Commun. math. Phys. 21, 164—170 (1971)

A Remark on Yukawa Plus Boson Selfinteraction in Two Space Time Dimensions*

ROBERT SCHRADER

Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts

Received October 20, 1970

Abstract. In this note we show how the results of Glimm and Jaffe [5, 6] on the Yukawa quantum field theory in two space time dimensions may be extended to the case where a boson selfinteraction term is added. Local fields are constructed which do not depend on any cut-off and which have the right (anti)-commutation properties for spacelike separated support of the test functions.

1. Introduction

Throughout this paper we shall employ the notations and definitions used in [7] to discuss the Yukawa interaction in two space dimensions. Let

$$H^Y(g,\varkappa) = H_0 + H_I(g,\varkappa) + c(g,\varkappa) \tag{1.1}$$

be the cutoff Hamilton operator describing the Yukawa interaction between a boson field φ and a fermion field ψ. g is a nonnegative space cutoff function and $\varkappa$ a momentum cutoff parameter. $c(g,\varkappa)$ is a renormalization counterterm. Set

$$:P(\varphi)(g): = \sum_{r=0}^{2n} a_r \int :\varphi(x)^{2n}: g(x)\,dx \tag{1.2}$$

where all a_r are real and $a_{2n} > 0$. Define

$$H(g,\varkappa) = H^Y(g,\varkappa) + :P(\varphi)(g): \tag{1.3}$$

Note that $:P(\varphi)(g):$ has only a space cutoff and no momentum cutoff. We shall be considering the family $H(g,\varkappa)$ as $\varkappa \to \infty$. It will be proved that the resolvents of $H(g,\varkappa)$ converge to the resolvent of a selfadjoint Hamilton operator $H(g)$, which locally gives rise to a finite propagation speed.

Further let

$$\psi(f) = \int e^{itH(g)}\,\psi(x)\,e^{-itH(g)} f(x,t)\,dx\,dt\,, \tag{1.4a}$$

$$\varphi(f) = \int e^{itH(g)}\,\varphi(x)\,e^{-itH(g)} f(x,t)\,dx\,dt\ (f \in C_0^\infty(\mathbb{R}^2)) \tag{1.4b}$$

---

* Supported in part by the Air Force Office of Scientific Research, Contract F 44620-70-C-0030.

be the smeared out field operators. Then $\psi(f)$ is a bounded operator and $\varphi(f)$ (f real) a selfadjoint operator. $\psi(f)$ and $\varphi(f)$ will be independent of g, if $g=1$ on a sufficiently large set. Finally these fields are shown to have the right (anti-)commutation properties for space like separation of the support of the test functions.

2. First and Second Order Estimates

The results of [2, 3, 9, 10] easily show the following

Theorem 2.1. *For each $\varkappa < \infty$, $H(g, \varkappa)$ is selfadjoint and e.s.a. (essentially selfadjoint) on $C^\infty(H_0)$. In fact any core for $(H_0)^n$ is a core for $H(g, \varkappa)$. Also*

$$N_\tau \leqq \text{const}(H(g, \varkappa) + \text{const}) \qquad \tau < 1 \tag{2.1}$$

with constants independent of $\varkappa$.

Hence we may choose $c(g, \varkappa)$ in such a way that

$$\inf \text{spectrum}\, H(g, \varkappa) = 0\,. \tag{2.2}$$

Also using (2.1) we may establish pull-through fourmulas similar to those given in [5] and [10]. This will give

Theorem 2.2. *For $\tau < \frac{1}{2}$ the following quadratic N_τ estimate holds*

$$N_\tau^2 \leqq \text{const}\,(H(g, \varkappa) + \text{const})^2 \tag{2.3}$$

with constants independent of $\varkappa$.

To prove inequality (2.3), the estimate

$$N_\tau^2 \leqq \text{const}(H^Y(g, \varkappa) + \text{const})^2 \qquad (\tau < \tfrac{1}{2}) \tag{2.4}$$

with constants independent of $\varkappa$, is necessary. The estimate (2.4) however, was established by Dimock [1].

The proof of Theorem 2.2 is quite long, but since no new techniques are needed, no details will be presented here.

3. The Local Hamiltonian

The estimates (2.1) and (2.3) permit the proof of the following

Theorem 3.1. *As $\varkappa \to \infty$ the resolvents $R_\varkappa(\zeta)$ of $H(g, \varkappa)$ converge uniformly to the resolvent $R(\zeta)$ of a selfadjoint nonnegative operator $H(g)$. Zero is an isolated eigenvalue of $H(g)$.*

Indeed we have

$$R_\varkappa(\zeta) - R_{\varkappa'}(\zeta) = -R_\varkappa(\zeta)\,\delta H\,R_{\varkappa'}(\zeta) \tag{3.1}$$

where

$$\begin{aligned}\delta H &= H(g,\varkappa) - H(g,\varkappa') \\ &= H^Y(g,\varkappa) - H^Y(g,\varkappa')\end{aligned} \tag{3.2}$$

and hence, using the pull through formulas, essentially the same arguments may be used as in the discussion of $H^Y(g,\varkappa)$ [5–7]. Also

$$N_{2\tau} \leqq \text{const}\,(H(g) + \text{const}) \qquad \tau < \tfrac{1}{2}, \tag{3.3}$$

$$(N_\tau)^2 \leqq \text{const}\,(H(g) + \text{const})^2\,. \tag{3.4}$$

Let $\mathscr{A}_f(B)$ (resp. $\mathscr{A}(B)$) be the time zero field algebra (resp. observable algebra) of [7], where B is a bounded open set in $\mathbb{R}$. Also let B_t be the set of points in R with distance less than $|t|$ from B.

Theorem 3.2. *For $g=1$ on B_t, the map $A \to A(t)$ with*

$$A(t) = e^{itH(g)}\,A\,e^{-itH(g)}$$

defines a g-independent automorphism

$$\sigma_t : \mathscr{A}_f(B) \to \mathscr{A}_f(B_t)\,,$$

$$\sigma_t : \mathscr{A}(B) \to \mathscr{A}(B)\,,$$

i.e. $H(g)$ satisfies the finite propagation speed property.

The proof is essentially as in [6], since $H_0 + {:}P(\varphi)\,(g){:}$ satisfies the finite propagation speed property [9].

4. The Field Operators

The operators $\psi(f)$ in (1.4a) make no real difficulties since they are fermion operators. They are bounded and independent of g, if $g=1$ on a sufficiently large set due to Theorem 3.2. Now let $f \in C_0^\infty(\mathbb{R}^2)$ be real and put

$$\varphi_g(f_t) = \int e^{itH(g)}\,\varphi(x)\,e^{-itH(g)}\,f(x,t)\,dx\,. \tag{4.1}$$

Using the estimates (3.3) and (3.4) and arguments of [4, 5], we have the

Theorem 4.1. *$\varphi_g(f_t)$ is a selfadjoint operator, e.s.a. on $D((H(g)+1)^{\frac{1}{2}})$ and for $\Theta \in D((H(g)+1)^{\frac{1}{2}})$, $\varphi_g(f_t)\,\Theta$ is strongly continuous in t, with the estimates*

$$\|\varphi_g(f_t)\,\Theta\| \leqq \text{const}\,\|f(\cdot,t)\|_2\,\|(H(g)+1)^{\frac{1}{2}}\,\Theta\|\,, \tag{4.2}$$

$$\|\varphi_g(f_t)^2\,\Theta\| \leqq \text{const}\,\|f(\cdot,t)\|_2^2\,\|(H(g)+1)\,\Theta\|\,. \tag{4.3}$$

Due to this theorem, the integral

$$\varphi_g(f)\,\Theta = \int \varphi_g(f_t)\,\Theta\,dt$$

exists and may be used to define $\varphi_g(f)$ as a closed symmetric operator

$$\varphi_g(f) = \left(\varphi_g(f)|_{D(H(g)+1)^{\frac{1}{2}})}\right)^{-} \tag{4.4}$$

and we have the

Theorem 4.2. *$\varphi_g(f)$ is selfadjoint and hence e.s.a. on any core of $(H(g)+1)^{\frac{1}{2}}$ since*

$$\|\varphi_g(f)\,\Theta\| \leqq \text{const} \int \|f(\cdot, t)\|_2\,dt\,\|(H(g)+1)^{\frac{1}{2}}\,\Theta\|\,. \tag{4.5}$$

For $g=1$ on a sufficiently large set depending on supp$f$, *$\varphi_g(f)$ is actually independent of g.*

The first part of the theorem may be proved as in [4]. For the second part, which is also a new result in the case of ordinary Yukawa coupling, we need two well known lemmata, which we state for convenience (see e.g. [8], pp. 429 and 432).

Lemma 4.3. *Let T_n and T be selfadjoint operators in a Hilbert-space and let D be a core of T such that $T_n u \to Tu$ as $n\to\infty$ for each $u \in D$. Then the resolvents of T_n converge strongly to the resolvent of T.*

Lemma 4.4. *Let the hypotheses be as in Lemma 4.3 and let $E_n(\lambda)$ resp. $E(\lambda)$ be the spectral family of T_n resp. T. Then*

$$\begin{aligned} &s - \lim E_n(\lambda') = E(\lambda)\\ &\lambda' \to \lambda\\ &n \to \infty \end{aligned} \tag{4.6}$$

if $E(\lambda) = E(\lambda - 0)$.

We want to apply these lemmata in the following way. Define the selfadjoint operators

$$\varphi(f_t) = \int \varphi(x)\,f(x,t)\,dx \tag{4.7}$$

such that $\varphi(f_t)$ are all e.s.a. on $D\left((H(g)+1)^{\frac{1}{2}}\right)$ and

$$\varphi_g(f_t) = e^{itH(g)}\,\varphi(f_t)\,e^{-itH(g)} \tag{4.8}$$

Let $E(\lambda, f_t)$ be the spectral family of $\varphi(f_t)$. Then due to the local properties of $H(g)$

$$E^t(\lambda, f_t) = e^{itH(g)}\,E(\lambda, f_t)\,e^{-itH(g)} \tag{4.9}$$

does not depend on g, if $g=1$ on a sufficiently large set $B = B(f)$. Also using the canonical commutation relations in Weyl form of the time zero boson field and its canonically conjugate field, it is easy to see that

$$E(\lambda, f_t) = E(\lambda - 0, f_t)$$

if $\varphi(f_t) \neq 0$ or $\lambda \neq 0$, i.e. $\varphi(f_t)$ is either zero or has no discrete eigenvalues. Therefore the above lemmata give

$$\underset{\substack{t' \to t \\ \lambda' \to \lambda}}{s - \lim} E(\lambda', f_{t'}) = E(\lambda, f_t)$$

if $\varphi(f_t) \neq 0$ or $\lambda \neq 0$. In particular

$$C_n(f) = \int_{|\lambda| \leqq n} \lambda \, dE^t(\lambda, f_t) \, dt \tag{4.10}$$

is a well defined bounded selfadjoint operator independent of g. We will show that the resolvents of $C_n(f)$ converge strongly to the resolvent of $\varphi_g(f)$. This will show that $\varphi_g(f)$ is actually independent of g. To prove the strong convergence of the resolvents, by the first part of the theorem and Lemma 4.3, it is sufficient to prove

$$s - \lim_{n \to \infty} C_n(f) u = \varphi_g(f) u$$

for $u \in D\big((H(g) + 1)\big)$. We have

$$\|(\varphi_g(f) - C_n(f) u)\| = \left\| \int e^{itH(g)} \left(\varphi(f_t) - \int_{|\lambda| \leqq n} \lambda \, dE(\lambda, f_t)\right) e^{-itH(g)} \, dt \, u \right\|$$

$$\leqq \int \left\| \int_{|\lambda| \geqq n} \lambda dE(\lambda, f_t) (\varphi(f_t)^2 + 1)^{-1} (\varphi(f_t)^2 + 1) \, (H(g) + 1)^{-1} (H(g) + 1) e^{-itH(g)} \, u \right\| dt$$

$$\leqq \sup_t \left\| \int_{|\lambda| \geqq n} \frac{\lambda}{\lambda^2 + 1} dE(\lambda, f_t) \right\| \int_{|t| \leqq T} (\varphi(f_t)^2 + 1) \, (H(g) + 1)^{-1} \| \, dt \, \|(H(g) + 1) \, u\|$$

where T is chosen so large that $f(\cdot, t) = 0$ for $|t| \geqq T$. Now

$$\sup_t \left\| \int_{|\lambda| \geqq n} \frac{\lambda}{\lambda^2 + 1} dE(\lambda, f_t) \right\| \leqq \frac{1}{n}$$

and

$$\int_{|t| \leqq T} \|(\varphi(f_t)^2 + 1) \, (H(g) + 1)^{-1}\| \, dt < \infty$$

by (4.3) so

$$\|(\varphi_g(f) - C_n(f)) \, u\| \leqq \frac{\text{const}}{n} \|(H(g) + 1) \, u\| \tag{4.11}$$

Letting $n \to \infty$ proves Theorem 4.2.

Remark 4.5. Inspection of the proof shows all that is needed is the validity of the two conditions

(a) first and second order N_τ estimates,

(b) finite propagation speed property of $H(g)$.

Therefore Theorem 4.2 will hold any time these conditions are satisfied.

We now write $\varphi(f)$ instead of $\varphi_g(f)$.

Theorem 4.6. *Let $f_i (i=1,2)$ be in $C_0^\infty(\mathbb{R}^2)$ such that f_1 and f_2 are real and have space-like separated support. Then $\varphi(f_1)$ commutes with $\varphi(f_2)$, and $\psi(f_2)$. Also $\psi(f_1)$ anticommutes with $\psi(f_2)$.*

The proof may be taken over from [4]. It also follows easily from the proof of Theorem 4.2, indeed we have e.g.

$$[C_n(f_1), C_m(f_2)] = 0 \tag{4.12}$$

for all n, m by construction and hence

$$e^{i\tau_1 C_n(f_1)} e^{i\tau_2 C_m(f_2)} = e^{i\tau_2 C_m(f_2)} e^{i\tau_1 C_m(f_1)} \tag{4.13}$$

for all n, m; $\tau_1, \tau_2 \in \mathbb{R}$.

Now due to the convergence of the resolvents

$$\operatorname*{s-lim}_{n\to\infty} e^{i\tau_1 C_n(f_1)} = e^{i\tau_1 \varphi(f_1)}$$

$$\operatorname*{s-lim}_{m\to\infty} e^{i\tau_2 C_m(f_2)} = e^{i\tau_2 \varphi(f_2)}$$

uniformly for τ_1, τ_2 in bounded sets of $\mathbb{R}$ (see e.g. [8] p. 502). Hence taking the strong limit of (4.13) proves

$$e^{i\tau_1 \varphi(f_1)} e^{i\tau_2 \varphi(f_2)} = e^{i\tau_2 \varphi(f_2)} e^{i\tau_1 \varphi(f_1)} \tag{4.14}$$

for all τ_1, τ_2, i.e. the spectral families of $\varphi(f_1)$ and $\varphi(f_2)$ commute. The other cases may be discussed in a similar way.

Acknowledgements. It is a pleasure to thank J. Dimock and Prof. A. Jaffe for helpful remarks.

References

1. Dimock, J.: Field equations for the Y_2 theory. Harvard University thesis. To appear.
2. Glimm, J.: The Yukawa coupling of quantum fields in two dimensions. Commun. math. Phys. **6**, 61–76 (1967).
3. — Boson fields with nonlinear selfinteraction in two dimensions. Commun. math. Phys. **8**, 18–28 (1967).
4. — Jaffe, A.: A $\lambda\Phi^4$ quantum field theory without cutoffs II. Ann. Math. **91**, 362–401 (1970).

5. — — Selfadjointness of the Y_2 Hamiltonian. Ann. Phys. **60**, 321—383 (1970).
6. — — The Yukawa$_2$ quantum field theory without cutoffs. To appear.
7. — — "Field Theory Models". Lecture notes delivered at the Summer School in Les Houches 1970. To appear.
8. Kato, T.: Perturbation theory of linear operators. Berlin-Heidelberg-New York: Springer 1966.
9. Rosen, L.: A $\lambda(\Phi^{2n})_2$ field theory without cutoffs. Commun. math. Phys. **16**, 157–183 (1970).
10. — The $(\Phi^{2n})_2$ quantum field theory. Higher order estimates. Preprint.

Robert Schrader
Lyman Laboratory of Physics
Harvard University
Cambridge, Mass. 02138