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Perturbation of Pseudoresolvents and Analyticity in $1/c$ in Relativistic Quantum Mechanics

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Abstract. The analytic functional calculus, relatively bounded and analytic perturbations of pseudoresolvents have been studied. As an application, the nonrelativistic limit of the Dirac and Klein-Gordon operator in the presence of an external static field has been considered. It has been proved that the resolvents of these operators have only a removable singularity at $c=\infty$. This implies the analyticity at $c=\infty$ of the eigenvalues and eigenvectors corresponding to the bound states of the mentioned operators.

In this paper we consider the behaviour of the Dirac and Klein-Gordon operators for a particle in an external static field as functions of the parameter c, where c is the velocity of light, when $c \to \infty$.

It turns out that the resolvents of the operators $T(c) \pm mc^2$, where $T(c)$ is the relativistic Hamiltonian, are analytic functions of the complex variable c, having only a *removable* singularity at $c=\infty$. The limit for $c \to \infty$ is a pseudoresolvent connected in a natural way with the resolvent of a Schrödinger operator with the same potential.

We show that the analytic functional calculus, the relatively bounded and analytic perturbation theory for pseudoresolvents hold equally well as in the case of closed operators (see e.g. [1, 3]).

As an application we obtain the following results: The eigenvalues and eigenvectors for the bound states of the relativistic Hamiltonians are analytic functions of c in some neighbourhood of $c=\infty$. For the Dirac operator this is a generalisation of an earlier work of Titchmarsh [6] in which similar results were obtained by other methods for spherically symmetric potentials.

It should be remarked that our results are only local, i.e., there are no estimates of the radii of convergence.

At the end we discuss some difficulties connected with the fact that the scalar products for the Klein-Gordon equation depend on c in a manner which is not analytic at $c=\infty$.

1. Definitions and Preliminary Results

A *pseudoresolvent* (see [2], p. 201) is a mapping $R(\cdot)$ of a nonvoid set ϱ' of complex numbers into a Banach algebra $\mathfrak{A}$ with an identity 1 such that

$$R(\mu)-R(\lambda)=(\lambda-\mu)\,R(\lambda)\,R(\mu)\,, \qquad \lambda,\mu\in\varrho' \tag{1}$$

or equivalently

$$\begin{aligned}&(1+(\lambda-\mu)\,R(\mu))\,(1+(\mu-\lambda)\,R(\lambda))\\ &\quad=(1+(\mu-\lambda)\,R(\lambda))\,(1+(\lambda-\mu)\,R(\mu))=1\,, \qquad \mu,\lambda\in\varrho'\,.\end{aligned} \tag{1'}$$

Thus, for $\lambda,\mu\in\varrho'$ we have $[1-(\mu-\lambda)\,R(\lambda)]^{-1}$ and

$$R(\mu)=R(\lambda)\,(1+(\mu-\lambda)\,R(\lambda))^{-1}\,. \tag{2}$$

It is well known (see [2], p. 205) that for any pseudoresolvent there is a unique maximal extension with the property (1). The domain of definition $\varrho\supseteq\varrho'$ of this extension will be called the *resolvent set* of the pseudoresolvent R. Its complement σ will be called the *spectrum* of R.

It is easy to show (see [2], p. 205) that

$$\varrho=\{\mu;(1+(\mu-\lambda)\,R(\lambda))^{-1}\in\mathfrak{A}\}\,, \qquad \lambda\in\varrho'\,, \tag{3}$$

where the right-hand side of (3) does not depend on $\lambda\in\varrho'$, and that the extension of R is given by (2). In what follows we shall always consider a pseudoresolvent as defined on its resolvent set unless stated otherwise.

If $\mathfrak{A}=L(X)$, where $L(X)$ denotes the algebra of bounded operators on a Banach space X, we shall say that R is a pseudoresolvent on a Banach space X. It is well known that in this case the range $\mathscr{R}(R(\lambda))$ and the nulspace $N(R(\lambda))$ of the operator $R(\lambda)$ do not depend on $\lambda\in\varrho$ and we simply write $\mathscr{R}$ and N, respectively.

1.1. Definition. A pseudoresolvent on a Hilbert space X is *symmetric* if for some pair $\lambda,\bar{\lambda}\in\varrho$

$$R(\lambda)=R(\bar{\lambda})^*\,. \tag{4}$$

It is called *J-symmetric* if there is $J=J^*=J^{-1}\in L(X)$ such that

$$R(\lambda)=JR(\bar{\lambda})^*\,J\,. \tag{5}$$

Consistently, a bounded operator A is called *J-symmetric* if

$$A=JA^*J\,. \tag{6}$$

From the resolvent Eq. (1) it follows that for a symmetric pseudoresolvent the set ϱ contains all nonreal numbers and that (4) holds for any pair $\lambda, \bar{\lambda} \in \varrho$ and that the operators $R(\lambda)$, $\lambda \in \varrho$ are normal. Thus, the subspace $\mathscr{R}$ is the orthogonal complement of N. Each symmetric pseudoresolvent can therefore be written in the form

$$R(\lambda) = R_0(\lambda)\,P\,, \tag{7}$$

where P is an orthogonal projection and $R_0(\lambda)$ is a resolvent of a uniquely determined selfadjoint operator T in the subspace PX. We then have

$$\|R(\lambda)\| \leqq 1/|\mathrm{Im}\,\lambda|\,. \tag{8}$$

2. Functional Calculus for Pseudoresolvents

It is almost plausible that a pseudoresolvent should possess a functional calculus with analytic functions in full analogy with that for closed operators in a Banach space (see, e.g.. [1], p. 600). For proofs of the theorems in this section we refer the reader to the corresponding proofs in [1]. Full proofs are also given in [3][1].

Here we consider complex analytic functions as defined in a complex sphere $K = C \cup \{\infty\}$ with the usual topology.

2.1. Definition. Let R be a pseudoresolvent on a Banach algebra $\mathfrak{A}$ with an identity 1 and let σ, ϱ be its spectrum and resolvent set, respectively. The set

$$\sigma' = \begin{cases} \sigma; \text{ if } R \text{ is a resolvent of some } a \in \mathfrak{A} \\ \sigma \cup \{\infty\}; \text{ otherwise} \end{cases} \tag{1}$$

will be called the *extended spectrum* of R.

By $\mathscr{F}$ we denote the set of all functions analytic on σ'.[2]

Let $V \subseteqq K$ be an open set containing σ' whose boundary Γ consists of finitely many of Jordan arcs, such that Γ is oriented positively with respect to V. For any f analytic on $V \cup \Gamma$ we define

$$T(f) = \delta f(\infty) + \frac{1}{2\pi i} \int_\Gamma f(\lambda)\, R(\lambda)\, d\lambda\,, \tag{2}$$

where

$$\delta = \begin{cases} 0; \ \infty \notin \sigma' \\ 1; \ \infty \in \sigma'\,. \end{cases}$$

[1] See also [5], where a functional calculus for pseudoresolvents with the properties a), b), c) of Theorem 2.4 below is given.

[2] Note that the analyticity on σ' means just the analyticity on some neighbourhood of σ'.

2.2. Theorem. *For any $f, g \in \mathscr{F}$ we have*

a) $T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$,

b) $T(fg) = T(f)\, T(g)$,

c) $T(f_0) = 1$, *where* $f_0(\lambda) = 1$,

d) $T(f_\mu) = R(\mu)$, *where* $f_\mu = 1/(\mu - \lambda)$, $\mu \in \varrho$,

e) $\sigma(T(f)) = f(\sigma')$.

2.3. Definition (cf. [1], p. 372). A set $\sigma_1 \subseteqq \sigma$ is called a spectral set if it is bounded in C and if it is simultaneously open and closed in σ'.

So, if $\sigma_1 \subseteqq \sigma$ is a spectral set, the element

$$p = \frac{1}{2\pi i} \int_{\Gamma_1} R(\lambda)\, d\lambda, \quad \Gamma_1 \text{ surrounds } \sigma_1 \text{ in the positive sense} \tag{3}$$

will be a projector.

If we take

$$d = \frac{1}{2\pi i} \int_{\Gamma_1} \lambda R(\lambda)\, d\lambda\,, \tag{4}$$

it is obvious that

$$(\lambda - d)^{-1} = R(\lambda)\, p + (1 - p)/\lambda\,, \qquad \lambda \in \varrho\,, \qquad \lambda \neq 0\,. \tag{5}$$

For $\mu \in \varrho$ we also have

$$pR(\lambda) = \frac{1}{2\pi i} \int_{\Gamma_1} \frac{R(\lambda)\, d\lambda}{\mu - \lambda} + \delta(\mu)\, R(\mu)\,, \tag{6}$$

where

$$\delta(\mu) = \begin{cases} 1;\ \mu \text{ inside } & \Gamma_1 \\ 0;\ \mu \text{ outside } & \Gamma_1\,. \end{cases}$$

This implies

$$\sigma(d) = \sigma_1 \cup \{0\}\,. \tag{7}$$

In the case $\mathfrak{A} = L(X)$, where X is a Banach space, we say that $x \in X$ is an eigenvector for R and λ_0 is the corresponding eigenvalue if

$$R(\lambda)\, x = \frac{1}{\lambda - \lambda_0}\, x\,, \quad \text{for all} \quad \lambda \in \varrho\,. \tag{8}$$

It is obvious that an eigenvalue belongs to the spectrum.

2.4. Theorem. *Let $\sigma_1 \subseteqq \sigma$ be a spectral set for a pseudoresolvent R on a Banach space X and let the corresponding spectral projection be finite dimensional. Then σ_1 consists of a finite number of poles for T which are eigenvalues of R.*

3. Relatively Bounded Perturbations

Let T be a closed operator in a Banach space X defined on $D(T)$ so that $\lambda_0 \in \varrho(T)$, and let V be a linear operator in X defined on $D(T)$ such that $1-VR(\lambda_0;T)$ is a regular element of $L(X)$.

Then, as it is well known, $T+V$ is a closed operator on $D(T)$, and the corresponding resolvents satisfy the identity

$$\begin{aligned} R(\lambda;T+V) &= R(\lambda;T)\,(1-VR(\lambda;T))^{-1} \\ &= R(\lambda;T)+R(\lambda;T)\,VR(\lambda;T+V) \end{aligned} \tag{1}$$

for λ from some neighbourhood of λ_0.

Here we take (1) as a definition of a perturbation of a pseudoresolvent.

3.1. Theorem. *Let R be a pseudoresolvent on a Banach space X. Let V be a linear operator in X defined on $\mathscr{R}=\mathscr{R}(R(\lambda))$ such that $1+VR(\lambda_0)$ is a regular element in $L(X)$ for some $\lambda_0 \in \varrho$. Then there is a unique pseudoresolvent Q such that*

$$Q(\lambda)=R(\lambda)+Q(\lambda)\,VR(\lambda)=R(\lambda)+R(\lambda)\,VQ(\lambda)=R(\lambda)\,(1-VR(\lambda))^{-1} \tag{2}$$

for λ from some neighbourhood of λ_0. The pseudoresolvent Q does not depend on the choice of $\lambda_0 \in \varrho$.

Proof. Let $|\lambda-\lambda_0|\ \|R(\lambda_0)\|<1$. Then 1 – (2) implies

$$\lambda \in \varrho\,, \qquad VR(\lambda)\in L(X)\,,$$

$$\|VR(\lambda)-VR(\lambda_0)\| \leqq \frac{\|VR(\lambda_0)\|\ |\lambda-\lambda_0|\ \|R(\lambda_0)\|}{1-|\lambda-\lambda_0|\ \|R(\lambda_0)\|}\,. \tag{3}$$

Since the set of all regular elements is open and the map $\lambda \mapsto R(\lambda)$ is continuous, there is an $r \in (0,1)$ such that $|\lambda-\lambda_0| \leqq r$ implies the regularity of $1-VR(\lambda)\in L(X)$.

Set

$$K_r=\{\lambda;\,|\lambda-\lambda_0|\leqq r\}$$

and define a map

$$K_r \ni \lambda \mapsto Q(\lambda)=R(\lambda)\,(1-VR(\lambda))^{-1}\in L(X)\,. \tag{4}$$

It is obvious that $Q(\cdot)$ satisfies the identity

$$Q(\lambda)=R(\lambda)+Q(\lambda)\,VR(\lambda)\,.$$

Furthermore, since $\lambda \in K_r$ implies $VQ(\lambda)\in L(X)$, we have

$$1+VQ(\lambda)=1+VR(\lambda)\,(1-VR(\lambda))^{-1}=(1-VR(\lambda))^{-1}\,, \qquad \lambda \in K_r\,,$$

which shows that

$$Q(\lambda)=R(\lambda)+R(\lambda)\,VQ(\lambda)\,, \qquad \lambda\in K_r\,.$$

Thus Q satisfies (2). Now we must show that $Q(\cdot)$ is a pseudoresolvent. We first remark that $1+Q(\lambda)V$ maps $\mathscr{R}$ into $\mathscr{R}$. For $\lambda, \mu \in K_r$, $\lambda \neq \mu$ we have

$$Q(\lambda)\,Q(\mu)=(1+Q(\lambda)\,V)\,R(\lambda)\,R(\mu)\,(1+VQ(\mu))=\frac{Q(\mu)-Q(\lambda)}{\lambda-\mu},$$

where we have used (2).

So, Q is a pseudoresolvent defined on K_r. To prove the uniqueness, suppose that $\lambda_1 \in \varrho$ and that $1-VR(\lambda_1)$ is regular in $L(X)$. Let K_s be a disk of radius $s<1$ around λ_1 and $Q_1(\lambda)$ the pseudoresolvent obtained in the same way as above. Then Q_1 satisfies the equalities (2), and for $\lambda \in K_r$, $\mu \in K_s$, $\lambda \neq \mu$ we have

$$Q(\lambda)\,Q_1(\mu)=\frac{1}{\lambda-\mu}(Q_1(\mu)-Q(\lambda))$$

$$Q_1(\mu)\,Q(\lambda)=\frac{1}{\lambda-\mu}(Q_1(\mu)-Q(\lambda)).$$

This implies

$$\begin{aligned}&(1+(\lambda-\mu)\,Q_1(\mu))\,(1+(\mu-\lambda)\,Q(\lambda))\\ &\quad=(1+(\mu-\lambda)\,Q(\lambda))\,(1+(\lambda-\mu)\,Q_1(\mu))=1.\end{aligned}$$

Hence

$$Q_1(\mu)=Q(\lambda)\,(1+(\mu-\lambda)\,Q(\lambda))^{-1},$$

which means that K_s is contained in the resolvent set of Q and that Q_1 is a restriction on K_s of the maximal extension of Q.

4. Analytic Perturbations

In this section we develop the theory of analytic perturbations for pseudoresolvents. The results as well as the methods are quite analogous to those for analytic families of closed operators in the sense of Kato (see [3], p. 367).

Let $s \mapsto R(\cdot, s)$ be a family of pseudoresolvents on a Banach algebra defined for s from some complex neighbourhood $\mathcal{O}$ of zero.

The corresponding resolvent sets and spectra are denoted by $\varrho(s)$ and $\sigma(s)$, respectively.

4.1. Definition. A family of pseudoresolvents $s \mapsto R(\cdot, s)$ is called *analytic at zero* if there is a complex number $\lambda_0 \in \varrho(0)$ and a neighbourhood $\mathcal{O}$ of zero such that $\lambda_0 \in \varrho(s)$, $s \in \mathcal{O}$ and that $s \mapsto R(\lambda_0, s)$ is an analytic function on $\mathcal{O}$ with values in $\mathfrak{A}$.

4.2. Proposition. *Let $R(\cdot, s)$ be analytic at zero. Then for any compact $\Gamma \subseteqq \varrho(0)$ there exists a neighbourhood $\mathcal{O}$ of zero and an open set $\mathcal{U}$ such that $\Gamma \subseteqq \mathcal{U} \subseteqq \varrho(0)$ and that the function*

$$\lambda, s \mapsto R(\lambda, s)$$

is analytic in two variables on $\mathcal{U} \times \mathcal{O}$.

Proof. Take $\Gamma \subseteqq \varrho(0)$. The function

$$\lambda \mapsto \|(1 + (\lambda - \lambda_0) R(\lambda_0, 0))^{-1}\|$$

is continuous and hence bounded on Γ by a constant $M(\Gamma)$.

Set

$$N(\Gamma) = \sup_{\lambda \in \Gamma} |\lambda - \lambda_0|$$

and take a neighbourhood $\mathcal{O}$ of zero such that for $s \in \mathcal{O}$

$$\|R(\lambda_0, s) - R(\lambda_0, 0)\| < 1/M(\Gamma) N(\Gamma).$$

This is possible, since $R(\lambda_0, s)$ is continuous in s. Then

$$\begin{aligned} &1 + (\lambda - \lambda_0) R(\lambda_0, s) \\ &= [1 + (\lambda - \lambda_0) R(\lambda_0, 0)] \{1 + [1 + (\lambda - \lambda_0) R(\lambda_0, 0)]^{-1} (R(\lambda_0, s) - R(\lambda_0, 0))\} \end{aligned}$$

is regular in $\mathfrak{A}$ for any $\lambda \in \Gamma$, $s \in \mathcal{O}$. Since the set of regular elements is open, there is an open set $\mathcal{U}$ such that $\Gamma \subseteqq \mathcal{U} \subseteqq \varrho(0)$ and that $1 + (\lambda - \lambda_0) \cdot R(\lambda_0, s)$ is invertible in $\mathfrak{A}$ for $\lambda \in \mathcal{U}$, $s \in \mathcal{O}$. Now, the function

$$R(\lambda_0, s) [1 + (\lambda - \lambda_0) R(\lambda_0, s)]^{-1} = R(\lambda, s)$$

is analytic on $\mathcal{U} \times \mathcal{O}$ in each variable separately. Since it is obviously jointly continuous on $\mathcal{U} \times \mathcal{O}$, it is also jointly analytic on $\mathcal{U} \times \mathcal{O}$. Q.E.D.

4.3. Corollary. *Let σ_1 be a spectral set of $\sigma(0)$ and Γ_1 a closed Jordan curve separating σ_1 from $\sigma(0) \backslash \sigma_1$. Let f be a function analytic on $\mathcal{U} \cup \Gamma_1$, where $\mathcal{U}$ is an open set containing σ_1 whose boundary is Γ_1. Then there is a neighbourhood $\mathcal{O}$ of zero such that $s \in \mathcal{O}$ implies $\Gamma_1 \subseteqq \varrho(s)$ and that the function*

$$\mathcal{O} \ni s \mapsto \frac{1}{2\pi i} \int_{\Gamma_1} f(\lambda) R(\lambda, s) \, d\lambda$$

is analytic.

In the following, together with an analytic family $R(\cdot, s)$, we consider a family $V(s)$ of perturbations and give a sufficient condition for the analyticity of perturbed pseudoresolvents $Q(\cdot, s)$ obtained by Theorem 3.1.

4.4. Theorem. *Let $s \mapsto R(\cdot, s)$ be a family of pseudoresolvents on a Banach space X which is analytic at zero. In addition, let $V(s)$ be a family* *of linear operators in X such that there is a complex number λ_0 and a neighbourhood $\mathcal{O}$ of $s=0$ with the properties*

a) $V(s)\,R(\lambda_0, s) \in L(x),\ s \in \mathcal{O}$,

b) $s \mapsto V(s)\,R(\lambda_0, s)$ *is analytic for* $s \in \mathcal{O}$,

c) $1 - V(0)\,R(\lambda_0, 0)$ *is regular in* $L(X)$.

Then there is a neighbourhood $\mathcal{O}_1$ of $s=0$ and a neighbourhood $\mathcal{O}_2$ of λ_0 such that the function

$$s \mapsto Q(\lambda, s), \qquad s \in \mathcal{O}_1$$

is analytic in s.

Proof. We first remark that there is a constant $r>0$ such that for $|\lambda-\lambda_0| \leqq r$ the operator $1 - V(0)\,R(\lambda, 0)$ is regular in $L(X)$.

By the analyticity of $s \mapsto R(\lambda_0, s)$ there is a neighbourhood $\mathcal{O}'$ of $s=0$ and $r'>0$ such that $s \mapsto R(\lambda, s)$ is analytic and

$$|\lambda-\lambda_0|\ \|R(\lambda, s)\| \leqq k < 1$$

for $|\lambda-\lambda_0| \leqq r_0$, $s \in \mathcal{O}'$. Hence the function

$$1 - V(s)\,R(\lambda, s) = 1 - V(s)\,R(\lambda_0, s)\,[1 + (\lambda-\lambda_0)\,R(\lambda_0, s)]^{-1}$$

is analytic in $s \in \mathcal{O}'$ for every λ such that $|\lambda-\lambda_0| \leqq r'$. Since the function $[1 + (\lambda-\lambda_0)\,R(\lambda_0, s)]^{-1}$ is jointly continuous in λ, s on $\{|\lambda-\lambda_0| \leqq r'\} \times \mathcal{O}'$, there exists an $r''>0$ and a neighbourhood $\mathcal{O}_1$ of $s=0$ such that for $s \in \mathcal{O}_1$, $|\lambda-\lambda_0| \leqq r''$

$$1 - V(s)\,R(\lambda, s)$$

is regular in $L(X)$. Thus the function

$$s \mapsto R(\lambda, s)\,(1 - V(s)\,R(\lambda, s))^{-1}$$

is analytic in s, since it is product of two analytic functions. Q.E.D.

In the following we shall discuss the behaviour of isolated eigenvalues of finite multiplicity for some classes of analytic families of pseudoresolvents. For later applications we consider not only symmetric pseudoresolvents but also J-symmetric ones. In order to do this, we use the standard method of Kato, which is powerful enough to give results even in the J-symmetric case. We include the full proof mainly for the sake of completeness.

We consider a family $s \mapsto R(\cdot, s)$ of pseudoresolvents on a Hilbert space X with the properties

(i) $R(\cdot, s)$ is analytic at zero.

(ii) $R(\cdot, s)$ is J-symmetric for real s and for some $J = J^{-1} = J^* \in L(X)$.

(iii) The spectrum $\sigma(0)$ of the pseudoresolvent $R(\cdot, 0)$ possesses an isolated real eigenvalue $\lambda_0 \neq 0$ such that the "root space" given by the projection

$$P(0)=\frac{1}{2\pi i}\int_{\Gamma} R(\lambda,0)\,d\lambda\,,\qquad \Gamma \text{ surrounds } \lambda_0\,,$$

has a dimension $m<\infty$.

(iv) The restriction of the form

$$\langle x,y\rangle=(Jx,y)$$

on the subspace $P(0)X$ is strictly positive. (Here $(\cdot,\cdot)$ is the scalar product in X.)

4.5. Theorem (cf. [3], p. 385). *Let $R(\cdot,s)$ be a family of pseudo-resolvents with the properties* (i)–(iv) *listed above. Then for any circle Γ isolating λ_0 from the rest of $\sigma(0)$ there is a neighbourhood $\mathcal{O}$ of $s=0$ such that*

a) $\Gamma\subseteqq\varrho(s)$, $s\in\mathcal{O}$.

b) *The part of $\sigma(s)$ inside Γ consists of $t\leqq m$ points $\lambda_1(s),\ldots,\lambda_t(s)$.*

c) *The functions $\lambda_i(s)$, $i=1,\ldots,t$, $s\in\mathcal{O}$ are analytic and such that $\lambda_i(s)$ is real for real s and $\lambda_i(0)=\lambda_0$.*

d) $\lambda_i(s)\neq\lambda_j(s)$, *for $s\in\mathcal{O}$, $i\neq j$, $s\neq 0$.*

e) *For each $i=1,\ldots,t$ there are vectors Φ_{ij}, $j=1,\ldots,n_i$, such that*

$$R(\lambda,s)\,\Phi_{ij}(s)=\frac{1}{\lambda-\lambda_i(s)}\,\Phi_{ij}(s),\ \lambda\in\varrho(s)\,,$$

$$n_1+n_2+\cdots+n_t=m\,,$$

$$\langle\Phi_{ij}(s)\,\Phi_{kn}(s)\rangle=\delta_{ik}\delta_{jn}\,,\qquad \textit{for real } s\,.$$

f) *The functions $\Phi_{ij}(s)$ are analytic in $s\in\mathcal{O}$ and the vectors $\Phi_{ij}(s)$ are linearly independent.*

Proof. Let $\mathcal{O}(\Gamma)$ be a neighbourhood of $s=0$ such that $s\in\mathcal{O}(\Gamma)$ implies $\Gamma\subseteqq\varrho(s)$.

Put

$$T(s)=\frac{1}{2\pi i}\int_{\Gamma}\lambda R(\lambda,s)\,d\lambda\in L(X)\,,\tag{1}$$

$$P(s)=\frac{1}{2\pi i}\int_{\Gamma}R(\lambda,s)\,d\lambda\in L(X)\,.\tag{2}$$

As a consequence of 4.2 $\mathcal{O}(\Gamma)$ can be chosen such that both $T(s)$ and $P(s)$ are analytic on $\mathcal{O}(\Gamma)$. By 2 – (5) the spectrum of $R(\cdot,s)$ and that of $T(s)$ coincide inside Γ. The same is true of all the corresponding spectral projections and eigenvectors. It is, therefore, sufficient to consider the family $T(s)$.

Since λ_0 is an isolated eigenvalue with a finite dimensional root space for $T(0)$, the identity (1), (2) implies that both $T(s)$ and $P(s)$ are *J-symmetric* for real s. This together with the fact that $\langle\cdot,\cdot\rangle | P(0)X$ is strictly positive implies

$$T(0)P(0) = P(0)T(0) = \lambda_0 P(0). \tag{3}$$

Since $P(s)$ is analytic, we have $\dim P(s) = m$, $s \in \mathcal{O}(\Gamma)$. On the other hand, there is a neighbourhood $\mathcal{O}' \subsetneqq \mathcal{O}(\Gamma)$ of zero such that $s \in \mathcal{O}'$ implies

$$\|P(s) - P(0)\| < 1.$$

Then, as it is well known, the following implications hold

$$\begin{aligned} &(x = P(0)x,\ P(s)x = 0) \Rightarrow x = 0 \quad \forall s \in \mathcal{O}' \\ &(x = P(s)x,\ P(0)x = 0) \Rightarrow x = 0 \quad \forall s \in \mathcal{O}'. \end{aligned} \tag{4}$$

Hence the identity

$$(T(s) - \lambda)x = 0, \qquad x \in X$$

is equivalent to

$$P(0)(T(s)P(s) - \lambda P(s))y = 0, \qquad x = P(s)y, \tag{5}$$

where y is uniquely given by $x = P(s)y$. The last equation defines a generalised eigenvalue problem in the space $P(0)X$. It can be written as

$$(A(s) - \lambda B(s))y = 0 \tag{6}$$

where

$$A(s) = P(0)T(s)P(s) | P(0)X, \tag{7}$$

$$B(s) = P(0)P(s) | P(0)X, \qquad A(s),\ B(s) \in L(P(0)X). \tag{8}$$

We also have

$$\left.\begin{aligned} \langle A(s)x, y\rangle &= \langle x, A(s)x\rangle \quad (9) \\ \langle B(s)x, y\rangle &= \langle x, B(s)y\rangle \quad (10) \end{aligned}\right\} \quad s \text{ real},$$

$$B(0) = 1, \qquad A(0) = \lambda_0, \tag{11}$$

where $A(s)$, $B(s)$ are analytic for $s \in \mathcal{O}'$.

It is well known that all that (see, e.g., [3], p. 416) implies the existence of a neighbourhood $\mathcal{O} \subsetneqq \mathcal{O}'$ of $s = 0$ and of the analytic functions $\lambda_i(s)$, $y_{ij}(s)$ (scalar, vector-valued) $i = 1, \ldots, t, j = 1, \ldots, n_i$, $n_1 + \cdots + n_t = m$; $s \in \mathcal{O}$ such that

$$(A(s) - \lambda_i(s)B(s))y_{ij}(s) = 0, \tag{12}$$

$$\langle B(s)y_{ij}(s), y_{kn}(s)\rangle = \delta_{ik}\delta_{jn}, \qquad s \text{ real}, \tag{13}$$

$$\lambda_i(s) \neq \lambda_j(s), \qquad j \neq i, \qquad s \neq 0. \tag{14}$$

Moreover, $\lambda_i(s)$ are real for real s and $y_{ij}(0)$ can be chosen $\langle\cdot,\cdot\rangle$-orthogonal.

Thus the analytic functions $\Phi_{ij}(s) = P(s)\, y_{ij}(s)$ satisfy

$$\langle \Phi_{ij}(s), \Phi_{kn}(s) \rangle = \delta_{ik}\delta_{jn}, \qquad s \text{ real},$$

and

$$R(\lambda, s)\,\Phi_{ij}(s) = R(\lambda, s)\,P(s)\,\Phi_{ij}(s) = R(\lambda, T(s))\,\Phi_{ij}(s) = \Phi_{ij}(s)/(\lambda - \lambda_j(s)).$$

This proves the statements a), b), c), d), e). Since $\Phi_{ij}(s)$ are analytic, $\mathcal{O}$ can be chosen in such a way that Φ_{ij} are linearly independent for $s \in \mathcal{O}$. This completes the proof.

5. The Dirac Operator

Take $X = [L_2(R^3)]^4$, i.e., as a Hilbert space of equivalence classes of all measurable spinor functions

$$\tilde{\psi} = \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \\ \tilde{\psi}_3 \\ \tilde{\psi}_4 \end{pmatrix}$$

on R^3, which are square integrable with the usual scalar product. The Dirac operator is formally given as

$$\tilde{T}'(s) = -\frac{i\hbar}{s}\alpha\nabla + \frac{\beta m}{s^2} + \tilde{V} = \tilde{S}'(s) + \tilde{V}, \tag{1}$$

where

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right),$$

$$\alpha = (\alpha_1, \alpha_2, \alpha_3), \qquad \alpha\nabla = \alpha_1\frac{\partial}{\partial x_1} + \alpha_2\frac{\partial}{\partial x_2} + \alpha_3\frac{\partial}{\partial x_3}.$$

Here s is the inverse of the velocity of light, α_i, β are Dirac matrices, and $\tilde{V}$ is the multiplication by a real measurable function $V(x)$.

It is well known that the operator $\tilde{S}'(s)$, as defined on $[C_0^\infty]^4$ has a unique selfadjoint extension $\tilde{S}(s)$. It is a result of Prosser [4] that under the condition

$$\int |V(x)|^p\, dx < \infty \qquad 3 < p < \infty \tag{2}$$

the operator V is $\tilde{S}$-bounded with the relative bound less than 1, which guarantees the existence and the uniqueness of the selfadjoint extension $\tilde{T}$ of $\tilde{T}$ [3].

[3] Recently J. C. Guillot bas obtained the same result under the condition

$$\sup_{x \in R^3} \int_{|x-y|<1} \frac{|V(y)|^2}{|x-y|^\nu}\, dy < \infty, \qquad \nu > 1.$$

The author is indebted to Dr. J. C. Guillot for kindly communicating his result.

By the Fourier transformation

$$\psi(p)=\frac{1}{(2\pi\hbar)^{3/2}}\int e^{-ipx/\hbar}\,\tilde{\psi}(x)\,dx\,, \tag{3}$$

the operator S' is going into

$$[S'(s)\,\psi]\,(p)=\frac{\alpha p}{s}\,\psi(p)+\frac{\beta m}{s^2}\,\psi(p)\,,\qquad \alpha p=\alpha_1 p_1+\alpha_2 p_2+\alpha_3 p_3\,. \tag{4}$$

For a real $s>0$ the resolvent of the operator $S'(s)-m/s^2$ is

$$[R(\lambda,s)\,\psi]\,(p)=\frac{s\alpha p+(\beta+1)\,m+\lambda s^2}{\lambda^2 s^2+2m\lambda-p^2}\,\psi(p)\,. \tag{5}$$

It is obvious that for any $\lambda\notin[0,\infty]$ there is a complex neighbourhood $\mathcal{O}_\lambda$ of $s=0$ such that for $s\in\mathcal{O}_\lambda$ the expression (5) defines a bounded operator. Thus, even for complex $s\in\mathcal{O}_\lambda$ formula (4) defines a closable operator on $[C_0^\infty]^4$ with a range which is dense in X, whose closure is denoted by $S(s)-m/s^2$ and whose resolvent is (5). Note that (5) makes sense even for $s=0$ when

$$R(\lambda,0)=\frac{\beta+1}{2}\left(\lambda-\frac{p^2}{2m}\right)^{-1}=(\lambda-S_0)^{-1}\,\frac{\beta+1}{2}\,, \tag{6}$$

where S_0 is the restriction of $p^2/2m$ on $\dfrac{\beta+1}{2}X$.

Thus, at least formally, the family of resolvents $R(\lambda,s)$ has a limit when $s\to 0$. This limit is a pseudoresolvent expressed as a product of the projection $(\beta+1)/2$ and the resolvent of a Schrödinger operator in the space $\mathcal{R}\,((\beta+1)/2)$.

5.1. Proposition. *The family* $R(\cdot,s)$ *defined in* (5) *is analytic at* $s=0$.

Proof. Take $\lambda\notin[0,\infty)$. The assertion follows from the identity

$$\begin{aligned}&[R(\lambda,s)\,\psi]\,(p)\\&=\{(s\alpha p+(\beta+1)\,m+\lambda s^2]\,(2m\lambda-p^2)^{-1}\,(1+\lambda^2 s^2(2m\lambda-p^2)^{-1})^{-1}\,\psi\}\,(p)\,,\end{aligned} \tag{7}$$

and Definition 4.1. Q.E.D.

5.2. Theorem *Let* $\lambda\notin[0,\infty)$ *and let* $V(\cdot)$ *be such that* $1-VR(\lambda,0)$ *is regular in* $L(X)$ *and that* $s\mapsto VR(\lambda,s)\in L(X)$ *is analytic at* $s=0$. *Then the family of pseudoresolvents* $Q(\cdot,s)$ *defined by Theorem 3.1 is analytic at zero. Moreover*

$$Q(\lambda,s)=R(\lambda;T(s)-m/s^2)\,,\qquad s\neq 0\,, \tag{8}$$

$$Q(\lambda,0)=R(\lambda;T_0)\,P\,, \tag{9}$$

where $T(s)=S(s)+V, \quad T_0=S_0+V|PX, \quad P=(\beta+1)/2$

and $R(\lambda; A)$ *denotes the resolvent of an operator* A.

Proof. It follows from Theorem 4.4 that $Q(\cdot, s)$ is analytic at $s=0$. The identity (8) is obvious, while (9) follows from

$$\begin{aligned} Q(\lambda,0) &= R(\lambda,0)\,(1-VR(\lambda,0))^{-1} \\ &= R(\lambda,0)\,P(1-VR(\lambda;S_0)\,P)^{-1} = R(\lambda;T_0)\,P\,, \end{aligned}$$

where we have used the fact that $VR(\lambda;S_0)\,P$ commutes with P. Q.E.D.

5.3. Corollary. *Let* V *be as in Theorem 5.2 and let* T_0 *have an isolated eigenvalue* $\lambda_0 \neq 0$ *of finite multiplicity. Then all assertions of Theorem 4.5 are true for* $J=1$ *and the numbers* $\lambda_i(s)$ *are eigenvalues for* $T(s)-m/s^2$.

Obviously, any bounded V will satisfy the conditions of Theorem 5.2. A result for relatively bounded potentials is given by the following theorem.

5.4. Theorem. *Let* V *be such that* $VR(\lambda_0, s_0) \in L(X)$ *for some* $s_0>0$ *and some* $\lambda_0<0$ *and that* $1-VR(\lambda_0,0)$ *is regular in* $L(X)$. *Then the function* $s \mapsto VR(\lambda, s)$ *is analytic for* $|s|<s_0$ *and* $1-VR(\lambda_0, s)$ *is a regular element in* $L(X)$ *for* s *from some neighbourhood* $\mathcal{O}$ *of* $s=0$.

Proof. We have

$$\begin{aligned} VR(\lambda_0, s) &= VR(\lambda_0, s_0)\,(\lambda - T(s_0) + m/s_0^2)\,R(\lambda_0, s) \\ &= VR(\lambda_0, s_0)\,(\lambda - T(s_0) + m/s_0^2)\,\frac{s\alpha p + (\beta+1)\,m + \lambda_0 s^2}{\lambda_0^2 s^2 + 2m\lambda_0 - p^2}\,. \end{aligned}$$

Since $\lambda_0<0$ is in the resolvent set of $R(\cdot, s_0)$, we have

$$\lambda_0 \in (-2m/s_0^2, 0)\,,$$

which, together with $|s| \leqq s_0$, implies

$$\frac{\lambda_0^2\,|s|^2}{2m|\lambda_0|+p^2} \leqq \frac{|\lambda_0|\,s_0^2}{2m} < 1\,. \tag{10}$$

Now, (7) implies that the family

$$\alpha p\,R(\lambda_0, s)$$

is analytic in s for $|s|<s_0$. The same, therefore, is true of $VR(\lambda_0, s)$. The remaining part of the assertion is obvious. Q.E.D.

5.5. Remark. Completely analogous results follow for the family $T(s)+m/s^2$. The "limiting" operator in this case is

$$(-S_0 + V|(1-P)\,X)\,(1-P)\,.$$

6. The Klein-Gordon Operator

In [7] we considered the Klein-Gordon operator which can be given in the form

$$T'(s) = S'(s) + V,$$

where

$$[S'(s)\,\psi](p) = \frac{m}{s^2}\begin{pmatrix} 0 & \mu(p,s)^2 \\ 1 & 0 \end{pmatrix}\psi(p), \tag{1}$$

$$(V\psi)(p) = V(p)_* \, \psi(p), \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{2}$$

$$\mu(p,s) = (1 + p^2 s^2/m^2)^{1/2}. \tag{3}$$

The scalar product is given by

$$(\psi, \phi)_s = \int \frac{\psi_1(p)\,\overline{\phi_1(p)}}{\mu(p,s)} + \mu(p,s)\,\psi_2(p)\,\overline{(\phi_2(p))}\,dp, \tag{4}$$

where $p \in R^3$ and dp is the Lebesgue measure.

In [7] we proved that the closure of $T'(s)$ is a scalar type operator with a real spectrum in a Hilbert space with the scalar product $(\cdot,\cdot)_s$.

The difficulty arising here is that the scalar product varies with s. While all scalar products for real $s \neq 0$ are mutually topologically equivalent, this ceases to be the case when $s = 0$.

Thus, we fix an arbitrary $s_0 > 0$ and consider the Hilbert space $X(s_0)$ of pairs of functions with the scalar product (4). As a linear topological space, $X(s_0)$ is not varying with s_0 for $s_0 > 0$. Obviously, $\mathcal{S}^2 \subseteqq X(s_0)$, where $\mathcal{S}^2$ denotes all pairs of rapidly decreasing smooth Schwartz functions.

The expression (1) defines a linear operator from $\mathcal{S}^2$ to $\mathcal{S}^2$. This operator is a restriction of a closed operator $S(s_0)$ given by the same formula on the domain

$$D = \{\psi; \int [\mu(p,s_0)\,|\psi_1(p)|^2 + \mu(p,s_0)^3\,|\psi_2(p)|^2]\,dp < \infty\}. \tag{5}$$

It is obvious that $S(s_0)$ is selfadjoint in $X(s_0)$. The expression (1) defines a linear operator from $\mathcal{S}^2$ to $\mathcal{S}^2$ even for complex $s \neq 0$.

For $\lambda \notin [0, \infty)$ and s from some complex neighbourhood $\mathcal{O}_\lambda$ of zero, the inverse of the operator $\lambda - S'(s) + m/s^2$ is given by

$$[R(\lambda,s)\,\psi](p) = \begin{pmatrix} \lambda s^2 + m & m\mu(p,s)^2 \\ m & \lambda s^2 + m \end{pmatrix} \frac{\psi(p)}{\lambda^2 s^2 + 2\lambda m - p^2}, \tag{6}$$

which is bounded in $X(s_0)$.

Since $R(\lambda, s)$ maps $\mathcal{S}^2$ into $\mathcal{S}^2$, it is the resolvent of a closed operator $S(s) - m/s^2$ which is a closure of $S'(s) - m/s^2$.

On the other hand, (6) makes sense even for $s=0$, when we have

$$R(\lambda, 0) = R(\lambda; S_0)\, P\,, \tag{7}$$

where

$$R(\lambda; S_0) = (\lambda - S_0)^{-1}\,, \tag{8}$$

$$[S_0 \psi](p) = \frac{p^2}{2m}\, \psi(p)\,, \qquad \psi = P\psi\,, \qquad P = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{9}$$

6.1. Proposition. *The family $s \mapsto R(\lambda, s)$ is analytic at zero.*

Proof. As in 5 – (7), we write

$$[R(\lambda, s)\,\psi](s)$$
$$= \begin{pmatrix} \lambda s^2 + m & m + p^2 s^2/m \\ m & \lambda s^2 + m \end{pmatrix} (2\lambda m - p^2)^{-1}\,(1 - \lambda^2 s^2 (2\lambda m - p^2)^{-1})^{-1}\, \psi(p)\,.$$

Since the operators

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (2\lambda m - p^2)^{-1}\,,$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} (m + p^2 s^2/m)\,(2\lambda m - p^2)^{-1}\,,$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} (2\lambda m - p^2)^{-1}\,,$$

$$(1 - \lambda^2 s^2 (2\lambda m - p^2)^{-1})^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

are bounded in $X(s_0)$ for $\lambda \neq [0, \infty)$ and s from some complex neighbourhood $\mathcal{O}_\lambda$ of zero, it follows that $R(\cdot, s)$ is analytic at $s=0$. Q.E.D.

6.2. Theorem. *Let $\lambda \neq [0, \infty)$ and V be such that $1 - VR(\lambda, 0)$ is regular in $L(X(s_0))$, and that $s \mapsto VR(\lambda, s) \in L(X(s_0))$ is analytic in some neighbourhood of zero. Then the family of pseudoresolvents $Q(\cdot, s)$ defined by 3 – (2) is analytic at zero. Moreover*

$$Q(\lambda, s) = R(\lambda; T(s) - m/s^2), \;\; s \neq c\,, \tag{10}$$

$$Q(\lambda, 0) = R(\lambda; T_0)\, P\,, \tag{11}$$

$$T_0 = S_0 + V \,|\, PX(s_0)\,, \qquad T_0 \text{ closed in } P(X(s_0))\,. \tag{12}$$

Proof. The proof follows the lines of that given in Theorem 5.2 and will not be repeated here.

6.3. Corollary. *Let V be as Theorem 6.2 and let $\sigma(T_0)$ have an isolated point $\lambda_0 \neq 0$ such that the corresponding spectral projection P_0 has a finite range. Then all assertions of Theorem 4.5 are true with*

$$J = \begin{pmatrix} 0 & \mu(p, s_0)^{-1/2} \\ \mu(p, s_0)^{1/2} & 0 \end{pmatrix} = J^* = J^{-1},$$

and $\lambda_i(s)$ are eigenvalues of the operator $T(s) - m/s^2$.

Proof. It is easy to see that the operators $T(s)$ are J-symmetric such that

$$J Q(\lambda, s) J = Q(\lambda, s)^*, \qquad s \text{ real}.$$

Since $\lambda_0 \neq 0$, we have $P_0 P = P P_0 = P_0$. Thus, to apply Theorem 4.5, it is sufficient to show that the restriction of the form $\langle \cdot, \cdot \rangle = (J \cdot, \cdot)_{s_0}$ on the subspace $PX(s_0)$ is strictly positive.

Since

$$\langle \psi, \phi \rangle = \int (\psi_1 \bar{\phi}_2 + \psi_2 \bar{\phi}_1)\, dp$$

and $P\psi = \psi$ implies

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_1 \end{pmatrix},$$

we have

$$\langle \psi_1, \psi_1 \rangle = \int 2 |\psi_1(s)|^2\, dp,$$

which is strictly positive. Q.E.D.

6.4. Remark. As a result of [7], V will satisfy the conditions of Theorem 6.2 if, for example,

$$\int |V(p)| \, (1 + |p|)^{1/2}\, dp < \infty.$$

6.5. Remark. As in the case of the Dirac operator, analogous results will be obtained for the family $T(s) + m/s^2$.

We see that there is no complete analogy between the Dirac and the Klein-Gordon case. This is a consequence of the fact that the operator $T_0 = S_0 + V | PX(s_0)$ in (12) is not the usual Schrödinger operator, because it is not selfadjoint in $X(s_0)$ [4].

It is of some help in this case to consider the whole process in the Hilbert space X defined by the scalar product

$$(\psi, \phi)_0 = \int (\psi_1 \bar{\phi}_1 + \psi_2 \bar{\phi}_2)\, dp.$$

Then $R(\lambda, s)$, defined as in (6), will again represent a family of pseudo-resolvents analytic at $s = 0$, and the statements 6.1, 6.2, 6.3, 6.5 will be true provided that V is $(\cdot, \cdot)_0$-bounded, for example.

[4] However, the isolated eigenvalues will be the same under certain conditions on V.

In this case, however, a new pathology arises. The operators $S(s)$ are not of scalar type in X! This is seen from the expression for the projection belonging to the positive spectrum of $S(s)$, $s > 0$

$$(P_+\psi)(p) = \frac{1}{(1+\mu(p,s))^{1/2}} \begin{pmatrix} \mu(p,s) & \mu(p,s) \\ 1 & 1 \end{pmatrix} \psi(p),$$

which is bounded in $X(s_0)$ but unbounded in X.

There is another possible procedure. We can consider that $(\cdot,\cdot)_s$, $T(s)$ changes "simultaneously". However, in this case the analyticity in $1/c$ is lost. A kind of strong convergence of resolvents can still be proved (see [7]).

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