

# Unitary Renormalization of $[\varphi^4]_{2+1}$ \*

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**Abstract.** The two-space dimensional  $\varphi^4$  interaction is renormalized by unitary transformation. A sequence of unitary operators is defined which transform a sequence of cut-off Hamiltonians, arranged in order of increasing cut-off energy, to a sequence of operators converging strongly on a dense set of states. The proof is outlined, calculations leading to required  $L^2$  estimates on the kernels of a finite number of diagrams are not here detailed.

In this paper the two-space dimensional  $\varphi^4$  theory is treated by the methods initiated in references [1] and [2]. This model has been renormalized by Glimm [3, 4] and essentially the present treatment, using unitary transformation, is a “minimal modification” of the method of Glimm giving rise to a unitary transformation. Arranging a sequence of cut-off Hamiltonians in order of increasing cut-off energy, we will construct a sequence of unitary operators, such that the transformed Hamiltonians converge strongly on a dense set of states. It is expected that the sequence of unitary operators do not converge, and give rise to the same representation of the field operators as the transformation of Glimm [5]. We also collect some estimates for the norm of polynomials in the field operators restricted to states with given numbers of particles; particularly obtaining expressions that are useful when the polynomials are momentum conserving. It is expected that these estimates will be useful in future work; here they replace estimates on the kernels of arbitrarily complicated graphs in Glimm’s procedure.

We start by presenting estimates on the norms of certain operator expressions. Let

$$O = \sum_{k_i, p_j} K(k_1, \dots, p_s) a_{k_1}^* \dots a_{k_R}^* \dots a_{p_1} \dots a_{p_s} \quad (1)$$

be an operator constructed from the annihilation and creation operators of boson and/or fermion fields. (The number of fermion operators among the creation and annihilation operators is assumed the same in each term.) There is a sum over momenta as the fields are constructed in a

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box of period 1 in each spatial dimension. Let  $|n\rangle$  denote a state of norm one and containing  $n$  particles.

*Estimate I.* This is the now classical estimate:

$$\|O|n\rangle\| \leq c|K|_2(n+1)^{R+S/2}. \quad (2)$$

*Estimate II.* Assume there is at most one particle allowed in any mode. This is automatic for fermions, for bosons it involves implicitly inserting projection operators onto the states with this property in all expressions.

$$\|O|n\rangle\| \leq c|K|_1. \quad (3)$$

*Estimate III.* Make the one particle per mode assumption of estimate II and in addition assume  $O$  is momentum preserving.

$$\|O|n\rangle\| \leq c|K|_2(n+1)^{R+S-1/2}. \quad (4)$$

*Estimate IV*<sup>1</sup>. Assume  $O$  is momentum preserving and write the kernel  $K(k_1, \dots, p_s)$  as  $K = \sum_l K_l$  where  $K_l$  is zero except when

$$k_1 + k_2 + \dots + k_R = l$$

then

$$\|O|n\rangle\| \leq C \operatorname{Sup}_l |K_l|_2(n+1)^{R+S/2}. \quad (5)$$

*Estimate V.* Assume  $O$  is momentum preserving. We assume  $R > S$  and  $O$  contains an even number of fermion operators. Let

$$f(n) \equiv \operatorname{Sup}_{(n)} \|O|n\rangle\|^2$$

Then:

$$f(n) = \operatorname{Sup}_{(n)} \|O^* O|n\rangle\|$$

Now:

$$O^* O|n\rangle = (O^* O - O O^*)|n\rangle + O O^*|n\rangle. \quad (6)$$

Now using

$$\operatorname{Sup}_{(n)} \|O O^*|n\rangle\| = \operatorname{Sup}_{(n-R+S)} \|O^* O|(n-R+S)\rangle\| \quad (7)$$

we obtain Estimate V:

$$\begin{aligned} \operatorname{Sup}_{(n)} \|O|n\rangle\|^2 &\leq \operatorname{Sup}_{(n)} \|(O^* O - O O^*)|n\rangle\| \\ &\quad + \operatorname{Sup}_{(n-R+S)} \|O|(n-R+S)\rangle\|^2 \end{aligned} \quad (8)$$

or

$$f(n) \leq \operatorname{Sup}_{(n)} \|(O^* O - O O^*)|n\rangle\| + f(n-R+S). \quad (9)$$

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<sup>1</sup> Private communication from A. Jaffe and J. Cannon.

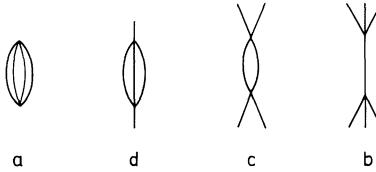


Fig. 1

To see that this estimate can improve the earlier estimates we consider the example

$$O = \sum K(k_1, \dots, k_4) a_{k_1}^* a_{k_2}^* a_{k_3}^* a_{k_4}^*. \quad (10)$$

Then (9) becomes

$$f(n) \leq f(n-4) + a + b(n+1)^3 + c(n+1)^2 + d(n+1) \quad (11)$$

where  $a, b, c, d$  are some constants times the  $L_2$  norms of the kernels of the contractions in  $O^* O - O O^*$  represented by a), b), c), and d) in Fig. 1.

Then we deduce from (11) that

$$f(n) \leq l[a(n+1) + b(n+1)^4 + c(n+1)^3 + d(n+1)^2] \quad (12)$$

for some constant  $l$ .

This can be compared to estimate I

$$f(n) \leq l' a(n+1)^4. \quad (13)$$

In the case of interest  $a$  will be much larger than  $b, c$ , or  $d$  – due to momentum conservation – and (12) will be a much better estimate than (13) due to the lower power of  $(n+1)$  multiplying  $a$ . This example illustrates the point of estimate V; the difference between (12) and (13) is enough to make the following construction of the unitary operators work. Estimate V was designed to replace by an operator language the estimates of (Eq. (4. 159)) in Ref. [5].

Turning now to the model Hamiltonian in two-space dimensions in a box of period one.

$$\begin{aligned} H &= H_0 + V \\ H_0 &= \int :[\frac{1}{2}\pi^2 + \frac{1}{2}M^2\varphi^2 + \frac{1}{2}(\nabla\varphi)^2]: dx \\ &= \sum \omega_k a_k^* a_k \\ V &= g \int :\varphi^4: dx. \end{aligned} \quad (14)$$

Corresponding to a cut-off momentum  $K$ , all  $a_p, a_p^*$  with  $|p| > K$  are discarded and counterterms  $\Delta_K$  (see Ref. [4]) added

$$\begin{aligned} H_K &= H_{oK} + V_K + \Delta_K \\ \Delta_K &= \Delta_K^{(2)} + \Delta_K^{(3)} + \frac{1}{2}(\delta M_K^{(2)}) \int : \varphi_K^2 : dx. \end{aligned} \quad (15)$$

We pick a sequence  $K_n$  with

$$K_n = n^\alpha \quad (16)$$

$\alpha$  to be later chosen. We desire to construct a sequence of unitary operators  $U_n$  such that

$$U_n^{-1} H_{K_n} U_n |\varphi\rangle \xrightarrow{\text{strongly}} \bar{H} |\varphi\rangle \quad (17)$$

for  $|\varphi\rangle$  the dense set of states with finite number of particles and momentum components in a bounded region,  $\bar{H}$  a symmetric operator with this domain.

$U_n$  is constructed as the product

$$U_n = e^{-A_n} e^{-A_{n-1}} \dots e^{-A_1}. \quad (18)$$

Define  $(V_n - V_{n-1})$  to be those terms in the sum of normal ordered products comprising  $V$  such that if the momenta in such a term  $k_1, k_2, k_3, k_4$  are arranged in order  $|k_1| \geq |k_2| \geq |k_3| \geq |k_4|$  then

$$K_{n-1} < |k_1| \leq K_n. \quad (19)$$

We define the operation  $\Gamma$  as in [2]. For example:

$$\begin{aligned} \Gamma a_{p_1}^* a_{p_2}^* a_{p_3} a_{p_4} &= \frac{1}{\omega_{p_1} + \omega_{p_2}} a_{p_1}^* a_{p_2}^* a_{p_3} a_{p_4} && \text{if } \omega_{p_1} + \omega_{p_2} > \omega_{p_3} + \omega_{p_4} \\ &= \frac{-1}{\omega_{p_3} + \omega_{p_4}} a_{p_1}^* a_{p_2}^* a_{p_3} a_{p_4} && \text{if } \omega_{p_3} + \omega_{p_4} > \omega_{p_1} + \omega_{p_2} \\ &= 0 && \text{otherwise.} \end{aligned} \quad (20)$$

Let  $P(a)$  be the projection onto states with  $a$  or less particles. Then

$$A_n = P(n^2) \Gamma (V_n - V_{n-1}) P(n^2) + B_n = C_n + B_n. \quad (21)$$

We specify  $B_n$  indirectly.

$$B_n = P(n^2) (\Gamma D_n) P(n^2) \quad (22)$$

$D_n$  consists of terms of the following six types

- a) a product of four creation operators,
- b) a product of four annihilation operators,
- c) a product of six creation operators,
- d) a product of six annihilation operators,
- e) a product of two annihilation operators,
- f) a product of two creation operators

with the maximum momentum in any term satisfying

$$K_{n-1} < |k_{\max}| \leq K_n$$

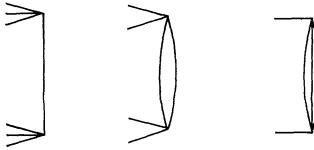


Fig. 2

and with coefficients chosen so that in the expansion  $U_n^{-1} H_{K_n} U_n$  the terms described by Fig. 2 are cancelled on states with few particles, between contributions from  $B_n$  and the  $A_n$ . The exact requirement will be clarified later.

If  $\alpha$  is picked large enough then Eq. (17) holds.

We outline the verification of (17). We begin by expanding  $U_n^{-1} H_{K_n} U_n$ . To simplify the expression we introduce the following notational points.

- 1) Write  $H_n, H_{on}, \delta M_n^2$  etc. for  $H_{K_n}, H_{oK_n}, \delta M_{K_n}^2$  etc.
- 2) The indices in all sums range from one to  $n$ , possibly subject to further indicated limitations.
- 3) Write  $(E, F), (E, F, G), (E, F, G, H)$  for  $[E, F], [E, [F, G]], [E, [F, [G, H]]]$ .
- 4) On commutators we allow five subscripts:  $T, F, NF, NT$ , and  $NFT$  such as  $(E, G)_{NF}$ , with the following meanings:

T) In the indicated commutator first discard all projection operators  $P(i^2)$  appearing in the  $B_i$  and  $C_i$  and in the normal ordered expansion of the commutator keep only terms with two creation and annihilation operators; terms with  $a^* a^*$ ,  $aa$ , and  $a^* a$  operator expressions. (In diagram language – there are two external lines.)

F) In the indicated commutator first discard all projection operators  $P(i^2)$  appearing in the  $B_i$  and  $C_i$  and in the normal ordered expansion of the commutator keep only terms with no creation and annihilation operators. (In diagram language – there are no external lines.)

$$NF) (E, G) = (E, G)_F + (E, G)_{NF},$$

$$NT) (E, G) = (E, G)_T + (E, G)_{NT},$$

$$NFT) (E, G) = (E, G)_T + (E, G)_F + (E, G)_{NFT}.$$

- 5) On a function of the variable  $t$  we use the operation  $l, (l \circ f)(t) = \int_0^t f(t) dt$  and write  $l^s(1) \circ f(t) = (l \circ \dots \circ l \circ f)(1)$ .

We now expand:

$$\begin{aligned} U_n^{-1} H_n U_n &= U_n^{-1} [H_{on} + V_n + A_n^{(2)} + A_n^{(3)} + \frac{1}{2} \delta M_n^{(2)} \int : \varphi_n^2 : dx] U_n \\ &= I + II + \dots + XII \end{aligned} \quad (23)$$

with:

$$\text{I} = H_{on}, \quad (24)$$

$$\text{II} = \sum_i (C_i, H_o) + V_n, \quad (25)$$

$$\text{III} = A_n^{(2)} + \sum_i (C_i, V_n)_F + \frac{1}{2} \sum_i (C_i, C_i, H_0)_F, \quad (26)$$

$$\begin{aligned} \text{IV} &= A_n^{(3)} + \left( \sum_{i=j} \frac{1}{2} + \sum_{j < i} \right) (C_j, C_i, V_n)_F \\ &\quad + \left( \sum_{i=j=l} \frac{1}{3!} + \sum_{l < i=j} \frac{1}{2!} + \sum_{l=j < i} \frac{1}{2!} + \sum_{l < j < i} \right) . \\ &\quad (C_l, C_j, C_i, H_0)_F + \sum_i (B_i, V_n)_F + \left( \sum_{i=j} \frac{1}{2!} + \sum_{j < i} \right) . \\ &\quad ((C_j, B_i, H_0)_F + (B_j, C_i, H_0)_F), \end{aligned} \quad (27)$$

$$\begin{aligned} \text{V} &= \frac{1}{2} \delta M_n^{(2)} \int : \varphi_n^2 : dx + \sum_i (B_i, H_0)_T \\ &\quad + \sum_i (C_i, V_n)_T + \sum_i \frac{1}{2} (C_i, C_i, H_0)_T, \end{aligned} \quad (28)$$

$$\text{VI} = \sum_i (B_i, H_0)_{NT} + \sum_j \left( C_j, \left[ V_n + \left( \sum_{i=j} \frac{1}{2} + \sum_{i>j} \right) (C_i, H_0) \right] \right)_{NFT}, \quad (29)$$

$$\begin{aligned} \text{VII} &= l^2(1) \circ \sum_i t^2 U_{i-1}^{-1} e^{A_it} (B_i, B_i, H_0) e^{-A_it} U_{i-1} \\ &\quad + l(1) \circ \sum_{j < i} t U_{j-1}^{-1} e^{A_jt} (B_j, B_i, H_0) e^{-A_jt} U_{j-1}, \end{aligned} \quad (30)$$

$$\begin{aligned} \text{VIII} &= l^2(1) \circ \sum_i t^2 U_{i-1}^{-1} e^{A_it} (C_i, B_i, H_0)_{NF} e^{-A_it} U_{i-1} \\ &\quad + l(1) \circ \sum_{j < i} t U_{j-1}^{-1} e^{A_jt} (C_j, B_i, H_0)_{NF} e^{-A_jt} U_{j-1}, \end{aligned} \quad (31)$$

$$\begin{aligned} \text{IX} &= l^2(1) \circ \sum_i t^2 U_{i-1}^{-1} e^{A_it} (B_i, C_i, H_0)_{NF} e^{-A_it} U_{i-1} \\ &\quad + l(1) \circ \sum_j t U_{j-1}^{-1} e^{A_jt} \left( B_j, \left[ \sum_{i>j} (C_i, H_0) + V_n \right] \right)_{NF} e^{-A_jt} U_{j-1}, \end{aligned} \quad (32)$$

$$\begin{aligned} \text{X} &= l(1) \circ \sum_l t U_{l-1}^{-1} e^{A_lt} \left( (B+C)_l, \left[ \frac{1}{2} \delta M_n^{(2)} \int : \varphi_n^2 : dx + \sum_{j>l} (C_j, V_n)_T \right. \right. \\ &\quad \left. \left. + \sum_{j>l} \frac{1}{2} (C_j, C_j, H_0)_T \right] \right) e^{-A_lt} U_{l-1}, \end{aligned} \quad (33)$$

$$\begin{aligned} \text{XI} &= l^3(1) \circ \sum_l t^3 U_{l-1}^{-1} e^{A_lt} [(C_l, C_l, H_0)_{NF} + (B_l, C_l, C_l, H_0)] e^{-A_lt} U_{l-1} \\ &\quad + l(1) \circ \sum_l t U_{l-1}^{-1} e^{A_lt} \sum_{j>l} \left[ \frac{1}{2} (C_l, (C_j, C_j, H_0)_{NT})_{NF} \right. \\ &\quad \left. + \frac{1}{2} (B_l, (C_j, C_j, H_0)_{NT}) \right] e^{-A_lt} U_{l-1}, \end{aligned} \quad (34)$$

$$\begin{aligned} \text{XII} &= \left( l^2(1) \circ \sum_{l=j} t^2 + l(1) \circ \sum_{l < j} t \right) U_{l-1} e^{A_lt} \\ &\quad \cdot \left[ \left( B_l, \left( C_j, \left[ V_n + \sum_{i>j} (C_i, H_0) \right] \right)_{NT} \right) \right. \\ &\quad \left. + \left( C_l, \left( C_j, \left[ V_n + \sum_{i>j} (C_i, H_0) \right] \right)_{NT} \right)_{NF} \right] e^{-A_lt} U_{l-1}. \end{aligned} \quad (35)$$

We now consider the convergence of the different terms. I clearly converges for vectors  $\varphi$  in  $\mathcal{D}$ , in fact  $H_{0,n}|\varphi\rangle = H_0|\varphi\rangle$  for  $n$  large enough. III and IV may by choice of  $A_n^{(2)}$  and  $A_n^{(3)}$  be picked identically zero. The two terms in II approximately cancel and their sum converges on vectors in  $\mathcal{D}$ . For the annihilation momentum in a compact subset of momentum space, the  $L_2$  norm of the creation momenta of the kernel of this sum is uniformly bounded. The approximate cancellation only occurs where the projection operators in  $C_i$  are 1 on the state considered. For  $\varphi$  in  $\mathcal{D}$  there is some maximum number of particles in the state so that for  $i$  large enough the  $P(i^2)$  are 1 on the state and the approximate cancellation occurs.).

In V the  $B_i$  may be picked so that no terms in  $a^*a^*$  or  $aa$  occur, and  $\delta M_n^{(2)}$  picked so that the coefficient of  $a_0^*a_0$  is zero. The remaining terms  $\Sigma a_k a_k^* a_k$  are fine on  $\varphi$  in  $\mathcal{D}$ . In VI the  $B_i$  are picked to make the terms in four creation and six creation operators to exactly cancel (if the projection operators are 1 on the state in question). The remaining terms in VI are helped along by the II-like approximate cancellation of the term in brackets.

In terms VII through XII terms individually converge except that in the second term of IX and both terms in XII the II-like approximate cancellation between the terms in square brackets is required, and in X the approximate cancellations are required between the three terms in the brackets.

We consider the second term in VII as an example. We write  $(B_j, B_i, H_0) = (B_j, B_i, H_0)_C + (B_j, B_i, H_0)_D$  where the first term indicates the connected part of the commutator. It is obtained by omitting all the  $P(i^2)$  terms in the  $B_i$  – omitting all projection operators. The other term may contain disconnected diagrams due to commutators between the projection operators and creation and annihilation operators. But note  $(B_j, B_i, H_0) = (B_j, B_i, H_0)_C$  acting on states with few particles, explicitly in this case for states with fewer than  $(j^2 - 12)$  particles.

Considering first the connected term. Suppose we deal with a state  $\varphi$  with fewer than  $r$  particles; then  $e^{-A_{jt}} U_{j-1} |\varphi\rangle$  contains fewer than  $(j^2 + r)$  particles. In certain other terms, but not here, it is necessary to observe that the momenta in  $e^{-A_{jt}} U_{j-1} |\varphi\rangle$  lie in a certain region of phase space. Let  $(B_j, B_i, H_0)_C^{(5,5)}$  be the term in the normal ordered expansion of the commutator containing five creation and five annihilation operators (for an explicit example) and  $K_{ij}$  the kernel of this term. Then on  $\varphi$  we see we have  $| (B_j, B_i, H_0)_C^{(5,5)} | \leq |K_{ij}|_2 (j^2 + r + 1)^5$ . Since  $U_{j-1}$ ,  $U_{j-1}^{-1}$ ,  $e^{-A_{jt}}$ , and  $e^{A_{jt}}$  are unitary and  $l(1) \circ t$  has norm 1 we get

$$\begin{aligned} & | l(1) \circ \sum_{j < i} t U_{j-1}^{-1} e^{A_{jt}} (B_j, B_i, H_0)_C^{(5,5)} e^{-A_{jt}} U_{j-1} |\varphi\rangle | \\ & \leq \sum_{j < i} |K_{ij}|_2 (j^2 + r + 1)^5 \cdot |\varphi\rangle . \end{aligned} \tag{36}$$

It is easy to see the convergence of this sum as  $n \rightarrow \infty$  implies the strong convergence of this connected part of the second term in VII on  $|\varphi\rangle$ . If  $\alpha$  is picked large enough the sum in (36) converges for all  $r$ .

Considering now the disconnected terms. Since  $U_{j-1}|\varphi\rangle$  has less than or equal to  $(j-1)^2$  particles for  $j$  large and  $(B_j, B_i, H_0) = (B_j, B_i, H_0)_C$  on states with fewer than  $(j^2 - 12)$  particles in the expansion of  $e^{-A_j t}$  only terms with at least  $\left(\frac{(j^2 - 12) - (j-1)^2}{6}\right)$  powers of  $A_j$  contribute here. Thus disconnected terms involve high powers of some  $A_j$ . If  $O$  is self adjoint

$$\begin{aligned} & \left| e^{iO} B e^{-iO} - B - i(O, B) - \frac{i^2}{2!} (O, O, B) \dots \frac{-i^{k-1}}{(k-1)!} \underset{k-1 \text{ terms}}{(O, \dots, O, B)} \right| \\ &= |l^k(1) \circ t^k e^{itO} (O, \dots, O, B) e^{-itO}| \leq \frac{1}{k!} 2^k |B| \cdot |O|^k \end{aligned} \quad (37)$$

since the norm of  $l^k(1)t^k$  is less than or equal  $1/k!$ .

Using estimates IV and V we have  $|A_n| \leq \lambda \sqrt[n]{n}$ . From which follows:

$$\sum_{n=1}^{\infty} \frac{1}{(n/4)!} \|P(n^2) V_n\| \cdot \|A_n\|^{n/4} < \infty \quad (38)$$

replacing (Eq. (4.153)), Ref. [5]. Estimate I would give  $\|A_n\| \leq \lambda n^{7/2}$  insufficient to give (38). Eq. (38) easily implies the convergence of disconnected terms.

This discussion reduces the complete proof of Eq. (17) to the study of the kernels of a *finite* number of finite commutators to check the analog of (36).

## References

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