

Irreducible Lie Algebra Extensions of the Poincaré Algebra

II. Extensions with Arbitrary Kernels

U. CATTANEO

University of Louvain, Centre de Physique Nucléaire, Heverlee/Belgium

Received July 24, 1970

Abstract. We analyse the extensions of the Poincaré algebra \mathcal{P} with arbitrary kernels. The main tool is a reduction theorem which generalizes the Hochschild-Serre theorem for $n = 2$. This reduction theorem is proved and used to investigate the structure of the Lie algebras obtained by extension.

We look particularly for the irreducible and \mathcal{R} -irreducible extensions of \mathcal{P} and we classify the types of irreducible extensions with arbitrary kernels.

Introduction

We pursue here the analysis of the irreducible extensions of the Poincaré algebra begun in [1]. In this II. Part we concentrate on the more complicated problem of extensions with arbitrary kernels.

The difficulties in the non-abelian case have their roots in the fact that the Chevalley-Eilenberg cohomology can not be directly used. As a consequence the set of extensions with fixed character of a given Lie algebra by a non-abelian Lie algebra can also be empty. But something of the Chevalley-Eilenberg cohomology subsists also if the kernel is non-abelian: we have a pseudocohomology (cohomology in Calabi's sense [2]) which allows us to generalize the results in [1]. This pseudocohomology is defined only if $n = 2$, since it is intimately related to the extension theory of Lie algebras. Then we are able to generalize the Hochschild-Serre theorem for $n = 2$ to a reduction theorem valid also in the non-abelian case.

Starting from this result it is possible to develop an extension theory of the Poincaré algebra with arbitrary kernels.

We introduce in Section I the ideas of prerepresentation and pseudocohomology [2]. We show how they are linked to the theory of Lie algebra kernels [2–5].

The preinessential extensions, which form the bridge between the extensions with arbitrary kernels and those with abelian kernels, are considered in Section II.

In Section III we prove the important reduction theorem which generalizes the Hochschild-Serre theorem for $n = 2$.

In Section IV the structure of the Lie algebras obtained by extending \mathcal{P} with arbitrary kernels is analysed.

The irreducible and \mathcal{R} -irreducible extensions of \mathcal{P} are studied in Section V. Furthermore, we classify the types of irreducible extensions of \mathcal{P} .

Some Conventions

The conventions of [1] are again used in this paper.

Particularly we recall that all Lie algebras, modules and vector spaces considered are of finite dimension over a field F of characteristic 0. This statement is tacitly understood throughout the paper. We use also the following new symbols:

$D(\mathcal{G})$: derivation Lie algebra of \mathcal{G} ;

$I(\mathcal{G})$: ideal of $D(\mathcal{G})$ consisting of the inner derivations of \mathcal{G} ;

$\Delta(\mathcal{G}) = D(\mathcal{G})/I(\mathcal{G})$;

$\mathcal{C}(\mathcal{G})$: center of \mathcal{G} ;

$\text{Hom}(\mathcal{G}, \mathcal{V})$: vector space of the Lie algebra homomorphisms of \mathcal{G} into \mathcal{V} ;

$i_{\mathcal{V}}$: natural monomorphism of $\text{Hom}(D(\mathcal{V}), D(\mathcal{C}(\mathcal{V})))$;

$j_{\mathcal{V}}$: natural monomorphism of $\text{Hom}(D(\mathcal{V}), \Delta(\mathcal{C}(\mathcal{V})))$;

$\Pi_{\mathcal{V}}$: canonical epimorphism $D(\mathcal{V}) \rightarrow \Delta(\mathcal{V})$.

I. Lie Algebra Kernels and Extensions with Arbitrary Kernels

I.1. Pseudocohomology of Degree 2 of Lie Algebras

Let \mathcal{G} and \mathcal{V} be Lie algebras and $h_2 \in A_2(G, V)$. A *prerepresentation* φ of \mathcal{G} into $D(\mathcal{V})$ associated with h_2 is a linear map $\varphi: \mathcal{G} \rightarrow D(\mathcal{V})$ such that

$$\varphi(g)\varphi(g') - \varphi(g')\varphi(g) = \varphi([g, g']) + \text{ad } h_2(g, g') \quad \forall g, g' \in \mathcal{G}.$$

If it is not important to specify the alternating map $h_2 \in A_2(G, V)$, to which the prerepresentation φ is associated, we say briefly that we have a prerepresentation $\varphi: \mathcal{G} \rightarrow D(\mathcal{V})$ (of \mathcal{G} into $D(\mathcal{V})$).

Given a prerepresentation $\varphi: \mathcal{G} \rightarrow D(\mathcal{V})$ we have a unique linear map $\Phi: \mathcal{G} \rightarrow \Delta(\mathcal{V})$ such that the following diagram is commutative:

$$\begin{array}{ccc} & \mathcal{G} & \\ \varphi \swarrow & \downarrow \Phi & \\ D(\mathcal{V}) & \xrightarrow{\Pi_{\mathcal{V}}} & \Delta(\mathcal{V}) \end{array} \quad \text{i.e. } \Pi_{\mathcal{V}} \circ \varphi = \Phi.$$

It follows immediately that $\Phi \in \text{Hom}(\mathcal{G}, \Delta(\mathcal{V}))$.

Conversely: each $\Phi \in \text{Hom}(\mathcal{G}, \Delta(\mathcal{V}))$ can be lifted to a prerepresentation $\varphi: \mathcal{G} \rightarrow D(\mathcal{V})$ by the canonical epimorphism $\Pi_{\mathcal{V}}$. We call φ a prerepresentation lifted over Φ .

Let $\varphi: \mathcal{G} \rightarrow D(\mathcal{V})$ be a prerepresentation. A linear operator

$$\delta_n(\varphi): A_n(G, V) \rightarrow A_{n+1}(G, V) \quad (n \in \mathbf{N})$$

is defined through

$$\begin{aligned} (\delta_n(\varphi)f_n)(g_1, \dots, g_{n+1}) &= \sum_{i=1}^{n+1} (-1)^{i+1} \varphi(g_i) f_n(g_1, \dots, \hat{g}_i, \dots, g_{n+1}) \\ &\quad + \sum_{j < k} (-1)^{j+k} f_n([g_j, g_k], g_1, \dots, \hat{g}_j, \dots, \hat{g}_k, \dots, g_{n+1}) \\ &\quad \forall (f_n \in A_n(G, V); g_1, \dots, g_{n+1} \in \mathcal{G}; n \in \mathbf{N}^+) \end{aligned} \quad (\text{I.1})$$

and

$$(\delta_0(\varphi)f_0)(g) = \varphi(g)f_0 \quad \forall (f_0 \in \mathcal{V}; g \in \mathcal{G}).$$

Then, given $\Phi \in \text{Hom}(\mathcal{G}, \Delta(\mathcal{V}))$, the following definitions are suitable:

$$\mathfrak{C}_{\Phi}^2(\mathcal{G}, \mathcal{V}) = \left\{ (\varphi, f_2) \left| \begin{array}{l} f_2 \in A_2(G, V); \varphi: \mathcal{G} \rightarrow D(\mathcal{V}) \text{ prerepresentation} \\ \text{associated with } f_2 \text{ and lifted over } \Phi \end{array} \right. \right\}$$

$$\mathfrak{Z}_{\Phi}^2(\mathcal{G}, \mathcal{V}) = \{ (\varphi, f_2) \mid (\varphi, f_2) \in \mathfrak{C}_{\Phi}^2(\mathcal{G}, \mathcal{V}); \delta_2(\varphi)f_2 = 0 \}.$$

The elements of $\mathfrak{C}_{\Phi}^2(\mathcal{G}, \mathcal{V})$ are called $(2; \mathcal{G}, \mathcal{V}, \Phi)$ -pseudocochains and the elements of $\mathfrak{Z}_{\Phi}^2(\mathcal{G}, \mathcal{V})$ are called $(2; \mathcal{G}, \mathcal{V}, \Phi)$ -pseudococycles.

We have an equivalence relation between $(2; \mathcal{G}, \mathcal{V}, \Phi)$ -pseudococycles which amounts to a generalization of the equivalence relation between cocycles of degree 2 in the Chevalley-Eilenberg cohomology.

Consider the $(2; \mathcal{G}, \mathcal{V}, \Phi)$ -pseudococycles (φ, f_2) and (φ', f'_2) . These two elements are set to be equivalent if there exists $f_1 \in L(G, V)$ such that:

$$\text{R} \left\{ \begin{array}{l} \varphi'(g) = \varphi(g) + \text{ad } f_1(g) \\ f'_2(g_1, g_2) = f_2(g_1, g_2) + (\delta_1(\varphi)f_1)(g_1, g_2) + [f_1(g_1), f_1(g_2)], \end{array} \right. \quad (\text{I.2})$$

$$\forall g, g_1, g_2 \in \mathcal{G}$$

where $[\cdot, \cdot]$ is the Lie product in \mathcal{V} .

One easily verifies that R is actually an equivalence relation between elements of $\mathfrak{Z}_{\Phi}^2(\mathcal{G}, \mathcal{V})$. Symbolically we write: $(\varphi, f_2) \sim (\varphi', f'_2) \pmod{\text{R}}$.

We note that, given a $(2; \mathcal{G}, \mathcal{V}, \Phi)$ -pseudococycle (φ, f_2) and any $f_1 \in L(G, V)$, the $(2; \mathcal{G}, \mathcal{V}, \Phi)$ -pseudocochain (φ', f'_2) obtained according to (I.2) is also a $(2; \mathcal{G}, \mathcal{V}, \Phi)$ -pseudococycle and obviously

$$(\varphi, f_2) \sim (\varphi', f'_2) \pmod{\text{R}}.$$

At this stage we can define the set of equivalence classes of $\mathfrak{Z}_{\Phi}^2(\mathcal{G}, \mathcal{V})$ with respect to the relation R:

$$\mathfrak{H}_{\Phi}^2(\mathcal{G}, \mathcal{V}) = \mathfrak{Z}_{\Phi}^2(\mathcal{G}, \mathcal{V}) / \text{R}.$$

The set $\mathfrak{H}_{\Phi}^2(\mathcal{G}, \mathcal{V})$ is referred to as the set of pseudocohomology of degree 2 of \mathcal{G} over \mathcal{V} associated with Φ .

A few comments are now necessary:

(1) When \mathcal{V} is an abelian Lie algebra $\text{D}(\mathcal{V}) \approx \Delta(\mathcal{V})$; so we can identify Φ and φ lifted over Φ with a representation $\mathcal{G} \rightarrow \text{End}_{\mathbf{F}}(V)$, which we will call even Φ . We have then the natural bijections

$$\mathfrak{C}_{\Phi}^2(\mathcal{G}, \mathcal{V}) \approx C^2(\mathcal{G}, V_{\Phi}),$$

$$\mathfrak{Z}_{\Phi}^2(\mathcal{G}, \mathcal{V}) \approx Z^2(\mathcal{G}, V_{\Phi})$$

and

$$\mathfrak{H}_{\Phi}^2(\mathcal{G}, \mathcal{V}) \approx H^2(\mathcal{G}, V_{\Phi}).$$

All the sets here considered carry a canonical structure of vector spaces.

(2) As $\mathcal{C}(\mathcal{V})$ is a characteristic ideal of \mathcal{V} we define $\psi = i_{\mathcal{V}} \circ \varphi$ and $\Psi = j_{\mathcal{V}} \circ \varphi$ (Ψ is independent of the choice of φ lifted over Φ).

$$\psi \in \text{Hom}(\mathcal{G}, \text{D}(\mathcal{C}(\mathcal{V})))$$

is lifted over $\Psi \in \text{Hom}(\mathcal{G}, \Delta(\mathcal{C}(\mathcal{V})))$; therefore we can define $\mathfrak{C}_{\Psi}^2(\mathcal{G}, \mathcal{C}(\mathcal{V}))$, $\mathfrak{Z}_{\Psi}^2(\mathcal{G}, \mathcal{C}(\mathcal{V}))$ and $\mathfrak{H}_{\Psi}^2(\mathcal{G}, \mathcal{C}(\mathcal{V}))$. According to (1), there exists a bijection of these sets into, respectively, $C^2(\mathcal{G}, C(\mathcal{V})_{\Psi})$, $Z^2(\mathcal{G}, C(\mathcal{V})_{\Psi})$ and $H^2(\mathcal{G}, C(\mathcal{V})_{\Psi})$.

1.2. Lie Algebra Extensions with Arbitrary Kernels

Let (\mathcal{E}, ϱ) be an extension of \mathcal{B} by \mathcal{A} [1, 6]. Consider the homomorphism $\text{ad}_{\mathcal{A}}: \mathcal{E}(\mathcal{B}, \mathcal{A}) \rightarrow \text{D}(\mathcal{A})$ defined by:

$$(\text{ad}_{\mathcal{A}} e)a = [e, a] \quad e \in \mathcal{E}, \quad \forall a \in \mathcal{A}.$$

The restriction $\text{ad}_{\mathcal{A}}|_{\mathcal{A}} = \text{ad}$ is then the epimorphism

$$\text{ad}: \mathcal{A} \twoheadrightarrow \text{I}(\mathcal{A}).$$

By passing to the quotients we obtain $\Phi \in \text{Hom}(\mathcal{B}, \Delta(\mathcal{A}))$ referred to as the *character* of the extension (\mathcal{E}, ϱ) .

$\text{ext}(\mathcal{B}, \mathcal{A}, \Phi)$ will stand for the set of equivalence classes of extensions of \mathcal{B} by \mathcal{A} with character Φ .

We choose a representative element (\mathcal{E}, ϱ) of the class $\{(\mathcal{E}, \varrho)\} \in \text{ext}(\mathcal{B}, \mathcal{A}, \Phi)$ and a section σ of (\mathcal{E}, ϱ) over \mathcal{B} [1]. There exists an element $f_2(b, b') \in A$ such that

$$[\sigma(b), \sigma(b')] = \sigma([b, b']) + f_2(b, b') \quad \forall b, b' \in \mathcal{B}. \quad (\text{I.3})$$

$f_2 \in A_2(B, A)$ is the *factor set* associated with the section σ . The Jacobi identity implies $\delta_2(\varphi)f_2 = 0$, where $\varphi = \text{ad}_{\mathcal{A}} \circ \sigma : \mathcal{B} \rightarrow D(\mathcal{A})$. It follows that $(\varphi, f_2) \in \mathcal{Z}_{\Phi}^2(\mathcal{B}, \mathcal{A})$ since φ is a prerepresentation of \mathcal{B} into $D(\mathcal{A})$ lifted over Φ and associated with f_2 , i.e.

$$\varphi(b)\varphi(b') - \varphi(b')\varphi(b) = \varphi([b, b']) + \text{ad} f_2(b, b') \quad \forall b, b' \in \mathcal{B}.$$

We call φ the *pseudococycle* of (\mathcal{E}, ϱ) determined by σ and (φ, f_2) the $(2; \mathcal{B}, \mathcal{A}, \Phi)$ -pseudococycle associated with (\mathcal{E}, ϱ) by σ .

Taking another section σ' of (\mathcal{E}, ϱ) over \mathcal{B} such that

$$\sigma'(b) - \sigma(b) = f_1(b) \in A \quad \forall b \in \mathcal{B}, \quad (\text{I.4})$$

where $f_1 \in L(B, A)$, we have a $(2; \mathcal{B}, \mathcal{A}, \Phi)$ -pseudococycle (φ', f'_2) associated with (\mathcal{E}, ϱ) by σ' , and

$$\begin{aligned} f'_2(b, b') &= f_2(b, b') + (\delta_1(\varphi)f_1)(b, b') + [f_1(b), f_1(b')] \\ \varphi'(b) &= \varphi(b) + \text{ad} f_1(b), \end{aligned} \quad \forall b, b' \in \mathcal{B} \quad (\text{I.5})$$

i.e. $(\varphi, f_2) \sim (\varphi', f'_2) \pmod{R}$.

Therefore the pseudococycles (φ, f_2) and (φ', f'_2) belong to the same equivalence class of $\mathcal{Z}_{\Phi}^2(\mathcal{B}, \mathcal{A})$.

If conversely there exists $(\varphi, f_2) \in \mathcal{Z}_{\Phi}^2(\mathcal{B}, \mathcal{A})$, we can determine a corresponding extension (\mathcal{E}, ϱ) of \mathcal{B} by \mathcal{A} with character Φ . This is performed following the lines of reference [1]: the elements of E are identified with the couples (b, a) ($b \in \mathcal{B}; a \in \mathcal{A}$) and we define ϱ and σ by $\varrho(b, a) = b$ and $\sigma(b) = (b, 0) \forall (a \in \mathcal{A}; b \in \mathcal{B})$. The Lie algebra product is now defined by the bilinear alternating map $\alpha : E \times E \rightarrow E$ such that:

$$\begin{aligned} \alpha((b_1, a_1), (b_2, a_2)) &= [(b_1, a_1), (b_2, a_2)] \\ &= ([b_1, b_2], [a_1, a_2] + \varphi(b_1)a_2 - \varphi(b_2)a_1 + f_2(b_1, b_2)) \end{aligned} \quad (\text{I.6})$$

$$\forall (b_1, b_2 \in \mathcal{B}; a_1, a_2 \in \mathcal{A}).$$

Then, even in the case of extensions with arbitrary kernels,

$$\text{ext}(\mathcal{B}, \mathcal{A}, \Phi) \approx \mathfrak{H}_{\Phi}^2(\mathcal{B}, \mathcal{A}); \quad (\text{I.7})$$

but, if \mathcal{A} is a non-abelian Lie algebra, the isomorphism is only a set isomorphism, i.e. a bijection. Moreover, for some Φ , $\text{ext}(\mathcal{B}, \mathcal{A}, \Phi)$ and $\mathfrak{H}_{\Phi}^2(\mathcal{B}, \mathcal{A})$ can also be the empty set (see Section I.3).

I.3. The Mori-Hochschild Theory of Lie Algebra Kernels [3, 4]

Let \mathcal{A} and \mathcal{B} be Lie algebras. A \mathcal{B} -kernel [3, 4] is a triple $(\mathcal{B}, \mathcal{A}, \Phi)$, where $\Phi \in \text{Hom}(\mathcal{B}, A(\mathcal{A}))$.

Each \mathcal{B} -kernel $(\mathcal{B}, \mathcal{A}, \Phi)$ induces a \mathcal{B} -kernel $(\mathcal{B}, \mathcal{C}(\mathcal{A}), \Psi)$. This is called the nucleus of $(\mathcal{B}, \mathcal{A}, \Phi)$ and is determined in the following way:

let $\Psi = j_{\mathcal{A}} \circ \varphi : \mathcal{B} \rightarrow \Delta(\mathcal{C}(\mathcal{A})) \approx D(\mathcal{C}(\mathcal{A}))$, where φ is a prerepresentation lifted over Φ . Then $\Psi \in \text{Hom}(\mathcal{B}, \Delta(\mathcal{C}(\mathcal{A})))$ and is independent of the choice of φ .

If (\mathcal{C}, ϱ) is an extension of \mathcal{B} by \mathcal{A} with character Φ , the triple $(\mathcal{B}, \mathcal{A}, \Phi)$ is a \mathcal{B} -kernel. A \mathcal{B} -kernel $(\mathcal{B}, \mathcal{A}, \Phi)$ is said to be extendable whenever there exists an extension (\mathcal{C}, ϱ) of \mathcal{B} by \mathcal{A} with character Φ . Alternatively (\mathcal{C}, ϱ) is said to be an extension of \mathcal{B} with the \mathcal{B} -kernel $(\mathcal{B}, \mathcal{A}, \Phi)$.

Let us consider the set of \mathcal{B} -kernels with the same nucleus $(\mathcal{B}, \mathcal{C}, \Psi)$. Mori [3] and Hochschild [4] introduced in this set an equivalence relation which allows the definition of a vector space structure $K(\mathcal{B}, \mathcal{C}, \Psi)$ on the set of equivalence classes. The extendable \mathcal{B} -kernels with nucleus $(\mathcal{B}, \mathcal{C}, \Psi)$ constitute the null vector of $K(\mathcal{B}, \mathcal{C}, \Psi)$. One can show that there is a vector space monomorphism

$$m : K(\mathcal{B}, \mathcal{C}, \Psi) \rightarrow H^3(\mathcal{B}, C_{\Psi})$$

defined by $m\{(\mathcal{B}, \mathcal{A}, \Phi)\} = \{\delta_2(\varphi)f_2\}$.

$(\mathcal{B}, \mathcal{A}, \Phi)$ is here a representative element of an equivalence class $\{(\mathcal{B}, \mathcal{A}, \Phi)\} \in K(\mathcal{B}, \mathcal{C}, \Psi)$, and $\delta_2(\varphi)f_2$, where $(\varphi, f_2) \in \mathfrak{C}_{\Phi}^2(\mathcal{B}, \mathcal{A})$, is a representative element of an equivalence class $\{\delta_2(\varphi)f_2\}$ of $Z^3(\mathcal{B}, C(\mathcal{A})_{\Psi})$.

In general the monomorphism m is not an isomorphism as in the case of extensions of abstract groups [7–9].

We have indeed [5, 10]

$\text{Im } m = \{0\}$ provided that \mathcal{B} is semisimple (hence all \mathcal{B} -kernels are extendable if \mathcal{B} is semisimple),

$\text{Im } m = H^3(\mathcal{B}, C_{\Psi})$ if \mathcal{B} is solvable.

Only by considering infinite-dimensional Lie algebras \mathcal{A} , we have $K(\mathcal{B}, \mathcal{C}, \Psi) \approx H^3(\mathcal{B}, C_{\Psi})$ for any \mathcal{B} .

$\delta_2(\varphi)f_2$ is called an *obstruction* of the \mathcal{B} -kernel $(\mathcal{B}, \mathcal{A}, \Phi)$ and one uses the notation $\{\delta_2(\varphi)f_2\} = \text{Obs}(\mathcal{B}, \mathcal{A}, \Phi) \in H^3(\mathcal{B}, C_{\Psi})$. It follows [3, 4] that a \mathcal{B} -kernel $(\mathcal{B}, \mathcal{A}, \Phi)$ is extendable if and only if $\text{Obs}(\mathcal{B}, \mathcal{A}, \Phi) = 0$.

It is obvious that, if \mathcal{A} is an abelian Lie algebra, $\text{Obs}(\mathcal{B}, \mathcal{A}, \Phi) = 0$ for any Lie algebra \mathcal{B} and any $\Phi \in \text{Hom}(\mathcal{B}, \Delta(\mathcal{A}))$.

II. Inessential and Preinessential Extensions

We first state the algebraic translation of a theorem proved by Michel [11] in the case of abstract groups.

Theorem 1. *There exists an inessential extension of \mathcal{B} by \mathcal{A} with character Φ if and only if there is a $\varphi \in \text{Hom}(\mathcal{B}, D(\mathcal{A}))$ lifted over Φ .*

Theorem 1 is easily proved by applying the results of I.2 [12].

The existence of $\varphi \in \text{Hom}(\mathcal{B}, \text{D}(\mathcal{A}))$ lifted over Φ allows us to equip A with the \mathcal{B} -module structure A_φ associated with the representation $\varphi: \mathcal{B} \rightarrow \text{D}(\mathcal{A}) \subseteq \text{End}_{\mathbf{F}}(A)$. In this case we have obviously $\text{Obs}(\mathcal{B}, \mathcal{A}, \Phi) = 0$.

Corollary. *The existence of a monomorphism $i \in \text{Hom}(\text{Im}\Phi, \Pi_{\mathcal{A}}^{-1}(\text{Im}\Phi))$ such that $(\Pi_{\mathcal{A}} \circ i)\Phi(b) = \Phi(b) \forall b \in \mathcal{B}$ is a sufficient condition in order to have an inessential extension of \mathcal{B} by \mathcal{A} with character Φ .*

The sufficient condition of this corollary is obviously satisfied if $\Phi = 0$, \mathcal{A} and \mathcal{B} being any Lie algebra, and hence $\text{Obs}(\mathcal{B}, \mathcal{A}, 0) = 0$. The condition is also satisfied, for each $\Phi \in \text{Hom}(\mathcal{B}, \Delta(\mathcal{A}))$ and each Lie algebra \mathcal{B} , if $(\text{D}(\mathcal{A}), \Pi_{\mathcal{A}})$ is an inessential extension of $\Delta(\mathcal{A})$ by $\text{I}(\mathcal{A})$. Therefore, in this case, $\text{Obs}(\mathcal{B}, \mathcal{A}, \Phi) = 0$ for any Lie algebra \mathcal{B} and any $\Phi \in \text{Hom}(\mathcal{B}, \Delta(\mathcal{A}))$.

The following exceptional cases for \mathcal{A} are then particularly interesting:

- i) \mathcal{A} abelian Lie algebra: $\text{D}(\mathcal{A}) \approx \Delta(\mathcal{A})$;
- ii) \mathcal{A} complete Lie algebra: $\text{D}(\mathcal{A}) = \text{I}(\mathcal{A})$, $\mathcal{C}(\mathcal{A}) = \{0\}$ (in particular: \mathcal{A} semisimple).

The inessential extensions belong to a remarkable family of Lie algebra extensions to which also belong the abelian extensions: the family of the preinessential extensions [2].

Definition 1. *We call an extension (\mathcal{E}, ϱ) of \mathcal{B} by \mathcal{A} preinessential if the induced extension $(\mathcal{E}/\mathcal{C}(\mathcal{A}), \varrho_q)$ of \mathcal{B} by $\mathcal{A}/\mathcal{C}(\mathcal{A})$, obtained by passing to the quotient, is inessential.*

If (\mathcal{E}, ϱ) is a preinessential extension, then any extension of the equivalence class $\{(\mathcal{E}, \varrho)\}$ is preinessential. We therefore speak of classes of preinessential extensions.

The following proposition is a straightforward consequence of Definition 1:

Proposition 1. *The extension (\mathcal{E}, ϱ) of \mathcal{B} by \mathcal{A} with character Φ is preinessential if and only if there exists a $(2; \mathcal{B}, \mathcal{A}, \Phi)$ -pseudococycle (φ, f_2) associated with (\mathcal{E}, ϱ) and such that $\text{Im}f_2 \subseteq \mathcal{C}(\mathcal{A})$.*

The following proposition provides another necessary and sufficient condition in order to prove that a given extension is preinessential:

Proposition 2. *The extension (\mathcal{E}, ϱ) of \mathcal{B} by \mathcal{A} with character Φ is preinessential if and only if there exists an inessential extension of \mathcal{B} by \mathcal{A} with character Φ .*

We prove first the following lemma:

Lemma 1. *Let (φ', f'_2) be a $(2; \mathcal{B}, \mathcal{A}, \Phi)$ -pseudococycle and $\varphi: \mathcal{B} \dashrightarrow \text{D}(\mathcal{A})$ a prerepresentation lifted over Φ . Then there exists a $(2; \mathcal{B}, \mathcal{A}, \Phi)$ -pseudococycle $(\varphi, f_2) \sim (\varphi', f'_2) \pmod{\mathbf{R}}$.*

Proof. We consider $\omega = \varphi - \varphi'$. ω is a linear map of \mathcal{B} into $I(\mathcal{A})$, since φ and φ' are lifted over Φ . We choose $f_1 \in L(\mathcal{B}, \mathcal{A})$ such that $\text{ad} \circ f_1 = \omega$ and we define $(\varphi, f_2) \sim (\varphi', f'_2) \pmod{R}$ by the following relations:

$$\begin{aligned} f_2(b, b') &= f'_2(b, b') + (\delta_1(\varphi') f_1)(b, b') + [f_1(b), f_1(b')] \\ \varphi(b) &= \varphi'(b) + \text{ad } f_1(b) = \varphi'(b) + \omega(b) \end{aligned} \quad \forall b, b' \in \mathcal{B}.$$

If we fix a prerepresentation $\varphi: \mathcal{B} \rightarrow D(\mathcal{A})$ lifted over Φ there exists, by Lemma 1, a $(2; \mathcal{B}, \mathcal{A}, \Phi)$ -pseudococycle (φ, f_2) (where f_2 is changeable) in each equivalence class of $\mathfrak{Z}_{\Phi}^2(\mathcal{B}, \mathcal{A})$.

Proof of Proposition 2. Necessity: Let (φ, f_2) be a $(2; \mathcal{B}, \mathcal{A}, \Phi)$ -pseudococycle such that $\text{Im } f_2 \subseteq C(\mathcal{A})$ and associated with the preinessential extension (\mathcal{E}, ϱ) . Then $\varphi \in \text{Hom}(\mathcal{B}, D(\mathcal{A}))$ and an inessential extension of \mathcal{B} by \mathcal{A} with character Φ exists by Theorem 1.

Sufficiency: If an inessential extension of \mathcal{B} by \mathcal{A} with character Φ exists, then there exists also a representation $\varphi \in \text{Hom}(\mathcal{B}, D(\mathcal{A}))$ lifted over Φ . By the preceding Lemma 1: If (\mathcal{E}, ϱ) is any extension of \mathcal{B} by \mathcal{A} with character Φ , there exists an associated $(2; \mathcal{B}, \mathcal{A}, \Phi)$ -pseudococycle (φ, f_2) . Therefore $\text{Im } f_2 \subseteq C(\mathcal{A})$, and (\mathcal{E}, ϱ) is a preinessential extension by Proposition 1.

Corollary. *Whenever there exists a preinessential extension of \mathcal{B} by \mathcal{A} with character Φ , any extension of \mathcal{B} by \mathcal{A} with character Φ is preinessential.*

By Proposition 2 all the extensions of a Lie algebra \mathcal{B} by a complete Lie algebra \mathcal{A} are preinessential. Moreover, they all are trivial as is well known. This is achieved here by the fact that the character Φ of any such extension is 0 and it is possible to choose a representation $\varphi = 0$ lifted over Φ . Lemma 1 states that there exists a $(2; \mathcal{B}, \mathcal{A}, 0)$ -pseudococycle $(0, f_2)$ in each equivalence class of $\mathfrak{Z}_{\Phi}^2(\mathcal{B}, \mathcal{A})$. Since $\text{Im } f_2 \subseteq C(\mathcal{A}) = \{0\}$, we have only trivial extensions. Let φ be a prerepresentation of \mathcal{B} into $D(\mathcal{A})$ lifted over Φ . Then, given a $(2; \mathcal{B}, \mathcal{A}, \Phi)$ -pseudococycle (φ', f'_2) , there exists a $(2; \mathcal{B}, \mathcal{A}, \Phi)$ -pseudococycle $(\varphi, f_2) \sim (\varphi', f'_2) \pmod{R}$. We choose a fixed $(2; \mathcal{B}, \mathcal{A}, \Phi)$ -pseudococycle (φ, g_2) and we consider the set

$$\{(\psi, f_2 - g_2) \mid (\varphi, f_2) \sim (\varphi', f'_2) \pmod{R}; (\varphi', f'_2) \in \mathfrak{Z}_{\Phi}^2(\mathcal{B}, \mathcal{A})\},$$

where $\psi = i_{\mathcal{A}} \circ \varphi$.

It is easily verified [12] that

$$\{(\psi, f_2 - g_2)\} = \mathfrak{Z}_{\Psi}^2(\mathcal{B}, \mathcal{C}(\mathcal{A})) \approx Z^2(\mathcal{B}, C(\mathcal{A})_{\Psi}), \quad (\text{II.1})$$

and that we have the bijections

$$\text{ext}(\mathcal{B}, \mathcal{A}, \Phi) \approx \mathfrak{H}_{\Phi}^2(\mathcal{B}, \mathcal{A}) \approx H^2(\mathcal{B}, C(\mathcal{A})_{\Psi}), \quad (\text{II.2})$$

where $\Psi = j_{\mathcal{A}} \circ \varphi$.

We note that, if no $(2; \mathcal{B}, \mathcal{A}, \Phi)$ -pseudococycle exists (i.e. $\text{Obs}(\mathcal{B}, \mathcal{A}, \Phi) \neq 0$), the set $\{(\psi, f_2 - g_2)\}$ is not defined.

If $\mathcal{C}(\mathcal{A}) = \{0\}$, then $\text{Obs}(\mathcal{B}, \mathcal{A}, \Phi) = 0$ for any Φ and any \mathcal{B} . In this case there exists one and only one equivalence class of extensions of \mathcal{B} with any \mathcal{B} -kernel $(\mathcal{B}, \mathcal{A}, \Phi)$.

III. A Reduction Theorem and the Extensions of the Poincaré Algebra

III.1. The Reduction Theorem

Let \mathcal{G} and \mathcal{V} be Lie algebras and φ a prerepresentation of \mathcal{G} into $D(\mathcal{V})$ associated with h_2 .

We define the following \mathcal{G} -multiplication in $A_n(G, V)$:

$$\begin{aligned} (g(\varphi) \cdot f_n)(g_1, \dots, g_n) &= \varphi(g) f_n(g_1, \dots, g_n) \\ &\quad - \sum_{i=1}^n f_n(g_1, \dots, g_{i-1}, [g, g_i], g_{i+1}, \dots, g_n) \quad (\text{III.1}) \\ &\quad \forall (f_n \in A_n(G, V); g, g_1, \dots, g_n \in \mathcal{G}). \end{aligned}$$

We consider also the linear maps [1]

$$i_n(g): A_n(G, V) \rightarrow A_{n-1}(G, V) \quad \forall (g \in \mathcal{G}; n \in \mathbf{N}^+),$$

such that

$$\begin{aligned} (i_n(g) f_n)(g_1, \dots, g_{n-1}) &= (f_n)_g(g_1, \dots, g_{n-1}) = f_n(g, g_1, \dots, g_{n-1}) \quad (\text{III.2}) \\ &\quad \forall (f_n \in A_n(G, V); g_1, \dots, g_{n-1} \in \mathcal{G}). \end{aligned}$$

If

$$n = 0: i_0(g) f_0 = 0 \quad \forall (g \in \mathcal{G}; f_0 \in V).$$

The following identities are easily proved:

$$\begin{aligned} (g(\varphi) \cdot f_n)_{g'} &= g(\varphi) \cdot (f_n)_{g'} - (f_n)_{[g, g']} \\ &\quad \forall (n \in \mathbf{N}; g, g' \in \mathcal{G}; f_n \in A_n(G, V)) \quad (\text{III.3}) \end{aligned}$$

$$\begin{aligned} (\delta_n(\varphi) f_n)_g &= g(\varphi) \cdot f_n - \delta_{n-1}(\varphi) (f_n)_g \\ &\quad \forall (n \in \mathbf{N}^+; g \in \mathcal{G}; f_n \in A_n(G, V)) \quad (\text{III.4}) \end{aligned}$$

and

$$(\delta_0(\varphi) f_0)_g = g(\varphi) \cdot f_0 \quad \forall (g \in \mathcal{G}; f_0 \in V).$$

These identities allow us to prove, by induction on n , the following lemma [12]:

Lemma 2. *Let φ be a prerepresentation of \mathcal{G} into $D(\mathcal{V})$ associated with h_2 and let $f_n \in A_n(G, V) \forall n \in \mathbb{N}$. Then:*

$$1) (g(\varphi) \cdot g'(\varphi) \cdot f_n)(g_1, \dots, g_n) - (g'(\varphi) \cdot g(\varphi) \cdot f_n)(g_1, \dots, g_n) \\ = ([g, g'](\varphi) \cdot f_n)(g_1, \dots, g_n) + (\text{ad } h_2(g, g')) f_n(g_1, \dots, g_n) \quad (\text{III.5})$$

$$2) (g(\varphi) \cdot (\delta_n(\varphi) f_n))(g_1, \dots, g_{n+1}) - (\delta_n(\varphi) (g(\varphi) \cdot f_n))(g_1, \dots, g_{n+1}) \\ = \sum_{i=1}^{n+1} (-1)^{i+1} (\text{ad } h_2(g, g_i)) f_n(g_1, \dots, \hat{g}_i, \dots, g_{n+1}) \quad (\text{III.6})$$

$$3) (\delta_{n+1}(\varphi) \delta_n(\varphi) f_n)(g_1, \dots, g_{n+2}) \\ = \sum_{j < k} (-1)^{j+k+1} (\text{ad } h_2(g_j, g_k)) f_n(g_1, \dots, \hat{g}_j, \dots, \hat{g}_k, \dots, g_{n+2}) \quad (\text{III.7})$$

$\forall g, g', g_1, \dots, g_{n+2} \in \mathcal{G}$.

Let \mathcal{G} and \mathcal{V} be Lie algebras, \mathcal{H} a subalgebra of \mathcal{G} and ω a representation of \mathcal{H} into $D(\mathcal{V})$. We call $f_n \in A_n(G, V)$ ω -orthogonal to \mathcal{H} if

$$h(\omega) \cdot f_n = 0 \quad \forall h \in \mathcal{H}, \\ (f_n)_h = 0 \quad \forall h \in \mathcal{H}. \quad (\text{III.8})$$

Then we can define the sets:

$$\mathfrak{C}_{\Phi, \omega}^2(\mathcal{G}, \mathcal{H}, \mathcal{V}) = \left\{ (\varphi, f_2) \left| \begin{array}{l} (\varphi, f_2) \in \mathfrak{C}_{\Phi}^2(\mathcal{G}, \mathcal{V}); \varphi|_{\mathcal{H}} = \omega; \\ f_2 \text{ } \omega\text{-orthogonal to } \mathcal{H} \end{array} \right. \right\}$$

and

$$\mathfrak{Z}_{\Phi, \omega}^2(\mathcal{G}, \mathcal{H}, \mathcal{V}) = \mathfrak{C}_{\Phi, \omega}^2(\mathcal{G}, \mathcal{H}, \mathcal{V}) \cap \mathfrak{Z}_{\Phi}^2(\mathcal{G}, \mathcal{V}).$$

If $(\varphi, f_2) \in \mathfrak{Z}_{\Phi, \omega}^2(\mathcal{G}, \mathcal{H}, \mathcal{V})$, let $f_1 \in L(G, V)$ be ω -orthogonal to \mathcal{H} and consider:

$$\varphi'(g) = \varphi(g) + \text{ad } f_1(g) \\ f_2'(g_1, g_2) = f_2(g_1, g_2) + (\delta_1(\varphi) f_1)(g_1, g_2) + [f_1(g_1), f_1(g_2)]. \quad (\text{III.9}) \\ \forall g, g_1, g_2 \in \mathcal{G}.$$

It is easy to prove [12] that $(\varphi', f_2') \in \mathfrak{Z}_{\Phi, \omega}^2(\mathcal{G}, \mathcal{H}, \mathcal{V})$ and that (III.9) defines an equivalence relation $R(\omega)$ in $\mathfrak{Z}_{\Phi, \omega}^2(\mathcal{G}, \mathcal{H}, \mathcal{V})$.

$$\mathfrak{H}_{\Phi, \omega}^2(\mathcal{G}, \mathcal{H}, \mathcal{V}) = \mathfrak{Z}_{\Phi, \omega}^2(\mathcal{G}, \mathcal{H}, \mathcal{V}) / R(\omega)$$

is then the set of relative pseudocohomology of degree 2 of $\mathcal{G} \bmod(\mathcal{H}, \omega)$ associated with Φ . If \mathcal{V} is an abelian Lie algebra we obtain the relative

cohomology of Chevalley-Eilenberg (after introduction of the natural vector space structures).

Theorem 2 (Reduction Theorem). *Let \mathcal{G} and \mathcal{V} be Lie algebras and $\Phi \in \text{Hom}(\mathcal{G}, \Delta(\mathcal{V}))$. Suppose that \mathcal{D} is an ideal of \mathcal{G} such that \mathcal{G}/\mathcal{D} is semi-simple, and let \mathcal{S} be a subalgebra of \mathcal{G} isomorphic to \mathcal{G}/\mathcal{D} by the canonical epimorphism $\mathcal{G} \rightarrow \mathcal{G}/\mathcal{D}$. There are then a representation ω of \mathcal{S} into $D(\mathcal{V})$ lifted over $\Phi|_{\mathcal{S}}$ and a bijection*

$$\mathfrak{H}_{\Phi}^2(\mathcal{G}, \mathcal{V}) \approx \mathfrak{H}_{\Phi, \omega}^2(\mathcal{G}, \mathcal{H}, \mathcal{V}). \quad (\text{III.10})$$

Proof. The following two lemmas are required in order to prove the theorem:

Lemma 3. *Let $\mathcal{G}, \mathcal{V}, \mathcal{D}, \mathcal{S}$ and Φ be as in Theorem 2. If $(\varphi, f'_2) \in \mathfrak{Z}_{\Phi}^2(\mathcal{G}, \mathcal{V})$ and $\text{Im}(f'_2)_s \subseteq C(\mathcal{V}) \ \forall s \in \mathcal{S}$, there exists $f_2 \in Z^2(\mathcal{G}, C(\mathcal{V})_{\Psi})$, $\Psi = j_{\mathcal{V}} \circ \varphi$, such that $(f'_2)_s = (f_2)_s \ \forall s \in \mathcal{S}$.*

Proof. Using Lemma 2, identity (III.6):

$$\delta_2(\varphi)(s(\varphi) \cdot h_2) = s(\varphi) \cdot (\delta_2(\varphi)h_2) \quad \forall (h_2 \in A_2(G, V); s \in \mathcal{S}).$$

Hence, by (III.5), we can equip the subspace $Z^2 = \{h_2 | h_2 \in A_2(G, V); \delta_2(\varphi)h_2 = 0\}$ of the vector space $A_2(G, V)$ with the \mathcal{S} -module structure Z_{Ω}^2 associated with the representation $\Omega: \mathcal{S} \rightarrow \text{End}_{\mathbf{F}}(Z^2)$ given by $\Omega(s) = s(\varphi)$ and (III.1). The set $\{\delta_1(\varphi)h_1 | h_1 \in L(G, C(\mathcal{V}))\}$ becomes an \mathcal{S} -submodule $(\delta_1(\varphi)L)_{\Omega}$ of Z_{Ω}^2 . Then, by Weyl's theorem on the semi-simplicity of semisimple Lie algebra modules [6], there is an \mathcal{S} -module H_{Ω}^2 such that

$$Z_{\Omega}^2 = H_{\Omega}^2 \oplus (\delta_1(\varphi)L)_{\Omega}.$$

In particular we have $f'_2 = h'_2 + \delta_1(\varphi)h'_1$, where $h'_2 \in H_{\Omega}^2$ and $h'_1 \in L(G, C(\mathcal{V}))$; hence $\text{Im}(h'_2)_s \subseteq C(\mathcal{V}) \ \forall s \in \mathcal{S}$. Using (III.4):

$$(\delta_2(\varphi)h'_2)_s = s(\varphi) \cdot h'_2 - \delta_1(\varphi)(h'_2)_s = 0 \quad \forall s \in \mathcal{S}$$

and therefore

$$s(\varphi) \cdot h'_2 = \delta_1(\varphi)(h'_2)_s = 0 \quad \forall s \in \mathcal{S}.$$

We define $f_2'' \in Z^2(\mathcal{G}, C(\mathcal{V})_{\Psi})$ as follows:

$$\begin{aligned} (f_2'')_s &= (h'_2)_s \quad \forall s \in \mathcal{S} \\ f_2''(d_1, d_2) &= 0 \quad \forall d_1, d_2 \in \mathcal{D}. \end{aligned}$$

It is easily verified that $\delta_2 f_2'' = 0$, since $(\delta_2 f_2'')_s = (\delta_2(\varphi)h'_2)_s = 0 \ \forall s \in \mathcal{S}$ by (III.4).

Therefore the definition $f_2 = f_2'' + \delta_1(\varphi)h'_1 = f_2'' + \delta_1 h'_1$ provides the quoted result.

Lemma 4. *Let $\mathcal{G}, \mathcal{V}, \mathcal{D}, \mathcal{S}, \Phi$ and ω be as in Theorem 2 and let $(\varphi, f_2), (\varphi'', f_2'') \in \mathfrak{Z}_{\Phi, \omega}^2(\mathcal{G}, \mathcal{S}, \mathcal{V})$ be such that $(\varphi, f_2) \sim (\varphi'', f_2'') \pmod{R}$. Then $(\varphi, f_2) \sim (\varphi'', f_2'') \pmod{R(\omega)}$.*

Proof. This lemma is proved in the same way as Lemma 3 [12].

We now prove Theorem 2.

Let (φ', f_2') be a $(2; \mathcal{G}, \mathcal{V}, \Phi)$ -pseudococycle. Consider $\Phi|_{\mathcal{S}}$ and the semisimple Lie algebra $\text{Im } \Phi|_{\mathcal{S}} \cdot (\Pi_{\mathcal{V}}^{-1}(\text{Im } \Phi|_{\mathcal{S}}), \Pi'_{\mathcal{V}})$, where $\Pi'_{\mathcal{V}} = \Pi_{\mathcal{V}}|_{\Pi_{\mathcal{V}}^{-1}(\text{Im } \Phi|_{\mathcal{S}})}$, is an inessential extension of $\text{Im } \Phi|_{\mathcal{S}}$. Thus we have a monomorphism $i: \text{Im } \Phi|_{\mathcal{S}} \rightarrow \Pi_{\mathcal{V}}^{-1}(\text{Im } \Phi|_{\mathcal{S}})$ such that $\Pi'_{\mathcal{V}} \circ i = \text{Id}_{\text{Im } \Phi|_{\mathcal{S}}}$ (the identity map of $\text{Im } \Phi|_{\mathcal{S}}$). There is the possibility to choose a prerepresentation $\varphi: \mathcal{G} \rightarrow \text{D}(\mathcal{V})$ lifted over Φ and such that $\omega = \varphi|_{\mathcal{S}} = i \circ \Phi|_{\mathcal{S}}$. $\omega \in \text{Hom}(\mathcal{S}, \text{D}(\mathcal{V}))$ since

$$\begin{aligned} [\omega(s), \omega(s')] &= [(i \circ \Phi)(s), (i \circ \Phi)(s')] = (i \circ \Phi)([s, s']) \\ &= \omega([s, s']) \quad \forall s, s' \in \mathcal{S}. \end{aligned}$$

Lemma 1 proves the existence of a $(2; \mathcal{G}, \mathcal{V}, \Phi)$ -pseudococycle $(\varphi, h_2') \sim (\varphi', f_2') \pmod{R}$, where $h_2'(s, s') \in C(\mathcal{V}) \quad \forall s, s' \in \mathcal{S}$. $\Pi_{\mathcal{V}}^{-1}(\text{Im } \Phi)$ carries the structure of a semisimple \mathcal{S} -module $\Pi_{\mathcal{V}}^{-1}(\text{Im } \Phi)_{\Sigma}$ associated with the representation $\Sigma: \mathcal{S} \rightarrow \text{End}_{\mathbf{F}}(\Pi_{\mathcal{V}}^{-1}(\text{Im } \Phi))$ given by

$$\Sigma(s)r = [\omega(s), r] \quad \forall (s \in \mathcal{S}; r \in \Pi_{\mathcal{V}}^{-1}(\text{Im } \Phi)),$$

where $[\cdot, \cdot]$ is the Lie product in $\text{D}(\mathcal{V})$. Obviously we have an \mathcal{S} -submodule $\Pi_{\mathcal{V}}^{-1}(\text{Im } \Phi|_{\mathcal{S}})_{\Sigma}$ of $\Pi_{\mathcal{V}}^{-1}(\text{Im } \Phi)_{\Sigma}$, and by Weil's theorem there is an \mathcal{S} -module L_{Σ} such that

$$\Pi_{\mathcal{V}}^{-1}(\text{Im } \Phi)_{\Sigma} = \Pi_{\mathcal{V}}^{-1}(\text{Im } \Phi|_{\mathcal{S}})_{\Sigma} \oplus L_{\Sigma}.$$

Moreover $\text{I}(\mathcal{V})_{\Sigma}$ is an \mathcal{S} -submodule of $\Pi_{\mathcal{V}}^{-1}(\text{Im } \Phi|_{\mathcal{S}})_{\Sigma}$, since $\text{I}(\mathcal{V})$ is an ideal of $\text{D}(\mathcal{V})$:

$$\Sigma(s)\text{ad } v = [\omega(s), \text{ad } v] = \text{ad } \omega(s)v \quad \forall (s \in \mathcal{S}; v \in \mathcal{V}).$$

$(\text{Im } \omega)_{\Sigma}$ is an \mathcal{S} -module too and $\text{I}(\mathcal{V})_{\Sigma} \cap (\text{Im } \omega)_{\Sigma} = \{0\}$. We have therefore the following \mathcal{S} -module decomposition:

$$\Pi_{\mathcal{V}}^{-1}(\text{Im } \Phi|_{\mathcal{S}})_{\Sigma} = \text{I}(\mathcal{V})_{\Sigma} \oplus (\text{Im } \omega)_{\Sigma},$$

which implies

$$\Pi_{\mathcal{V}}^{-1}(\text{Im } \Phi)_{\Sigma} = \text{I}(\mathcal{V})_{\Sigma} \oplus (\text{Im } \omega)_{\Sigma} \oplus L_{\Sigma}.$$

The prerepresentation φ can now be completely fixed by choosing $(\text{Im } \varphi|_{\mathcal{D}})_{\Sigma} = L_{\Sigma}$. Then:

$$[\varphi(s), \varphi(d)] - \varphi([s, d]) = \text{ad } h_2'(s, d) = 0 \quad \forall (s \in \mathcal{S}; d \in \mathcal{D}),$$

as $\text{ad } h'_2(s, d) \in I(\mathcal{V})_{\mathcal{S}}$ and $\text{ad } h'_2(s, d) \in (\text{Im } \varphi | \mathcal{D})_{\mathcal{S}}$. Hence $\text{Im}(h'_2)_s \subseteq C(\mathcal{V}) \forall s \in \mathcal{S}$.

Lemma 3 gives us an $h_2 \in Z^2(\mathcal{G}, C(\mathcal{V})_{\Psi})$, $\Psi = j_{\mathcal{V}} \circ \varphi$, satisfying $(h_2)_s = (h'_2)_s \forall s \in \mathcal{S}$, and the Hochschild-Serre theorem [1] provides us with $h''_2 \in Z^2(\mathcal{G}, \mathcal{S}, C(\mathcal{V})_{\Psi})$ and $f_1 \in C^1(\mathcal{G}, C(\mathcal{V})_{\Psi})$ such that $h_2 = h''_2 + \delta_1 f_1$.

It follows that $f_2 = h'_2 - \delta_1 f_1 = h'_2 - \delta_1(\varphi) f_1$ satisfies $(f_2)_s = 0 \forall s \in \mathcal{S}$ and $(\varphi', f'_2) \sim (\varphi, h'_2) \sim (\varphi, f_2) \pmod{\text{R}}$.

Further $(\delta_2(\varphi) f_2)_s = 0 \forall s \in \mathcal{S}$ gives directly $s(\omega) \cdot f_2 = 0 \forall s \in \mathcal{S}$, i.e. $(\varphi, f_2) \in \mathfrak{Z}_{\Phi, \omega}^2(\mathcal{G}, \mathcal{S}, \mathcal{V})$.

We have thus established a surjection of $\mathfrak{H}_{\Phi, \omega}^2(\mathcal{G}, \mathcal{S}, \mathcal{V})$ into $\mathfrak{H}_{\Phi}^2(\mathcal{G}, \mathcal{V})$ such that $\{(\varphi', f'_2)\} \pmod{\text{R}}$ is the image of $\{(\varphi, f_2)\} \pmod{\text{R}(\omega)}$. This surjection is actually a bijection as follows from Lemma 4. It is possible to give another proof of Theorem 2 by using the bijection $\mathfrak{H}_{\Phi}^2(\mathcal{G}, \mathcal{V}) \approx \text{ext}(\mathcal{G}, \mathcal{V}, \Phi)$ [12].

A meaning of the theorem is the following: Let (\mathcal{E}, ϱ) be an extension of \mathcal{B} by \mathcal{A} with character Φ . Suppose that \mathcal{D} is an ideal of \mathcal{B} such that $\mathcal{B}/\mathcal{D} \approx \mathcal{S} \subseteq \mathcal{B}$ by ϱ , and \mathcal{B}/\mathcal{D} is semisimple. Then there exists, associated with (\mathcal{E}, ϱ) , a $(2; \mathcal{B}, \mathcal{A}, \Phi)$ -pseudococycle $(\varphi, f_2) \in \mathfrak{Z}_{\Phi, \omega}^2(\mathcal{B}, \mathcal{S}, \mathcal{A})$, i.e. such that

$$(f_2)_s = 0 \quad \forall s \in \mathcal{S}$$

$$\omega(s) f_2(d_1, d_2) = f_2([s, d_1], d_2) + f_2(d_1, [s, d_2]) \quad \forall (s \in \mathcal{S}; d_1, d_2 \in \mathcal{D}).$$

Obviously the preceding proof is also correct in the (exceptional) abelian case: in which case the result coincides with the Hochschild-Serre theorem for $n = 2$ (after introduction of the natural vector space structures).

Corollary. *Let $\mathcal{G}, \mathcal{V}, \mathcal{D}, \mathcal{S}, \Phi$ be as in Theorem 2 and let $\text{Obs}(\mathcal{G}, \mathcal{V}, \Phi) = 0$. Then there is a prerepresentation φ of \mathcal{G} into $\text{D}(\mathcal{V})$ lifted over Φ such that $\varphi | \mathcal{S} \in \text{Hom}(\mathcal{S}, \text{D}(\mathcal{V}))$ and $[\varphi(s), \varphi(d)] = \varphi([s, d]) \forall (s \in \mathcal{S}; d \in \mathcal{D})$.*

III.2. Extensions of the Poincaré Algebra \mathcal{P} with Arbitrary Kernels

We consider now the extensions of \mathcal{P} with character Φ by an arbitrary kernel \mathcal{K} . We suppose $\text{Obs}(\mathcal{P}, \mathcal{K}, \Phi) = 0$. Then, given a representative element of an equivalence class of $\mathfrak{Z}_{\Phi}^2(\mathcal{P}, \mathcal{K})$, we can determine a representative element of the corresponding class of equivalent extensions. By Theorem 2 there is a representation ω of \mathcal{L} into $\text{D}(\mathcal{K})$ lifted over $\Phi | \mathcal{L}$ such that we can construct in every equivalence class, according to (I.6), one extension whose associated $(2; \mathcal{P}, \mathcal{K}, \Phi)$ -pseudococycle belongs to $\mathfrak{Z}_{\Phi, \omega}^2(\mathcal{P}, \mathcal{L}, \mathcal{K})$.

K_ω will denote the semisimple \mathcal{L} -module defined on K by a pre-representation φ of \mathcal{P} into $D(\mathcal{H})$ lifted over Φ and such that

$$\omega = \varphi|_{\mathcal{L}} \in \text{Hom}(\mathcal{L}, D(\mathcal{H})), \quad [\varphi(l), \varphi(t)] = \varphi([l, t]) \quad \forall (l \in \mathcal{L}; t \in \mathcal{T}).$$

Such a prerepresentation φ exists by the Corollary to Theorem 2. There is then a $(\varphi, f_2) \in \mathfrak{Z}_{\Phi, \omega}^2(\mathcal{P}, \mathcal{L}, \mathcal{H})$ in every equivalence class of $\mathfrak{Z}_{\Phi}^2(\mathcal{P}, \mathcal{H})$ and on $\text{Im} f_2$ the structure of an \mathcal{L} -submodule of K_ω .

In analogy with [1], $(\mathcal{E}, \tau)_{\varphi, f_2}$ will stand for the extension (\mathcal{E}, τ) of \mathcal{P} by \mathcal{H} considered with the factor set f_2 and the pseudocharacter φ such that $(\varphi, f_2) \in \mathfrak{Z}_{\Phi, \omega}^2(\mathcal{P}, \mathcal{L}, \mathcal{H})$.

If $f_2 \neq 0$ we have on $\text{Im} f_2$ the structure of a simple \mathcal{L} -module $K_{\mathfrak{Z}\{1, 0\}}$. As in the abelian case we will refer to this simple \mathcal{L} -module as the *fundamental \mathcal{L} -module $K(f_2)$ of $(\mathcal{E}, \tau)_{\varphi, f_2}$* .

If $f_2 = 0$, then $K(f_2) = \{0\}$ too, and all extensions of the equivalence class $\{(\mathcal{E}, \tau)_{\varphi, f_2}\}$ are inessential.

We note that, in general, \mathcal{H} does not induce a Lie algebra structure $\mathcal{H}(f_2)$ on $K(f_2)$ (see Section IV, Corollary to Theorem 5).

The following proposition generalizes Proposition 2 of [1]:

Proposition 3. *Let $\text{Obs}(\mathcal{P}, \mathcal{H}, \Phi) = 0$ and $(\varphi, f_2) \in \mathfrak{Z}_{\Phi, \omega}^2(\mathcal{P}, \mathcal{L}, \mathcal{H})$. If the semisimple \mathcal{L} -module K_ω contains no simple component $K_{\mathfrak{Z}\{1, 0\}}$, then $f_2 = 0$.*

Corollary. *Let $\text{Obs}(\mathcal{P}, \mathcal{H}, \Phi) = 0$. Then any extension (\mathcal{E}, τ) of \mathcal{P} by \mathcal{H} with character Φ is inessential, provided that the semisimple \mathcal{L} -module K_ω does not contain any simple component $K_{\mathfrak{Z}\{1, 0\}}$.*

If we consider in \mathcal{T} the standard basis $\{t_0, t_1, t_2, t_3\}$ we obtain two simple \mathcal{L} -modules $K_{\mathfrak{Z}\{1\frac{1}{2}, \frac{1}{2}\}}$ generated by

$$t'_\varrho = \epsilon_{\varrho\sigma\mu\nu}(t^\sigma)f_2(t^\mu, t^\nu)$$

and

$$t''_\varrho = \varphi(t^\mu)f_2(t_\mu, t_\varrho) \quad (\text{as in [1]}).$$

In order to have $\delta_2(\varphi)f_2 = 0$ we must require $t'_\varrho = 0 \quad \forall \varrho \in \{0, 1, 2, 3\}$.

IV. The Structure of $\mathcal{E}(\mathcal{P}, \mathcal{H})$

IV.1. The Levi Decomposition of $\mathcal{E}(\mathcal{P}, \mathcal{H})$

The following generalizations of Theorem 2 and Theorem 3 of [1] are obvious:

Theorem 3. *Let (\mathcal{E}, τ) be an extension of \mathcal{P} . The Lie algebra \mathcal{E} then contains a subalgebra isomorphic to \mathcal{L} by τ .*

Theorem 4. Let (\mathcal{E}, τ) be an extension of \mathcal{P} by \mathcal{K} . There exists an inessential extension (\mathcal{E}, τ') of \mathcal{L} such that τ' factors uniquely through τ .

The following theorem gives the structure of the algebra $\mathcal{E}(\mathcal{P}, \mathcal{K})$ obtained by extending \mathcal{P} by \mathcal{K} :

Theorem 5. Let (\mathcal{E}, τ) be an extension of \mathcal{P} by \mathcal{K} and let $\mathcal{K} = \mathcal{S}_1 \oplus \mathcal{R}_1$ be a Levi decomposition of \mathcal{K} , where \mathcal{R}_1 is the radical of \mathcal{K} and $D^n \mathcal{R}_1 = \{0\}$. Then $\mathcal{E} = (\mathcal{L}' \oplus \mathcal{S}_1) \oplus \mathcal{R}$ is a Levi decomposition of \mathcal{E} , where $\mathcal{L}' \approx \mathcal{L}$ by τ . The radical \mathcal{R} of \mathcal{E} is such that $D^{n+1} \mathcal{R} = \{0\}$.

Proof. $(\mathcal{E}/\mathcal{R}_1, \tau_q)$ is an extension of \mathcal{P} by $\mathcal{K}/\mathcal{R}_1 \approx \mathcal{S}_1$ which is trivial since \mathcal{S}_1 is semisimple. Therefore

$$\mathcal{E}/\mathcal{R}_1 \approx \mathcal{S}_1 \oplus \mathcal{P} = (\mathcal{S}_1 \oplus \mathcal{L}) \oplus \mathcal{T},$$

where $[\mathcal{S}_1, \mathcal{T}] = \{0\}$. We obtain a short exact sequence:

$$\mathcal{R}_1 \hookrightarrow \mathcal{E} \xrightarrow{\tau^*} (\mathcal{S}_1 \oplus \mathcal{L}) \oplus \mathcal{T} = \mathcal{P}^*,$$

where $\tau^*|_{\mathcal{S}_1} = \mathcal{I}_{\mathcal{S}_1}$. Using a trivial generalization of Theorem 3 we see that \mathcal{E} contains a subalgebra $\mathcal{S}_1 \oplus \mathcal{L}' \approx \mathcal{S}_1 \oplus \mathcal{L}$ by τ^* . Let $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{L}$. We have a Lie product on $E \approx P^* \times R_1$, defined by the bilinear alternating map $\alpha'' : E \times E \rightarrow E$ such that:

$$\begin{aligned} \alpha''((p_1^*, 0), (p_2^*, 0)) &= (\alpha(p_1^*, p_2^*), f_2^*(p_1^*, p_2^*)) \quad \forall p_1^*, p_2^* \in P^* \\ \alpha''((0, r_1), (0, r_2)) &= (0, \alpha'(r_1, r_2)) \quad \forall r_1, r_2 \in R_1 \\ \alpha''((p^*, 0), (0, r)) &= (0, \varphi^*(p^*)r) \quad \forall (p^* \in P^*; r \in R_1), \end{aligned} \quad (\text{IV.1})$$

where $\alpha : P^* \times P^* \rightarrow P^*$ and $\alpha' : R_1 \times R_1 \rightarrow R_1$ are, respectively, the bilinear alternating maps which define the Lie products on P^* and on R_1 . (φ^*, f_2^*) is a $(2; \mathcal{P}^*, \mathcal{R}_1, \Phi^*)$ -pseudococycle associated with the extension (\mathcal{E}, τ^*) of \mathcal{P}^* by \mathcal{R}_1 (with character $\Phi^* = \Pi_{\mathcal{R}_1} \circ \varphi^*$).

We select $(\varphi^*, f_2^*) \in \mathfrak{Z}_{\Phi^*, \omega^*}^2(\mathcal{P}^*, \mathcal{S}, \mathcal{R}_1)$ and we look for the extension $(\mathcal{E}', \tau^*)_{\varphi^*, f_2^*}$ in $\{(\mathcal{E}, \tau^*)\}$, constructed according to (IV.1), identifying \mathcal{E}' and \mathcal{E} .

There is $T' \approx T$ by τ such that α'' induces on $R = R_1 \oplus T'$ ($R \subset E$) the structure of a solvable ideal of \mathcal{E} . $\mathcal{E}/\mathcal{R} \approx \mathcal{L} \oplus \mathcal{S}_1$ and thus \mathcal{R} is the radical of \mathcal{E} . $\mathcal{E} = (\mathcal{S}_1 \oplus \mathcal{L}') \oplus \mathcal{R}$ is a Levi decomposition of \mathcal{E} , where $\mathcal{L}' \approx \mathcal{L}$ by τ . Here $D^1 \mathcal{R} = [\mathcal{R}, \mathcal{R}] \subseteq \mathcal{R}_1$, i.e. $D^{n+1} \mathcal{R} = \{0\}$.

Theorem 5 implies the existence of a section σ of (\mathcal{E}, τ) over \mathcal{P} such that:

- 1) $\sigma|_{\mathcal{L}} \in \text{Hom}(\mathcal{L}, \mathcal{E})$ and $\sigma(\mathcal{L}) = \mathcal{L}'$, $[\sigma(\mathcal{L}), \mathcal{S}_1] = \{0\}$;
- 2) $[\sigma(\mathcal{T}), \mathcal{S}_1] = \{0\}$;
- 3) the $(2; \mathcal{P}, \mathcal{K}, \Phi)$ -pseudococycle (φ, f_2) associated with (\mathcal{E}, τ) by σ belongs to $\mathfrak{Z}_{\Phi, \omega}^2(\mathcal{P}, \mathcal{L}, \mathcal{K})$ and $\text{Im } f_2 \subseteq R_1$, \mathcal{R}_1 being the radical of \mathcal{K} .

In the following, whenever we will consider an extension $(\mathcal{E}, \tau)_{\varphi, f_2}$ of \mathcal{P} by \mathcal{K} , we mean that $(\varphi, f_2) \in \mathfrak{Z}_{\Phi, \omega}^2(\mathcal{P}, \mathcal{L}, \mathcal{K})$ is associated with (\mathcal{E}, τ) by such a σ .

Corollary. *Given the extension $(\mathcal{E}, \tau)_{\varphi, f_2}$ of \mathcal{P} by \mathcal{K} , if \mathcal{K} induces a Lie algebra structure $\mathcal{K}(f_2)$ on $K(f_2)$, then $\mathcal{K}(f_2)$ is abelian.*

Proof. As $\text{Im } f_2 \subseteq R_1$, \mathcal{R}_1 being the radical of \mathcal{K} , $\mathcal{K}(f_2)$ is a solvable Lie algebra. Moreover $D^1 \mathcal{K}(f_2)$ is an \mathcal{L} -submodule of $K(f_2)$. But $K(f_2)$ is $\{0\}$ or a simple \mathcal{L} -module and therefore $D^1 \mathcal{K}(f_2) = \{0\}$, i.e. $\mathcal{K}(f_2)$ is an abelian Lie algebra.

We remark that, contrary to the abelian case, the extensions (\mathcal{E}, τ) of \mathcal{P} by arbitrary kernels \mathcal{K} give Lie algebras \mathcal{E} with radical \mathcal{R} not necessarily nilpotent. Indeed, if (\mathcal{E}, τ) is an extension of \mathcal{P} by a solvable and non-nilpotent \mathcal{K} , then $\mathcal{K} \subset \mathcal{R}$ is impossible if \mathcal{R} is nilpotent.

In general we can study the series $\{D^i \mathcal{R}\}_{i=0}^{n+1}$ of the derived ideals of \mathcal{R} : $D^1 \mathcal{R} = [\mathcal{R}, \mathcal{R}] \subseteq \mathcal{R}_1$, ..., $D^i \mathcal{R} = [D^{i-1} \mathcal{R}, D^{i-1} \mathcal{R}] \subseteq D^{i-1} \mathcal{R}_1$, ... The ideals $D^i \mathcal{R}$ are nilpotent if $i \in N^+$. If \mathcal{R}^* is the biggest nilpotent ideal of \mathcal{R} , then $D^i \mathcal{R} \subseteq \mathcal{R}^* \ \forall i \in N^+$.

IV.2. Extensions of \mathcal{P} by \mathcal{K} with a Simple or a Trivial \mathcal{L} -Module Structure

We consider the extensions (\mathcal{E}, τ) of \mathcal{P} by \mathcal{K} with character Φ and with an associated simple \mathcal{L} -module K_ω .

Theorem 6. *Let \mathcal{K} be a non-abelian Lie algebra. There are no essential extensions (\mathcal{E}, τ) of \mathcal{P} by \mathcal{K} with character Φ if K_ω is simple.*

Proof. Let $(\mathcal{E}, \tau)_{\varphi, f_2}$ be an essential extension of \mathcal{P} by a non-abelian Lie algebra \mathcal{K} such that K_ω is simple. It follows from the Corollary to Proposition 3 that $K_\omega = K(f_2)$. This is impossible by the Corollary to Theorem 5.

Therefore, even in the case of extensions of \mathcal{P} by arbitrary Lie algebras \mathcal{K} , the abelian extensions of \mathcal{P} by a simple \mathcal{P} -module $K_{\mathfrak{Z}\{1,0\}}$ will be called *minimal essential extensions* [1].

Now consider a \mathcal{P} -kernel $(\mathcal{P}, \mathcal{K}, 0)$. There is, in any equivalence class of $\mathfrak{Z}_0^2(\mathcal{P}, \mathcal{K})$, a pseudococycle $(\varphi, f_2) \in \mathfrak{Z}_{0,0}^2(\mathcal{P}, \mathcal{L}, \mathcal{K})$, i.e. the \mathcal{L} -module K_ω is trivial ($\omega = 0$). Therefore $f_2 = 0$ and $\varphi = 0$. We arrive at the same result of Michel [13] and Galindo [10]: all extensions of \mathcal{P} by \mathcal{K} with character $\Phi = 0$ are trivial.

IV.3. Extensions of \mathcal{P} by a Reductive Lie Algebra

Let \mathcal{K} be a reductive Lie algebra, i.e. a Lie algebra such that $\text{ad } \mathcal{K}$ is completely reducible (in particular: \mathcal{K} compact Lie algebra) [14]. Then $\mathcal{K} = \mathcal{S} \oplus \mathcal{A}$, where \mathcal{S} is a semisimple and \mathcal{A} an abelian Lie algebra [6, 14, 15].

If \mathcal{K} is reductive we have $\text{Obs}(\mathcal{P}, \mathcal{K}, \Phi) = 0$ for any character Φ . Indeed $I(\mathcal{K}) \approx \mathcal{S}$ and $(D(\mathcal{K}), \Pi_{\mathcal{K}})$ is a trivial extension of $\Delta(\mathcal{K})$ by $I(\mathcal{K})$. We consider the extension $(\mathcal{E}, \tau)_{\varphi, f_2}$ of \mathcal{P} by \mathcal{K} with character Φ . $\text{Im } f_2 \subseteq \mathcal{A}$, hence the extension is preinessential and $\varphi \in \text{Hom}(\mathcal{P}, D(\mathcal{K}))$. We have on \mathcal{K} the \mathcal{P} -module structure K_{φ} .

By Theorem 5 we obtain: $\mathcal{E} = (\mathcal{S} \oplus \mathcal{L}') \oplus \mathcal{R} = \mathcal{S} \oplus (\mathcal{L}' \oplus \mathcal{R})$, where $\mathcal{L}' \approx \mathcal{L}$ by τ , $D^2 \mathcal{R} = \{0\}$. Therefore it is sufficient to study the induced abelian extension $(\mathcal{L}' \oplus \mathcal{R}, \tau|_{\mathcal{L}' \oplus \mathcal{R}})$ of \mathcal{P} by A_{φ} , and all the results of [1] can be easily extended to the case where \mathcal{K} is a reductive Lie algebra. In particular we have:

i) If the \mathcal{L} -module A_{ω} contains no simple component $A_{\mathfrak{A}\{1, 0\}}$, then there exists no essential extension of \mathcal{P} by \mathcal{K} with character Φ .

ii) If the \mathcal{L} -module A_{ω} is simple, then the extensions of \mathcal{P} by \mathcal{K} with character Φ can be essential only if $A_{\omega} = A_{\mathfrak{A}\{1, 0\}}$.

V. The Irreducible and \mathcal{R} -Irreducible Extensions of \mathcal{P}

V.1. Irreducibility and \mathcal{R} -Irreducibility of Extensions

The irreducibility of an arbitrary extension (\mathcal{E}, ϱ) of \mathcal{B} is defined as in the abelian case:

Definition 2. Let (\mathcal{E}, ϱ) be an extension of \mathcal{B} . We say that (\mathcal{E}, ϱ) is irreducible if there is no proper subalgebra $\mathcal{E}' \subset \mathcal{E}$ such that

$$\varrho'(\mathcal{E}') = \varrho|_{\mathcal{E}'}(\mathcal{E}') = \mathcal{B}.$$

If (\mathcal{E}, ϱ) is irreducible (reducible), then all extensions of the equivalence class $\{(\mathcal{E}, \varrho)\}$ are irreducible (reducible). We can prove the following theorem pursuing the same procedure as for the proof of Theorem 5 (Necessity) in [1]:

Theorem 7 (Irreducibility Criterion). In order that an extension (\mathcal{E}, ϱ) of \mathcal{B} by \mathcal{A} be irreducible a necessary condition is that the induced extensions $(\mathcal{E}/\mathcal{A}^*, \varrho_{\mathcal{A}})$ of \mathcal{B} by $\mathcal{A}/\mathcal{A}^*$ are essential for every proper ideal $\mathcal{A}^* \subset \mathcal{A}$ of \mathcal{E} . If \mathcal{A} is an abelian Lie algebra this condition is also sufficient.

The proof of the following proposition is now obvious:

Proposition 4. Let (\mathcal{E}, ϱ) be a preinessential extension of \mathcal{B} by \mathcal{A} . If (\mathcal{E}, ϱ) is irreducible, then \mathcal{A} is abelian.

Using an immediate generalization of Theorem 5 we see that any extension (\mathcal{E}, ϱ) of \mathcal{B} by a non-solvable Lie algebra \mathcal{A} is reducible. But we will define a weak form of irreducibility, such that the semisimple Lie algebras are not disregarded.

Definition 3. An extension (\mathcal{E}, ϱ) of \mathcal{B} is \mathcal{R} -irreducible if there is no proper subalgebra $\mathcal{E}' \subset \mathcal{E}$ such that:

- i) $\varrho(\mathcal{E}') = \mathcal{B}$;
- ii) there exists one Levi subalgebra of \mathcal{E} which is also a Levi subalgebra of \mathcal{E}' .

(\mathcal{E}, ϱ) is \mathcal{R} -reducible if there is a proper subalgebra $\mathcal{E}' \subset \mathcal{E}$ which satisfies i) and ii).

Clearly, as in the case of irreducibility, if a representative element of an equivalence class of extensions is \mathcal{R} -irreducible (\mathcal{R} -reducible), then all extensions of this class are \mathcal{R} -irreducible (\mathcal{R} -reducible).

An \mathcal{R} -irreducibility criterion, analogous to the irreducibility criterion of Theorem 7, can be easily proved [12]:

Theorem 8 (\mathcal{R} -Irreducibility Criterion). Let (\mathcal{E}, ϱ) be an extension of \mathcal{B} by \mathcal{A} and let \mathcal{R}_1 be the radical of \mathcal{A} . In order that (\mathcal{E}, ϱ) be \mathcal{R} -irreducible a necessary condition is that the induced extensions $(\mathcal{E}/\mathcal{A}^*, \varrho_q)$ of \mathcal{B} by $\mathcal{A}/\mathcal{A}^*$ are essential for every proper ideal $\mathcal{A}^* \subset \mathcal{R}_1$ of \mathcal{E} . If \mathcal{R}_1 is an abelian Lie algebra this condition is also sufficient.

Note also the following definition (according to [1]):

Definition 4. The extensions (\mathcal{E}, ϱ) of \mathcal{B} by \mathcal{A} and (\mathcal{E}', ϱ') of \mathcal{B} by \mathcal{A}' are of the same type if $\mathcal{E} \approx \mathcal{E}'$.

In Section V.4 we shall give a classification of the types of irreducible extensions of \mathcal{P} .

V.2. Irreducible and \mathcal{R} -Irreducible Extensions of \mathcal{P}

We are going to study the irreducible and \mathcal{R} -irreducible extensions of \mathcal{P} .

If \mathcal{K} is non-solvable, then the extension (\mathcal{E}, τ) of \mathcal{P} by \mathcal{K} is reducible (Theorem 5).

Proposition 5. Let (\mathcal{E}, τ) be an \mathcal{R} -irreducible or irreducible extension of \mathcal{P} by \mathcal{K} . Then the radical \mathcal{R} of \mathcal{E} is nilpotent (the biggest nilpotent ideal).

Proof. We examine only the case of \mathcal{R} -irreducibility, since, if the statement is right in this instance, it is a fortiori right for the irreducibility.

We suppose that \mathcal{R} is non-nilpotent, then the biggest nilpotent ideal \mathcal{R}^* of \mathcal{R} is a proper subalgebra of \mathcal{R} .

We consider $(\mathcal{E}, \tau)_{\varphi, f_2}, (\varphi, f_2)$ being associated with (\mathcal{E}, τ) by σ . Let $\mathcal{K} = \mathcal{S}_1 \oplus \mathcal{R}_1$ be a Levi decomposition of \mathcal{K} and let $\omega' \in \text{Hom}(\mathcal{L}, \text{D}(\mathcal{R}))$ be defined by $\omega'(\mathcal{L}) = \text{ad}_{\mathcal{R}} \sigma(\mathcal{L})$, where $\omega'(\mathcal{L})|_{\mathcal{R}_1} = \omega(\mathcal{L})$, $\omega = \varphi|_{\mathcal{L}}$. $R_{\omega'}$ is an \mathcal{L} -module and we have the decomposition $R_{\omega'} = R_{\omega'}^* \oplus R'_{\omega'}$, where $R'_{\omega'}$ is a trivial \mathcal{L} -module. Furthermore R contains a vector space $T' = \sigma(T)$ which becomes an \mathcal{L} -module $T'_{\omega'}$ such that $T'_{\omega'} \cap R'_{\omega'} = \{0\}$.

Now, if $f_2 = 0$, $(\mathcal{E}, \tau)_{\varphi, f_2}$ is an inessential extension which is \mathcal{R} -reducible if \mathcal{K} is non-semisimple. If \mathcal{K} is semisimple, T' becomes an abelian Lie algebra $\mathcal{T}' = \mathcal{R}$ and so \mathcal{R} is nilpotent.

If $f_2 \neq 0$, then $K(f_2) = K_{\mathfrak{L}(1,0)}$ and $K(f_2) \cap R'_{\omega'} = \{0\}$. Hence there is $(\mathcal{E}', \tau') \subset (\mathcal{E}, \tau)$, where $\mathcal{E}' = (\sigma(\mathcal{L}) \oplus \mathcal{S}_1) \oplus \mathcal{R}^*$ and $\tau' = \tau|_{\mathcal{E}'}$. This is contrary to the assumption and we infer that $\mathcal{R} = \mathcal{R}^*$.

Any subalgebra of a nilpotent Lie algebra is nilpotent, thus:

Corollary. *All extensions of \mathcal{P} by a solvable, non-nilpotent Lie algebra \mathcal{K} are reducible (and \mathcal{R} -reducible).*

If $(\mathcal{E}, \tau)_{\varphi, f_2}$ is an irreducible extension of \mathcal{P} by \mathcal{K} we have

$$\mathcal{E} = \mathcal{L}' \oplus \mathcal{R}, \quad (\text{V.1})$$

where $\mathcal{L}' \approx \mathcal{L}$, $T' \approx T$ by τ , and $R = T' \oplus K$ (Theorem 5). \mathcal{K} is a nilpotent Lie algebra. We will now limit ourselves to extensions of \mathcal{P} by a nilpotent Lie algebra \mathcal{K} . Let $(\mathcal{E}, \tau)_{\varphi, f_2}$ be a representative element of a class $\{(\mathcal{E}, \tau)\}$ of equivalent extensions.

We will say that $K(f_2)$ is *maximal* in K_{ω} if $K(f_2) \neq K_{\omega}$ and there is no ideal \mathcal{K}' of \mathcal{E} such that $K(f_2) \subseteq K'_{\omega} \subset K_{\omega}$. We have the following irreducibility criterion:

Theorem 9. *The extension $(\mathcal{E}, \tau)_{\varphi, f_2}$ of \mathcal{P} by \mathcal{K} is irreducible if and only if $K(f_2) = K_{\omega}$ or $K(f_2)$ is maximal in K_{ω} .*

Proof. The necessity of the condition $K(f_2) = K_{\omega}$ or $K(f_2)$ maximal in K_{ω} for the irreducibility of $(\mathcal{E}, \tau)_{\varphi, f_2}$ is easily proved following the lines of the proof of Theorem 7 (Necessity) of [1]. This condition is also sufficient, as follows from the fact that K_{ω} has to be spanned by elements of the form $\varphi(t_1) \varphi(t_2) \dots \varphi(t_n) k(t_1, \dots, t_n \in \mathcal{T}; k \in K(f_2))$.

Corollary. *If $K(f_2) \neq K_{\omega}$ and $\varphi(t)k = 0 \ \forall (t \in \mathcal{T}; k \in K(f_2))$, then the extension $(\mathcal{E}, \tau)_{\varphi, f_2}$ of \mathcal{P} by \mathcal{K} is reducible.*

Recall the following definitions:

$$\varphi^i(\mathcal{T}) K(f_2)$$

$$= \overline{\{\varphi^{i_1}(t_1) \varphi^{i_2}(t_2) \dots \varphi^{i_j}(t_j) k \mid t_1, \dots, t_j \in \mathcal{T}; i_1 + \dots + i_j = i; k \in K(f_2)\}}$$

$$\forall i \in \mathbb{N}^+,$$

where $\overline{\{\}} \}$ means the vector space spanned by $\{\}$, and $\varphi^0(\mathcal{T})K(f_2) = K(f_2)$.

The nilpotency of \mathcal{R} requires the existence of $n \in N$ such that $\varphi^n(\mathcal{T})K(f_2) = \{0\}$.

We consider the vector space $K' = \sum_{i=0}^{\infty} \varphi^i(\mathcal{T})K(f_2)$ and the \mathcal{L} -module K'_ω .

Proposition 6. *Given the extension $(\mathcal{E}, \tau)_{\varphi, f_2}$ of \mathcal{P} by \mathcal{K} , \mathcal{K} induces a Lie algebra structure \mathcal{K}' on the vector space $K' = \sum_{i=0}^{\infty} \varphi^i(\mathcal{T})K(f_2)$, where*

$$[\varphi^i(\mathcal{T})K(f_2), \varphi^j(\mathcal{T})K(f_2)] \subseteq \varphi^{i+j+2}(\mathcal{T})K(f_2) \quad \forall i, j \in N. \quad (\text{V.2})$$

Proof. We prove $[\varphi^i(\mathcal{T})K(f_2), \varphi^j(\mathcal{T})K(f_2)] \subseteq \varphi^{i+j+2}(\mathcal{T})K(f_2)$ by induction.

First:

$$[K(f_2), K(f_2)] \subseteq \varphi^2(\mathcal{T})K(f_2)$$

since

$$\begin{aligned} \varphi(t_1)\varphi(t_2)f_2(t'_1, t'_2) - \varphi(t_2)\varphi(t_1)f_2(t'_1, t'_2) &= [f_2(t_1, t_2), f_2(t'_1, t'_2)] \\ &\quad \forall t_1, t_2, t'_1, t'_2 \in \mathcal{T}. \end{aligned}$$

For the same reason it is also $[K(f_2), \varphi^j(\mathcal{T})K(f_2)] \subseteq \varphi^{j+2}(\mathcal{T})K(f_2) \quad \forall j \in N$. Let $i \in N^+$, then we make the induction hypothesis

$$[\varphi^{i-1}(\mathcal{T})K(f_2), \varphi^j(\mathcal{T})K(f_2)] \subseteq \varphi^{i+j+1}(\mathcal{T})K(f_2) \quad \forall j \in N.$$

Therefore:

$$\begin{aligned} [\varphi^i(\mathcal{T})K(f_2), \varphi^j(\mathcal{T})K(f_2)] &= [[\sigma(\mathcal{T}), \varphi^{i-1}(\mathcal{T})K(f_2)], \varphi^j(\mathcal{T})K(f_2)] \\ &\subseteq [[\varphi^{i-1}(\mathcal{T})K(f_2), \varphi^j(\mathcal{T})K(f_2)], \sigma(\mathcal{T})] \\ &\quad + [[\varphi^j(\mathcal{T})K(f_2), \sigma(\mathcal{T})], \varphi^{i-1}(\mathcal{T})K(f_2)] \\ &\subseteq [\varphi^{i+j+1}(\mathcal{T})K(f_2), \sigma(\mathcal{T})] \\ &\quad + [\varphi^{j+1}(\mathcal{T})K(f_2), \varphi^{i-1}(\mathcal{T})K(f_2)] \\ &\subseteq \varphi^{i+j+2}(\mathcal{T})K(f_2) \quad \forall j \in N, \end{aligned}$$

where σ is the section by which (φ, f_2) is associated with (\mathcal{E}, τ) .

Theorem 9 can now be stated: $(\mathcal{E}, \tau)_{\varphi, f_2}$ is an irreducible extension of \mathcal{P} by \mathcal{K} if and only if:

$$K_\omega = \sum_{i=0}^{n-1} \varphi^i(\mathcal{T})K(f_2); \quad \varphi^n(\mathcal{T})K(f_2) = \{0\}. \quad (\text{V.3})$$

Therefore we have again that $(\mathcal{E}, \tau)_{\varphi, f_2}$ is reducible if the \mathcal{L} -module K_ω contains simple components $K_{\mathfrak{Z}(j_1, j_2)}$ where $j_1 + j_2$ is half integer.

If only \mathcal{R} -irreducibility is required we have the following theorem:

Theorem 10. *The extension $(\mathcal{E}, \tau)_{\varphi, f_2}$ of \mathcal{P} by \mathcal{K} is \mathcal{R} -irreducible if and only if $K(f_2) = R_{1_\omega}$ or $K(f_2)$ is maximal in R_{1_ω} , where \mathcal{R}_1 is the radical of \mathcal{K} .*

Proof. The theorem is proved in like manner as Theorem 9, but applying the \mathcal{R} -irreducibility criterion (Theorem 8).

We can also express the \mathcal{R} -irreducibility condition of Theorem 10 by:

$$R_{1_\omega} = \sum_{i=0}^{n-1} \varphi^i(\mathcal{T}) K(f_2); \quad \varphi^n(\mathcal{T}) K(f_2) = \{0\}. \quad (\text{V.4})$$

Corollary. *If the extension $(\mathcal{E}, \tau)_{\varphi, f_2}$ of \mathcal{P} by \mathcal{K} is \mathcal{R} -irreducible, then there is a Levi decomposition $\mathcal{K} = \mathcal{S}_1 \oplus \mathcal{R}_1$ of \mathcal{K} , where \mathcal{R}_1 is nilpotent and \mathcal{S}_1 semisimple.*

Proof. Using the notation of Theorem 5 we have $\mathcal{E} = (\mathcal{L}' \oplus \mathcal{S}_1) \oplus \mathcal{R}$, where $\mathcal{L}' = \sigma(\mathcal{L})$ and $[\sigma(\mathcal{T}), \mathcal{S}_1] = \{0\}$, σ being the section by which (φ, f_2) is associated with (\mathcal{E}, τ) . Let $r_i \in \varphi^i(\mathcal{T}) K(f_2) \subseteq \mathcal{R}_1$, then there exist $m \in \mathbb{N}^+$ and $\{t_1^j, t_2^j, \dots, t_{i+2}^j\}_{j=1}^m \subset \mathcal{T}$ such that

$$r_i = \sum_{j=1}^m \alpha_j [\sigma(t_1^j), [\sigma(t_2^j), [\dots [\sigma(t_{i+1}^j), \sigma(t_{i+2}^j)] \dots]]]$$

($\alpha_j \in \mathbb{R}$ and $[\cdot, \cdot]$ is the Lie product in \mathcal{E}). Therefore: $[r_i, s] = 0 \quad \forall s \in \mathcal{S}_1$.

The \mathcal{R} -irreducibility of $(\mathcal{E}, \tau)_{\varphi, f_2}$ implies $r = \sum_{i=0}^{n-1} r_i \quad \forall r \in \mathcal{R}_1$ and thus $[r, s] = 0 \quad \forall (r \in \mathcal{R}_1; s \in \mathcal{S}_1)$. The nilpotency of \mathcal{R}_1 follows from Proposition 5.

If $(\mathcal{E}, \tau)_{\varphi, f_2}$ is an \mathcal{R} -irreducible extension of \mathcal{P} by $\mathcal{K} = \mathcal{S}_1 \oplus \mathcal{R}_1$ we have

$$\mathcal{E} = \mathcal{S}_1 \oplus (\mathcal{L}' \oplus \mathcal{R}), \quad (\text{V.5})$$

where $\mathcal{L}' = \sigma(\mathcal{L})$, $\mathcal{R} = \sigma(\mathcal{T}) \oplus \mathcal{R}_1$, σ being the section by which (φ, f_2) is associated with (\mathcal{E}, τ) . \mathcal{R}_1 is a nilpotent Lie algebra. If $\mathcal{E}' = \mathcal{L}' \oplus \mathcal{R}$ and $\tau' = \tau|_{\mathcal{E}'}$, then (\mathcal{E}', τ') is an irreducible extension of \mathcal{P} by \mathcal{R}_1 .

It follows easily that any extension $(\mathcal{E}, \tau)_{\varphi, f_2}$ of \mathcal{P} by a reductive Lie algebra $\mathcal{K} = \mathcal{S} \oplus \mathcal{A}$ is \mathcal{R} -irreducible if and only if the induced abelian extension of \mathcal{P} by A_φ is irreducible.

V.3. Examples

We look for the nilpotent and non-abelian Lie algebras \mathcal{K} of low dimension such that the extensions $(\mathcal{E}, \tau)_{\varphi, f_2}$ of \mathcal{P} by \mathcal{K} are irreducible.

Let $\{t_0, t_1, t_2, t_3\}$ be the standard basis in \mathcal{T} . We choose the basis $\{k_{\mu_1 \mu_2 \dots [\mu_i + 1 \mu_i + 2]}\}$ of $\varphi^i(\mathcal{T}) K(f_2)$, where $\mu_1, \dots, \mu_{i+2} \in \{0, 1, 2, 3\}$, defined by:

$$k_{\mu_1 \mu_2 \dots \mu_i [\mu_{i+1} \mu_{i+2}]} = \varphi(t_{\mu_1}) \varphi(t_{\mu_2}) \dots \varphi(t_{\mu_i}) f_2(t_{\mu_{i+1}}, t_{\mu_{i+2}}). \quad (\text{V.6})$$

Therefore:

$$\begin{aligned} \varphi(l) k_{\mu_1 \mu_2 \dots \mu_i [\mu_{i+1} \mu_{i+2}]} &= \varphi(l) \varphi(t_{\mu_1}) \varphi(t_{\mu_2}) \dots \varphi(t_{\mu_i}) f_2(t_{\mu_{i+1}}, t_{\mu_{i+2}}) \\ &= \sum_{j=1}^{i+2} \varphi(t_{\mu_j}) \dots \varphi([l, t_{\mu_j}]) \dots f_2(t_{\mu_{i+1}}, t_{\mu_{i+2}}) \\ &\quad \forall l \in \mathcal{L}. \end{aligned}$$

The symbol $[\zeta v]$ means antisymmetrization of ζv .

a) Let $K_\omega = K(f_2) \oplus K_{\mathfrak{D}\{1, 0\}} \oplus K_{\mathfrak{D}\{\frac{1}{2}, \frac{1}{2}\}}$. In order to obtain an irreducible extension of \mathcal{P} by a non-abelian \mathcal{K} we must have (Proposition 6):

$$\varphi(\mathcal{T}) K(f_2) = K_{\mathfrak{D}\{\frac{1}{2}, \frac{1}{2}\}}, \quad \varphi(\mathcal{T}) K_{\mathfrak{D}\{\frac{1}{2}, \frac{1}{2}\}} = K_{\mathfrak{D}\{1, 0\}}$$

and

$$[K(f_2), K(f_2)] = K_{\mathfrak{D}\{1, 0\}},$$

all other commutators of \mathcal{K} vanishing. $\{k_{[\mu]_2}^\mu\}$ is a basis of $K_{\mathfrak{D}\{\frac{1}{2}, \frac{1}{2}\}}$ (see Section III.2) and we can define

$$\begin{aligned} [f_2(t_\mu, t_\nu), f_2(t_\varrho, t_\sigma)] &= g_{\mu\varrho} g^{\alpha\beta} k_{[\alpha\nu][\beta\sigma]} + g_{\mu\sigma} g^{\alpha\beta} k_{[\alpha\nu][\varrho\beta]} \\ &\quad + g_{\nu\varrho} g^{\alpha\beta} k_{[\mu\alpha][\beta\sigma]} + g_{\nu\sigma} g^{\alpha\beta} k_{[\mu\alpha][\varrho\beta]} \\ &\quad \forall \mu, \nu, \varrho, \sigma \in \{0, 1, 2, 3\}, \end{aligned}$$

where g is the metric tensor. \mathcal{K} is nilpotent and $C^1 \mathcal{K} = \mathcal{K}_{\mathfrak{D}\{1, 0\}}$, $C^2 \mathcal{K} = \{0\}$. We obtain an irreducible extension $(\mathcal{E}_3^{[1]}, \tau_3^{[1]})_{\varphi, f_2}$, where $\dim \mathcal{E}_3^{[1]} = 26$.

b) Let $K_\omega = K(f_2) \oplus K_{\mathfrak{D}\{1, 0\}} \oplus K_{\mathfrak{D}\{\frac{1}{2}, \frac{1}{2}\}}$. We have a result analogous to that of case a): an irreducible extension $(\mathcal{E}_3^{[2]}, \tau_3^{[2]})_{\varphi, f_2}$, where $\dim \mathcal{E}_3^{[2]} = 38$.

c) If $K_\omega = K(f_2) \oplus K_{\mathfrak{D}\{\frac{1}{2}, \frac{1}{2}\}} \oplus K_{\mathfrak{D}\{1, 1\}}$ we must have by Proposition 6:

$$\varphi(\mathcal{T}) K(f_2) = K_{\mathfrak{D}\{\frac{1}{2}, \frac{1}{2}\}}, \quad \varphi(\mathcal{T}) K_{\mathfrak{D}\{\frac{1}{2}, \frac{1}{2}\}} = K_{\mathfrak{D}\{1, 1\}}$$

and $[K(f_2), K(f_2)] = K_{\mathfrak{D}\{1,1\}}$, all other commutators of \mathcal{K} vanishing. $\{k_{[\mu\varrho]}^\mu\}$ is a basis of $K_{\mathfrak{D}\{\frac{1}{2}, \frac{1}{2}\}}$ and we can define

$$\begin{aligned} [f_2(t_\mu, t_\nu), f_2(t_\varrho, t_\sigma)] &= 2k_{[\mu\nu][\varrho\sigma]} - g_{\mu\varrho}g^{\alpha\beta}k_{[\alpha\nu][\beta\sigma]} \\ &\quad - g_{\mu\sigma}g^{\alpha\beta}k_{[\alpha\nu][\varrho\beta]} - g_{\nu\varrho}g^{\alpha\beta}k_{[\mu\alpha][\beta\sigma]} \\ &\quad - g_{\nu\sigma}g^{\alpha\beta}k_{[\mu\alpha][\varrho\beta]} \quad \forall \mu, \nu, \varrho, \sigma \in \{0, 1, 2, 3\}. \end{aligned}$$

\mathcal{K} is nilpotent and $C^1\mathcal{K} = \mathcal{K}_{\mathfrak{D}\{1,1\}}$, $C^2\mathcal{K} = \{0\}$. We obtain an irreducible extension $(\mathcal{E}_3^{[3]}, \tau_3^{[3]})_{\varphi, f_2}$, where $\dim \mathcal{E}_3^{[3]} = 29$.

d) Let $K_\omega = K(f_2) \oplus K_{\mathfrak{F}\{\frac{3}{2}, \frac{1}{2}\}} \oplus K_{\mathfrak{F}\{1,1\}}$: this case is analogous to case c) and the extension $(\mathcal{E}_3^{[4]}, \tau_3^{[4]})_{\varphi, f_2}$ so obtained is such that $\dim \mathcal{E}_3^{[4]} = 41$.

The irreducible extensions of the class $\{(\mathcal{E}_3^{[1]}, \tau_3^{[1]})_{\varphi, f_2}\}$ are the minimal essential extensions of \mathcal{P} by a non-abelian \mathcal{K} .

V.4. Types of Irreducible Extensions of \mathcal{P}

We consider the descending central series $\{C^i\mathcal{R}\}$, where \mathcal{R} is the radical of \mathcal{E} in the extension $(\mathcal{E}, \tau)_{\varphi, f_2}$ of \mathcal{P} by \mathcal{K} . We know that \mathcal{R} and \mathcal{K} are nilpotent if $(\mathcal{E}, \tau)_{\varphi, f_2}$ is irreducible. Even in the case of extensions of \mathcal{P} with arbitrary kernels it follows:

Theorem 11. *Given the extension $(\mathcal{E}, \tau)_{\varphi, f_2}$ of \mathcal{P} by \mathcal{K} , let \mathcal{R} be the radical of \mathcal{E} . Then $(\mathcal{E}, \tau)_{\varphi, f_2}$ is irreducible if and only if $C^1\mathcal{R} = \mathcal{K}$ and \mathcal{R} is nilpotent.*

We can now classify the types of irreducible extensions with arbitrary kernels following the procedure of [1].

Let $\mathfrak{F}_n = \{(n, [i])\} \supseteq \mathfrak{F}_n$ ($i \in I_n$ and $n \in N$) be the set of all types $(n, [i])$ of irreducible extensions (\mathcal{E}, τ) of \mathcal{P} for which the radical \mathcal{R} of \mathcal{E} satisfies

$$C^{n+1}\mathcal{R} = \{0\} \quad \text{and} \quad C^j\mathcal{R} \neq \{0\} \quad \forall (j \in N; j \leq n). \quad (\text{V.7})$$

I_n is an index set of the types of irreducible extensions of \mathcal{P} satisfying (V.7) and \mathfrak{F}_n is the subset of types of irreducible abelian extensions.

The family \mathfrak{F}_0 contains only the truly trivial extension type; the family \mathfrak{F}_1 contains only one element too: the type (1) of the minimal essential extensions. The family \mathfrak{F}_2 contains 3 elements.

We note that $\mathfrak{F}_i = \mathfrak{F}_j$ if $i = 0, 1, 2$, but $\mathfrak{F}_j \supset \mathfrak{F}_j \forall j \geq 3$. In effect there exist non-abelian irreducible extensions for $j = 3$ (see Section V.3), and also for $j > 3$ \mathfrak{F}_j contains always non-abelian extension types.

After some obvious notational changes, Proposition 6 of [1] and its Corollary are also right if \mathcal{K} is arbitrary. Using Theorems 9 and 11 it is easy to prove [12] the following result which generalizes Theorem 9 of [1]:

Theorem 12. *The extension $(\mathcal{E}, \tau)_{\varphi, f_2}$ of \mathcal{P} by \mathcal{K} is irreducible of type $(n, [i])$, where $n \in \mathbf{N}^+$, if and only if:*

$$\varphi^n(\mathcal{T}) K(f_2) = \{0\}; \quad \varphi^j(\mathcal{T}) K(f_2) \neq \{0\} \quad \forall (j \in \mathbf{N}; j < n)$$

and

$$K_\omega = \sum_{j=0}^{n-1} \varphi^j(\mathcal{T}) K(f_2).$$

If $n=0$, $\mathcal{K} = \{0\}$ is the necessary and sufficient condition for the irreducibility.

For the construction of a representative element of the equivalence classes of type $(n, [i])$ ($n \in \mathbf{N}^+$; $i \in I_n$) we can build K_ω starting from $K(f_2)$: by Theorem 12 $K_\omega = \sum_{j=0}^{n-1} \varphi^j(\mathcal{T}) K(f_2)$, where

$$\varphi(t_1) \varphi(t_2) - \varphi(t_2) \varphi(t_1) = \text{ad } f_2(t_1, t_2) \quad \forall t_1, t_2 \in \mathcal{T},$$

and $\varphi^2(\mathcal{T}) K_\omega^* \cap K_\omega^* = \{0\}$ for any simple \mathcal{L} -submodule K_ω^* of K_ω . It is then possible to equip K with a Lie algebra structure \mathcal{K} following Proposition 6.

We end with a few comments about the extensions of \mathcal{P}_c , the complexification of \mathcal{P} . The remarks in the Appendix of [1] fit also in the case of extensions with arbitrary kernels.

We have $\mathfrak{F}_i = \mathfrak{F}_i$, $i = 0, 1, 2$, but $\mathfrak{F}_j \supset \mathfrak{F}_j \forall j \geq 3$ too.

It can easily be seen that we have 2 types of irreducible extensions by a non-abelian \mathcal{K} of lower dimension. They belong to \mathfrak{F}_3 and if $(\mathcal{E}_3^{[i]}, \tau_3^{[i]})_{\varphi, f_2}$, $i = 1, 2$, is one of these extensions, then $\dim \mathcal{E}_3^{[i]} = 24$.

Acknowledgement. I wish to thank Prof. D. Speiser for his constant interest in this work. Warm thanks are due to Dr. T. Janssen for many critical remarks. I am also indebted to Prof. F. Cerulus and to Drs. F. Halzen and H. Stern for discussions.

References

1. Cattaneo, U.: Commun. Math. Phys. **13**, 226—245 (1969).
2. Calabi, L.: Ann. Mat. Pura Appl. **32**, 295—370 (1951).
3. Mori, M.: J. Math. Soc. Japan **5**, 171—183 (1953).
4. Hochschild, G.: Am. J. Math. **76**, 698—716 (1954).
5. — Am. J. Math. **76**, 763—778 (1954).
6. Bourbaki, N.: Groupes et algèbres de Lie, Ch. 1, ASI 1285. Paris: Hermann 1960.
7. Eilenberg, S., Mac Lane, S.: Ann. Math. **48**, 326—341 (1947).
8. Cartan, H., Eilenberg, S.: Homological algebra. Princeton: Princeton Univ. Press 1956.
9. Mac Lane, S.: Homology. Berlin-Göttingen-Heidelberg: Springer 1963.
10. Galindo, A.: J. Math. Phys. **8**, 768—774 (1967).

11. Michel, L.: Lectures on "Relativistic Invariance and Internal Symmetries" at the Brandeis University Summer Institute 1965, Vol. I Axiomatic field theory. London: Gordon and Breach 1966.
12. Cattaneo, U.: *Invariance relativiste, symétries internes et extensions d'algèbres de Lie*. Louvain Doctoral Thesis 1970.
13. Michel, L.: Lectures on "Group extensions and quantum mechanics" at the Istanbul Summer School 1962. London: Gordon and Breach 1965. Nucl. Phys. **57**, 356—386 (1964).
14. Helgason, S.: Differential geometry and symmetric spaces. London: Academic Press 1962.
15. Jacobson, N.: Lie algebras. New York: Interscience Publ. Inc. 1962.

Dr. U. Cattaneo
Centre de Physique Nucléaire
Parc d'Arenberg
Heverlee/Belgium.
Present address:
Instituut voor Theoretische Fysica
Katholieke Universiteit
Driehuizerweg 200
Nijmegen/Netherlands