Commun. math. Phys. 18, 319—326 (1970)

Non-factor Quasi-free States of the CAR-algebra

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Received April 10, 1970

Abstract. A necessary and sufficient condition is given in order that a quasi-free state on the Clifford algebra $\overline{\mathscr{A}(H, s)}$ build on a real separable Hilbert space (H, s) be a factor state.

I. Introduction

Let (H, s) be a real Hilbert space which is separable (i.e. H is a real vector space and s a real scalar product on H). Let $\overline{\mathscr{A}(H, s)}$ be the CAR-algebra constructed on (H, s) i.e. it is the C^*-algebra generated by the elements $B(\psi)$ where $\psi \to B(\psi)$ is a real linear map of H into $\overline{\mathscr{A}(H, s)}$ satisfying the anticommutation relations

$$[B(\psi), B(\varphi)]_+ = 2s(\psi, \varphi) I$$

for all ψ and φ of H; I is the unit element in $\overline{\mathscr{A}(H, s)}$.

The quasi-free states ω_A on $\overline{\mathscr{A}(H, s)}$ are those states which are completely determined by an operator A on H such that for all $\psi, \varphi \in H$

$$\omega_A(B(\psi) B(\varphi)) = s(\psi, \varphi) + i s(A\psi, \varphi), \tag{1}$$

$$s(A\psi, \varphi) = -s(\psi, A\varphi) \quad \text{or} \quad A^+ = -A, \tag{2}$$

$$\|A\| \leqq 1. \tag{3}$$

For more details see (1).

A state on a C^*-algebra is called factor state if it induces a factor G.N.S. representation. In this note we prove that ω_A is not a factor state if and only if the dimension of the kernel of A is odd and

$$\mathrm{Tr}[1 - (A^* A)^{\frac{1}{2}}] < \infty$$

II. The Theorem

Among the set of quasi-free states ω_A we distinguish two cases: let $\mathfrak{M}_A$ be the kernel of the operator A, then:

1. dimension of $\mathfrak{M}_A$ is even or infinite,
2. dimension of $\mathfrak{M}_A$ is odd.

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a) First case: dimension $\mathfrak{M}_A$ is even or infinite.

Let $A = U|A|$ be the polar decomposition of A on (H, s), then on $\mathfrak{M}_A^\perp = H \ominus \mathfrak{M}_A : U^2 = -1$, $U^+ = -U$. Since A is a normal operator, U and $|A|$ commute, and since $\dim \mathfrak{M}_A$ is even or infinite we can extend U to H such that its extension J satisfies:

$$J^+ J = J J^+ = 1; \quad J^+ = -J; \quad A = J|A|.$$

The operator J is a complex structure on (H, s) such that $[A, J]_- = 0$. Hence the quasi-free state ω_A determined by the operator A is J-gauge invariant [1]. It has been proved by several authors [2, 3] that gauge invariant quasi-free states are factor states. We summarize:

Proposition 1. *Every quasi-free state* ω_A *on the CAR-algebra* $\overline{\mathscr{A}(H, s)}$ *such that* $\dim \mathfrak{M}_A$ *is even or infinite, is a factor state.*

b) Second case: $\dim \mathfrak{M}_A$ is odd.

From the work on product states (see e.g. [4]) it follows that without loss of generality we can restrict ourselves to the case that $\dim \mathscr{M}_A = 1$. The operator A leaves invariant $\mathfrak{M}_A$ and $\mathfrak{M}_A^\perp$, therefore [4] ω_A is a product state

$$\omega_A = \omega_0 \otimes \omega_C$$

where $\omega_0 = \omega_A | \mathscr{A}(\mathfrak{M}_A, s)$ (it is the central quasi-free state of $\mathscr{A}(\mathfrak{M}_A, s)$) and $\omega_C = \omega_A | \mathscr{A}(\mathfrak{M}_A^\perp, s)$ (where $C = A | \mathfrak{M}_A^\perp$). Let $(\pi_0, \Omega_0, \mathscr{H}_0)$ and $(\pi_C, \Omega_C, \mathscr{H}_C)$ be the G.N.S. representations, cyclic vectors and representation spaces of ω_0, respectively ω_C. Let $(\pi_A, \Omega_A, \mathscr{H}_A)$ be the G.N.S. representation, cyclic vector and representation space induced by the state ω_A, then one verifies that

$$\mathscr{H}_A = \mathscr{H}_0 \,\bar{\otimes}\, \mathscr{H}_C \tag{4}$$

($\bar{\otimes}$ is the completed tensor product of Hilbert spaces),

$$\Omega_A = \Omega_0 \otimes \Omega_C, \tag{5}$$

$$\left.\begin{aligned} \pi_A(B(\psi_0)) &= \pi_0(B(\psi_0)) \otimes \theta \quad \text{for} \quad \psi_0 \in \mathfrak{M}_A \\ \pi_A(B(\psi)) &= I_0 \otimes \pi_C(B(\psi)) \quad \text{for} \quad \psi \in \mathfrak{M}_A^\perp \end{aligned}\right\} \tag{6}$$

where I_0 is the unit operator on $\mathscr{H}_0$; θ is the unique unitary involutive operator on $\mathscr{H}_C$ anticommuting with every element of the form $\pi_C(B(\psi))$, $\psi \in \mathfrak{M}_A^\perp$ such that $\theta\, \Omega_C = \Omega_C$ (the existence of the operator θ is a consequence of the fact that the state ω_C is invariant under the *-automorphism γ defined by $\gamma(B(\psi)) = -B(\psi)$, $\psi \in \mathfrak{M}_A^\perp$).

It is proved in (2) and (3. Rideau) that each gauge invariant quasi-free state ω_A induces a G.N.S. representation which is quasi-equivalent to the representation induced by a quasi-free state ω_D, where $|D|$ has a pure point spectrum. The generalisation of this result to all quasi-free states is immediate. Because two quasi-equivalent representations are factor representations if and only if one of them is a factor (5; prop. 5.3.4), from now on we can suppose that $|A|$ has a pure point spectrum. Hence there exists an orthonormal basis $\{\psi_i\}_{i=1,2,\ldots}$ of $\mathfrak{M}_A^\perp$ such that

$$\text{if} \quad C = J|C|$$

then

$$\psi_{2i} = J\,\psi_{2i-1},$$

$$|C|\,\psi_{2i-1} = a_i \psi_{2i-1},$$

$$|C|\,\psi_{2i} = a_i \psi_{2i},$$

and

$$0 < a_i \leqq 1; \qquad i = 1, 2, 3, \ldots .$$

Using (4) the state ω_C is a product state

$$\omega_C = \bigotimes_{i=1,2,\ldots} \omega_i$$

where ω_i is the restriction of ω_C to $\mathscr{A}(H_i, s)$; H_i is the subspace generated by $\{\psi_{2i-1}, \psi_i\}$.

The states ω_i induce a G.N.S. representation determined by the triplet $(\pi_i, \Omega_i, \mathscr{H}_i)$.

Let

$$\theta_j = i\,\pi_j(B(\psi_{2j-1})\,B(\psi_{2j}))$$

then

$$\pi_C(B(\psi)) = \theta_1 \otimes \theta_2 \otimes \cdots \otimes \theta_{i-1} \otimes \pi_i(B(\psi)) \otimes I_{i+1} \otimes \cdots \tag{7}$$

for all $\psi \in H_i$; I_j is the unit operator on $\mathscr{H}_j$ and

$$\Omega_C = \bigotimes_{i=1}^{\infty} \Omega_i, \tag{8}$$

$$\mathscr{H}_C = \bigotimes_{i=1}^{\infty} \mathscr{H}_i \left(\text{associated to } \bigotimes_{i=1}^{\infty} \Omega_i\right). \tag{9}$$

Proposition 2. *Let* $\dim \mathfrak{M}_A = 1$ *and* $\mathrm{Tr}(1 - |A|) < \infty$, *then* ω_A *is not a factor state.*

Proof[1]. It is proved in [4; 2.3.5] that if ω_A is not a factor state, its center is generated by a hermitian, odd element Z, such that $Z^2 = 1$. Now we construct explicitly this element.

The operator $|A|$ has a pure point spectrum, hence we can use the infinite product form of the representation induced by ω_C given by (7), (8), and (9). Define the operator

$$\zeta = \bigotimes_{i=1}^{\infty} \pi_i(B(\psi_{2i-1})\,B(\psi_{2i}))$$

[1] For a shorter proof, see appendix.

on $\mathscr{H}_C$. Now

$$\sum_{i=1}^{\infty}(1-|(\Omega_i,\pi_i(B(\psi_{2i-1})B(\psi_{2i}))\Omega_i)|)$$
$$=\sum_{i=1}^{\infty}(1-a_i)$$
$$=\mathrm{Tr}(I-|C|).$$

Because $\mathrm{Tr}(I-|A|)<\infty$, we have that $\mathrm{Tr}(I-|C|)<\infty$.Hence the operator ζ is a bounded operator on $\mathscr{H}_C$ [7].

We prove further that $\zeta\in\pi_C(\mathscr{A}(\overline{\mathfrak{M}_A^\perp,s}))''$. We prove that ζ is the weak limit of the sequence $\{\zeta_n\}_{n=1,2,\ldots}$

$$\zeta_n=\pi_C\left(\prod_{i=1}^{n}B(\psi_{2i-1})B(\psi_{2i})\right).$$

The sequence $\{\zeta_n\}_{n=1,2\ldots}$ is a uniformly bounded sequence, hence it is sufficient to prove the convergence on the set of vectors

$$\left\{\Omega_C,\prod_{j=1}^{k}\pi_C(B(\psi_{i_j}))\Omega_C|i_j\in\mathbb{N}\quad\text{and}\quad k\in\mathbb{N}\right\}.$$

Let

$$\Psi_{(1,\ldots,i_l)}=\prod_{j=1}^{l}\pi_C(B(\psi_{i_j}))\Omega_C$$

then we have to prove that

$$|(\Psi_{(i_1,\ldots,i_l)};(\zeta-\zeta_n)\Psi_{(r_1\ldots r_k)})|\to 0 \tag{10}$$

as n tends to infinity, for all finite $(i_1,\ldots,i_l)$ and $(r_1\ldots r_k)$. We take $n>i_l$ and r_k.

Because

$$\prod_{j=1}^{l}\pi_C(B(\psi_{i_j}))$$

commutes or anticommutes with both ζ and ζ_n depending on whether l is even or odd, (10) is equivalent with

$$|(\Omega_C,(\zeta-\zeta_n)\Psi_{(r_1\ldots r_k)})|\to 0\quad\text{as}\quad n\to\infty. \tag{11}$$

The state ω_C is even, hence if k is odd, (11) is trivially satisfied, otherwise put $k=2m$. It is easily verified that the only non trivial case happens when

$$\Psi_{(2i_1-1,2i_1,\ldots)}=\prod_{j=1}^{m}\pi_C(B(\psi_{2i_j-1})B(\psi_{2i_j}))\Omega_C.$$

In this case

$$|(\Omega_C, (\zeta - \zeta_n) \cdot \Psi_{(2i_1-1, 2i_1, \ldots)})|$$

$$= \left|\prod_{i=1}^{\infty} a_i - \prod_{i=1}^{n} a_i\right| \left(\prod_{j=1}^{m} a_{i_j}\right)^{-1}$$

$$= \left(\prod_{i=1}^{n} a_i\right)\left(1 - \prod_{j=n+1}^{\infty} a_j\right)\left(\prod_{j=1}^{m} a_{i_j}\right)^{-1}$$

$$\leqq 1 - \prod_{j=n+1}^{\infty} a_j$$

which vanishes when n tends to infinity [7].

Hence

$$\zeta \in \pi_C(\overline{\mathscr{A}(\mathfrak{M}_A^{\perp}, s)})''.$$

Now we have two operators ζ and θ both anticommuting with the generators

$$\pi_C(B(\psi)), \quad \psi \in \mathfrak{M}_A^{\perp}$$

of

$$\pi_C(\overline{\mathscr{A}(\mathfrak{M}_A^{\perp}, s)})''$$

and $I_0 \otimes \zeta \in \pi_A(\overline{\mathscr{A}, H, s})''$. One easily verifies that

$$Z = \pi_A(B(\psi_0))(I_0 \otimes \zeta) = \pi_0(B(\psi_0)) \otimes \theta\, \zeta \in \pi_A(\overline{\mathscr{A}(H, s)})'' \cap \pi_A(\overline{\mathscr{A}(H, s)})'.$$

Hence ω_A is not a factor state. Q.E.D.

Proposition 3. *Suppose that* $\dim \mathfrak{M}_A = 1$ *and that* $1 - |A|$ *is not a trace class operator, then the quasi-free state* ω_A *is a factor state.*

Proof. Let E_{2n} be the subspace of H generated by $\{\psi_0, \psi_1, \ldots, \psi_{2n-1}\}$. The algebra $\overline{\mathscr{A}(H, s)}$ is a UHF algebra and

$$\{\mathscr{M}_n = \mathscr{A}(E_{2n} s)\}_n$$

is an incrasing sequence of $(2^n \times 2^n)$-matrix algebras which generates $\mathscr{A}$. A state ω on $\mathscr{A}$ induces a factor representation of $\mathscr{A}$ if and only if for each $x \in \mathscr{A}$ there is an integer $n > 0$ depending only on x such that

$$|\omega(xy) - \omega(x)\,\omega(y)| \leqq \|y\| \tag{12}$$

for all $y \in \mathscr{M}_n^C$ [6] ($\mathscr{M}_n^C$ consists of all elements of $\mathscr{A}$ commuting with $\mathscr{M}_n$).

Let

$$x = x_e + x_0$$

be an arbitrary element of $\mathscr{A}$ such that

$$x_e = \tfrac{1}{2}(\gamma(x) + x),$$

$$x_0 = \tfrac{1}{2}(x - \gamma(x)).$$

We prove that there exists an integer n such that for all $y \in \mathscr{M}_n^C$:

$$|\omega_A(x_e y) - \omega_A(x_e)\,\omega_A(y)| \leqq \tfrac{1}{2}\,\|y\| \tag{13}$$

and

$$|\omega_A(x_0 y) - \omega_A(x_0)\,\omega_A(y)| \leqq \tfrac{1}{2}\,\|y\|\,. \tag{14}$$

1) Because the set $\{B(\psi_i) | i \in \mathbb{N} = \text{the non-negative integers}\}$ generates the algebra $\overline{\mathscr{A}(H, s)}$, there exists an integer p_e such that

$$x_e = s_1 + s_2\,,$$

$$\|s_2\| \leqq \tfrac{1}{4}\,,$$

and

$$s_1 \in \mathscr{A}_e(E_{p_e}, s)\,.$$

We note by $\mathscr{A}_e$ and $\mathscr{A}_0$, the even, respectively the odd part of $\mathscr{A}$. Further for all $q \geqq p_e$ and all

$$y \in \mathscr{A}(E_q, s)^C = \overline{\mathscr{A}_e(E_q^\perp, s)} \oplus \theta_q \overline{\mathscr{A}_0(E_q^\perp, s)}$$

([4], a) 2.3.1) where $\theta_q = i^q B(\psi_0) \ldots B(\psi_{2q-1})$,

$$y = y_e + \theta_q y_0$$

and we get

$$\begin{aligned} |\omega_A(x_e y) - \omega_A(x_e)\,\omega_A(y)| &\leqq |\omega_A(s_1 y) - \omega_A(s_1)\,\omega_A(y)| \\ &\quad + |\omega_A(s_2 y) - \omega_A(s_2)\,\omega_A(y)| \\ &\leqq |\omega_A(s_1 y_e) - \omega_A(s_1)\,\omega(y_e)| + \tfrac{1}{2}\,\|y\| = \tfrac{1}{2}\,\|y\| \end{aligned}$$

because firstly ω_A is an even state implying $\omega_A(s_1 \theta_q y_0) = 0$ and secondly ω_A is a product state implying $\omega_A(s_1 y_e) = \omega_A(s_1)\,\omega_A(y_e)$ (for more details see [4], a) 2.1). This proves (13).

2) By the same remark as in 1), there exists an integer $p_0 \in \mathbb{N}$ such that

$$x_0 = t_1 + t_2\,,$$

$$\|t_2\| \leqq \tfrac{1}{4}\,,$$

and

$$t_1 \in \mathscr{A}_0(E_{p_0}, s)\,.$$

Let r be an integer such that $p_0 < r$ and

$$y \in \mathscr{A}(E_r, s)^C = \overline{\mathscr{A}_e(E_r^\perp, s)} \oplus \theta_r \overline{\mathscr{A}_0(E_r^\perp, s)}$$

then

$$y = y_e + \theta_r y_0\,.$$

Hence

$$\begin{aligned}|\omega_A(x_0 y) \quad \omega_A(x_0)\,\omega_A(y)| &\leqq |\omega_A(t_1 y) - \omega_A(t_1)\,\omega_A(y)| + \tfrac{1}{4}\,\|y\| \\ &= |\omega_A(t_1 \theta_r y_0)| + \tfrac{1}{4}\,\|y\|\end{aligned}$$

Now $t_1 \in \mathscr{A}_0(E_{p_0}, s)$ and ω_A being a product state

$$|\omega_A(t_1 \theta_r y_0)| = K \cdot L_r \cdot M$$

where

$$\begin{aligned}K &= |\omega_A(t_1 B(\psi_0) \ldots B(\psi_{2q}))|\,, \\ L_r &= |\omega_A(B(\psi_{2q+1}) \ldots B(\psi_{2r-2}))|\,, \\ M &= |\omega_A(B(\psi_{2r-1})\, y_0)|\,.\end{aligned}$$

Only the value of K depends on x; by explicit computation

$$L_r = \prod_{i=q+1}^{r-1} a_i$$

and by increasing the value of r, the value of L_r can be made as small as we like $\left(\text{remember that } \operatorname{Tr}(1-|A|) = \infty \text{ implying } \prod_{i=1}^{\infty} a_i = 0\right)$. Hence we can take r such that

$$K\,L_r < \tfrac{1}{4}\,,$$

finally ω_A being an even state

$$\begin{aligned}M &= |\omega_A(B(\psi_{2r-1})\, y_0)| \\ &= |\omega_A(B(\omega_0) \ldots B(\psi_{2r-2})\, y)| \leqq \|y\|\,.\end{aligned}$$

Hence

$$|\omega_A(x_0 y) - \omega_A(x_0)\,\omega_A(y)| \leqq \tfrac{1}{4}\,\|y\| + \tfrac{1}{4}\,\|y\| = \tfrac{1}{2}\,\|y\|\,.$$

This proves (14).

By taking

$$n \geqq \max(p_e, r)$$

p_e as defined in 1) and r as in 2) it is satisfied to (12). Q.E.D.

As an immediate consequence of Propositions 1, 2, and 3 we may now formulate the main result.

Theorem. *In order that a quasi-free state ω_A on $\overline{\mathscr{A}(H, s)}$ is not a factor state, it is necessary and sufficient that the operator A en H satisfies*

(i) dim(kernel A) *is odd,*

(ii) $\operatorname{Tr}(I - (A^+ A)^{\frac{1}{2}}) < \infty$.

Appendix

D. Testard informed us about a simpler proof of proposition 2. With the same notations as above it goes as follows; the state $\omega = \omega_0 \otimes \omega_J$ is not a factor state because the operator θ belongs to the von Neumann algebra $\pi_J(\mathscr{A}(\mathfrak{M}_A^\perp, s))''$ (π_J being irreducible), hence

$$Z = \pi_\omega(B(\psi_0)) \cdot (I \otimes \theta) = \pi_0(B(\psi_0)) \otimes I$$

belongs to the center of π_ω.

Because $\mathrm{Tr}(I - |C|) < \infty$ we have that ω_C is quasi-equivalent to ω_J [8], hence ω_A is quasi-equivalent to ω, implying that ω_A is not factorial.

We reproduced this proof because of its elegance. However we prefer to keep the proof given above, because it contains the explicit construction of the generator of the center.

Acknowledgements. The authors thank M. Sirugue, D. Testard and M. Winnink for fruitful discussions; one of us (A.V.) thanks Professor N. M. Hugenholtz for his kind hospitality at Groningen University (the Netherlands) where this work began and the other (J. M.) Professor F. Cerulus of the Instituut voor Theoretische Natuurkunde, Leuven University (Belgium) where this work was completed.

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