prod_{j=1}^{k} \exp(-\beta_j H_n) A_n \exp(\beta_j H_n) \right. \\ \left. - \prod_{j=1}^{k} \exp(-\beta_j H_m) A_m \exp(\beta_j H_m) \right\| \end{aligned} \tag{5}$$

uniformly in $\beta \in [0, \beta_C)$ and all $n, m = 1, 2, \cdots$.

From (4) and the joint continuity in norm of the product operation, it is clear that $T_{\beta_0, n}$ is Cauchy. From II therefore, the first factor on the *RHS* of (3) converges in the w^*-topology[2] as $n \to \infty$. In particular, inserting I, the identity, in (3) and noting that the *LHS* is identically 1 and that the second factor on the *RHS* is bounded from (2), we see that this second factor in fact has a limit, F. And now III follows. q.e.d.

Appendix II. The X—Y Model

In order to account for the different conventions in our references, we make the following bijection of the one-sided chain $\mathbb{N}$ (the natural numbers) used in [11] with the usual two-sided chain $\mathbb{Z}$:

$$j \in \mathbb{Z} \to \begin{cases} 2|j| \in \mathbb{N} & j > 1 \\ 2|j| + 1 \in \mathbb{N} & j \leqq 0 . \end{cases}$$

Since the spin algebra and the full *CAR* algeara are naturally isomorphic [21] it is clear from [11] that the class of quadratic Fermi interactions considered in [22] contains the one-dimensional $X - Y$ model with arbitrary anisotropy parameter and free-end boundary conditions. It is also clear that as far as the infinite volume time development on local quantities is concerned, the a-cyclic and free-end conditions are equivalent. From [8] the free-end canonical ensemble $\varrho_{\beta,F.E.}$ exists for all $\beta > 0$ in the infinite volume limit, and is extremal lattice invariant. Since any w^*-convergent subsequence of the cutoff a-cyclic canonical ensembles is lattice invariant, a simple application of Proposition I shows that it must coincide with $\varrho_{\beta,F.E.}$. Since the set of all states is sequentially compact[12] we see that the cutoff, a-cyclic canonical ensembles also converge to $\varrho_{\beta,F.E.}$, and so we drop the extra subscript from ϱ_β.

In [8] it is shown that ϱ_β is extremal *KMS* and primary. From [24] ϱ_β has short range correlations and so is uniformly clustering. Define

[12] Since our algebra is separable, this follows from [V.4.2, V.5.1 and I.6.13; 23].

$\varrho_\beta|$ as the restriction of ϱ_β to $\mathfrak{A}_e$, the so called "even subalgebra" generated by the even products of Fermi operators. Since the proofs of [Props. 2.2, 2.3; 24] also go through for $\mathfrak{A}_e$, and since $\varrho_\beta|$ is obviously uniformly clustering on $\mathfrak{A}_e$, $\varrho_\beta|$ is primary. Now it is shown in [22] that the time development is norm asymptotic abelian on $\mathfrak{A}_e$, and therefore $\varrho_\beta|$ is extremal time invariant [25]. From [4], and denoting by α_t^* the Schrödinger time development, we have that

$$\frac{1}{T}\int_0^T \alpha_t^* \varrho_{\beta,A}|\,dt$$

has a w^*-limit $\widetilde{\varrho_{\beta,A}}|$ as $T\to\infty$ which is time invariant, and which from Proposition I satisfies[13] $\widetilde{\varrho_{\beta,A}}| \leqq e^{2\beta C}$ for some C. Since $\varrho_\beta|$ is extremal time invariant, it follows as in § 3 that $\widetilde{\varrho_{\beta,A}}| = \varrho_\beta|$. To extend this to the desired result it is only necessary to note the following fact. If γ^* is the dual of the automorphism γ associated[14] with $\mathfrak{A}_e$, then it is easy to show that ϱ_β, $\varrho_{\beta,A}$ and $\widetilde{\varrho_{\beta,A}}$ are all fixed points of γ^*, i.e. they all satisfy $\gamma^*\varrho=\varrho$. But the restriction to $\mathfrak{A}_e$ of such a fixed point determines the full state:

$$\varrho[A]=\varrho[\gamma(A)]=\varrho|\,[\gamma(A)]\,.$$

Therefore, since $\widetilde{\varrho_{\beta,A}}| = \widetilde{\varrho_{\beta,A}|}$, we have

$$\widetilde{\varrho_{\beta,A}}=\varrho_\beta\,. \quad \text{q.e.d.}$$

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[13] The existence of C for condition I is shown in [8].

[14] Specifically, by γ we mean the map α_I defined on page 107 of [26].

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