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On the Mathematical Structure of Einstein's Equations in Mixed Initial and Boundary Value Problems

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Abstract. It is shown that, also in the mixed initial and boundary value problem, Einstein's equations may be replaced by the two subsystems $T_l{}^m//_m = 0$ and $R_{\alpha\beta} = -\varkappa\left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}\right)$, provided that the initial data verify the consistency conditions $G_i^4 = -\varkappa T_i^4$ and that the analogous relations

$$(G_{lm} + \varkappa T_{lm}) N^m = 0$$

are imposed on the boundaries of the given domain.

Introduction

Einstein's gravitational equations[1]

$$G_{lm} = -\varkappa T_{lm} \tag{1}$$

imply the four relations

$$T_l^m //_m = 0 . \tag{2}$$

Conversely, LICHNEROWICZ [1] (see also [2, 3]) has proved that, in the initial value problem, Eq. (1) may be replaced by (2) and by the system

$$R_{\alpha\beta} = -\varkappa_{\alpha\beta}\left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}\right) \tag{3}$$

provided that the initial data verify the consistency conditions

$$G_i^4 = -\varkappa T_i^4 . \tag{4}$$

The importance of this result lies in the fact that it singles out Eq. (2) as a subsystem of Eq. (1). Now, since Eq. (2) have a definite physical meaning by themselves, the above result allows us to study their role in the determination of the solution of the gravitational problem.

For the same reason, an analogous possibility is also desirable in more general cases, e.g. in the mixed initial and boundary value problem. In fact, as we shall prove in a later paper, the analysis of Eq. (2) in a spatially finite domain of space-time (e.g. in a four-dimensional "world-

[1] Latin indices run from 1 to 4. Greek indices run from 1 to 3. The metric tensor is assumed to have the signature $+2$. A comma indicates partial derivative; a double stroke [like in Eq. (2)] indicates covariant derivative.

tube") plays an important role in the study of the inertial phenomena. Therefore, the purpose of the present work is to extend LICHNEROWICZ'S result to the mixed initial and boundary value problem.

In § 1 we shall briefly indicate the hypothesis required, as well as the exact formulation of the problem.

In § 2 we shall prove the desired result: Eq. (1) may be replaced by the system (2), (3) also in the mixed initial and boundary value problem, provided that the initial data verify the consistency conditions (4), and that the analogous relations

$$(G_{lm} + \varkappa T_{lm}) N^m = 0 \tag{5}$$

are imposed on the boundaries of the given domain. [In (5), as usual, N^m denotes the normal to the boundary itself.]

§ 1

We recall the following definitions [1—4]: the spacetime continuum V_4 is a twice continuously differentiable manifold on which there is defined a metric

$$ds^2 = g_{lm}\, dx^l\, dx^m$$

of normal hyperbolic type everywhere with signature $(+ + + -)$. A coordinate system is physically acceptable if one of its variables is time like, and the other three space-like. On then has $g_{44} < 0$ and the reciprocal quadratic forms

$$g^{\alpha\beta} \quad \text{and} \quad g_{\alpha\beta} - \frac{g_{4\alpha}\, g_{4\beta}}{g_{44}}$$

are positive definite.

A metric is said regular if it is continuously differentiable (C_1, C_3 piecewise).

Since the manifold V_4 admits of an everywhere hyperbolic metric, it possesses a vector field oriented in time, and therefore a global system of time lines. In V_4 let the domain Ω verify the properties:

i) Ω is the topological product of a manifold V_3 and the real straight line R (the mappings of R in V_4 oriented in time, mappings of V_3 everywhere oriented in space);

ii) the spatial sections, mappings of V_3, are bounded manifolds, homeomorphic to spatially finite regions of the Euclidean space R_3.

In Ω we assume a sufficiently smooth distribution of matter, described by a non-zero stress tensor T_{lm}. Then we have the Einstein field equations

$$G_{lm} = R_{lm} - \frac{1}{2} R g_{lm} = - \varkappa T_{lm}\,, \tag{1}$$

where R_{lm} is the Ricci tensor of the normal hyperbolic four-dimensional space-time metric.

Let B denote a spatial section of Ω (the local equation of B we take as $x^4 = 0$). Moreover, let S denote the boundary of Ω (the local equation of S we take as $f(x^1 x^2 x^3 x^4) = 0$). We define

$$N_m = \frac{f,_m}{\sqrt{g^{ij} f,_i f,_j}} \qquad N^m = g^{mk} N_k .$$

Then, as a consequence of the definitions, $N^m N_m > 0$ on S. The initial and boundary value problem for Eq. (1) consists in the determination of all those solutions of the Einstein equations consistent with the topological requirements stated above, and assuming definite values on B and on S respectively. These values, however, cannot be arbitrary: in fact, Eq. (1) and the regularity of the metric imply

$$G_i^{\,4} + \varkappa T_i^{\,4} = 0 \quad \text{on} \quad B , \tag{4}$$

$$(G_{lm} + \varkappa T_{lm}) N^m = 0 \quad \text{on} \quad S , \tag{5}$$

since the left-hand-sides of Eqs. (4) and (5) depend only on the *first* normal derivatives of the metric tensor on B and on S respectively.

Equation (4) represent four consistency conditions for the Cauchy data on B. They are known as "the problem of initial conditions" [1–4]. Similarly, Eq. (5) are consistency conditions for the data on S. In this case, however, since the latter do not imply the knowledge of both g_{lm} and $g_{lm,k}$, Eq. (5) cannot be considered as relations among a-priori known quantities. Therefore, they may simply be included among the given boundary conditions: this, of course, restricts the number of independent quantities which may be freely prescribed on S.

We shall now prove that, once the consistency conditions (4) and (5) have been imposed on the boundaries of the domain Ω, Eq. (1) may be replaced by the equivalent system

$$T_l^{\,m} /\!/_m = 0 , \tag{2}$$

$$R_{\alpha\beta} = -\varkappa \left(T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}\right) \tag{3}$$

provided that $g^{44} \neq 0$ in Ω.

This result provides the stated extension of LICHNEROWICZ's method to mixed initial and boundary value problems.

§ 2

In order to prove the equivalence between Eq. (1) and the system (2), (3) we shall extend a procedure indicated by SYNGE (see Ref. [5], p. 212).

Let W_{ij} be any symmetric tensor field in space-time with metric tensor g_{ij}. We define the conjugate tensor field by

$$W^*_{ij} = W_{ij} - \frac{1}{2} W g_{ij}, \qquad W = g^{lm} W_{lm}, \tag{6}$$

definition (6) implies

$$(W^*_{ij})^* = W_{ij}. \tag{6'}$$

We have the following

Lemma 1. *Provided* $g^{44} \neq 0$, *the mixed components* W_j^i *may always be expressed in terms of* $W^*_{\alpha\beta}$ *and* W_k^4 *in the linear form*

$$W_j^i = A_j^{i\alpha\beta} W^*_{\alpha\beta} + B_j^{ik} W_k^4 \tag{7}$$

the coefficients being linear and quadratic functions of g^{lm}, *divided by* g^{44}. *In particular*

$$B_j^{ik} = \frac{1}{g^{44}} [g^{i4} \delta_j^k - g^{4k} \delta_j^i + g^{ik} \delta_j^4]. \tag{8}$$

The proof of (7) is given in Ref. [5], and will be omitted here. The explicit form (8) for the coefficients B_j^{ik} is shown in the Appendix.

Lemma 2. *Provided* $g^{44} \neq 0$ *in* Ω, *the following statements are mathematically equivalent*

$$(\alpha)\ W_{lm} = 0 \quad \text{in} \quad \Omega$$

$$(\beta) \begin{cases} W^*_{\alpha\beta} = 0 \\ W_l^m /\!/_m = 0 \end{cases} \text{in} \quad \Omega, \quad \text{with} \quad \begin{cases} W_i^4 = 0 & \text{on} \quad B \\ W_{lm} N^m = 0 & \text{on} \quad S. \end{cases}$$

Proof. Obviously $(\alpha) \Rightarrow (\beta)$. We want to prove that $(\beta) \Rightarrow (\alpha)$. We assume (β). Then, by Lemma 1 and the first condition in (β)

$$W_j^i = B_j^{ik} W_k^4. \tag{9}$$

Moreover, by the second condition in (β)

$$W_j^i /\!/_i = W_j^i{}_{,i} + \Gamma_{il}^i W_j^l - \Gamma_{ij}^l W_l^i = 0. \tag{10}$$

By (9), this may be written in the form

$$B_j^{ik} W_k^4{}_{,i} + E_j^k W_k^4 = 0 \tag{11}$$

where the coefficients E_j^k depend only on the metric tensor and its first derivatives.

We have in (11) a system of four linear homogeneous differential equations of the first order for the four components W_j^4. The coefficients B_j^{ik} and E_j^k are by hypothesis continuous in the region considered, as well as across the hypersurfaces B and S. The last two conditions in (β) provide the initial data

$$W_j^4 = 0 \quad \text{on} \quad B \tag{12}$$

and the boundary conditions [see (9)]

$$B_j^{ik}\, W_k^4\, N_i = 0 \quad \text{on} \quad S\,. \tag{13}$$

It remains only to prove that the initial and boundary value problem (11), (12), (13) admits the unique solution $W_j^4 = 0$ in Ω. In fact, by (9), this implies $W_{ji} = 0$, so that the equivalence between (β) and (α) is proved.

To this purpose, we first reduce the system (11) to a symmetric hyperbolic system (see Ref. [8], p. 593; see also [9–11]): we introduce the symmetric matrix

$$U^{jl} = g^{jl} - 2\,\frac{g^{j4}\, g^{l4}}{g^{44}}\,. \tag{14}$$

The matrix (14) is clearly non singular, since the quadratic form

$$\sum_{jl} U^{jl}\,\lambda_j\lambda_l = \left(g^{jl} - \frac{2g^{j4}\,g^{l4}}{g^{44}}\right)\lambda_j\lambda_l = \left(g^{\alpha\beta} - \frac{g^{\alpha 4}\,g^{\beta 4}}{g^{44}}\right)\lambda_\alpha\lambda_\beta + \left(\frac{g^{j4}\,\lambda_j}{\sqrt{-g^{44}}}\right)^2$$

is positive definite $\Big($notice that the 3×3 matrix $\left\{g^{\alpha\beta} - \frac{g^{\alpha 4}\,g^{\beta 4}}{g^{44}}\right\}$ is the inverse of the matrix $\{g_{\alpha\beta}\}$, which is positive definite (see e.g. Ref. [6], p. 235)$\Big)$. Therefore, the system (11) is mathematically equivalent to the system

$$U^{jl}(B_l^{ik}\, W_{k\,,\,i}^{4} + E_l^k\, W_k^4) = 0\,. \tag{15}$$

By (8), (14) we have

$$U^{jl}\, B_l^{ik} = \frac{1}{g^{44}}\,(g^{i4}\,g^{jk} - g^{4k}\,g^{ji} - g^{ik}\,g^{j4})\,, \tag{16}$$

so that, for fixed i, the coefficients of the derivatives in (15) are symmetric in j and k.

Moreover, when $i = 4$, the matrix $U^{jl}\,G_l^{4k}$ is positive definite, since, by (16), we have

$$U^{jl}\, B_l^{4k} = \frac{1}{g^{44}}\,(g^{44}\,g^{jk} - 2g^{4k}\,g^{4j}) = U^{jk}\,. \tag{17}$$

Therefore, the system (15) is symmetric and hyperbolic. For such systems we have a general uniqueness theorem[2] which, in the actual case, reads

"the initial and boundary value problem for Eq. (15) possesses the unique solution $W_j^4 = 0$ provided that $W_j^4 = 0$ on B, and that the characteristic quadratic form

$$U^{jl}\, B_l^{ik}\, N_i\, W_j^4\, W_k^4 \tag{18}$$

be non negative on S".

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[2] See Ref. [8], p. 656–658. The proof of the uniqueness theorem is given there only in the particular case $\partial f/\partial x^4 = 0$ (where $f(x^1x^2x^3x^4) = 0$ is the equation of the hypersurface S). The same proof, however, may be trivially extended to the general case.

Both the conditions are fulfilled in view of (12) and (13) [with the quadratic form (18) identically zero on S]. This completes the proof of Lemma 2.

It is now quite simple to prove the equivalence between Eq. (1) and the system (2), (3) together with the consistency conditions (4) and (5).

To this purpose we set

$$W_{lm} = G_{lm} + \varkappa T_{lm} \tag{19}$$

and apply Lemma 2 to the symmetric tensor field (19). This completes the proof of LICHNEROWICZ's result also in the case of mixed initial and boundary value problems.

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Appendix

Proof of (8) § 2

We recall the following relations (see Ref. [5], Lemma 1, p. 213)

$$W^*_{\beta 4} = \frac{1}{g^{44}} [W_\beta{}^4 - g^{\alpha 4} W^*_{\alpha\beta}] , \tag{A, 1}$$

$$W^*_{44} = \frac{1}{g^{44}} [g^{\alpha\beta} W^*_{\alpha\beta} + 2 W_4{}^4] . \tag{A, 2}$$

We have [see (6′)]

$$\begin{aligned} W_j{}^i = W^*{}_j{}^i - \frac{1}{2} W^* \delta_j{}^i &= (g^{i\alpha} W^*_{\alpha\beta} + g^{i4} W^*_{4\beta}) \delta_j{}^\beta \\ &+ (g^{i\alpha} W^*_{\alpha 4} + g^{i4} W^*_{44}) \delta_j{}^4 \\ &- \frac{1}{2} \delta_j{}^i (g^{\alpha\beta} W^*_{\alpha\beta} + 2g^{\alpha 4} W^*_{\alpha 4} + g^{44} W^*_{44}) = C_j{}^{i\alpha\beta} W^*_{\alpha\beta} \\ &+ (g^{i4} \delta_j{}^\alpha + g^{i\alpha} \delta_j{}^4 - g^{\alpha 4} \delta_j{}^i) W^*_{\alpha 4} \\ &+ \left(g^{i4} \delta_j{}^4 - \frac{1}{2} g^{44} \delta_j{}^i\right) W^*_{44} , \end{aligned}$$

where the $C_j{}^{i\alpha\beta}$ are defined by the above relation. By (A, 1), (A, 2), this may be written in the form

$$\begin{aligned} W_j{}^i &= A_j{}^{i\alpha\beta} W^*_{\alpha\beta} \\ &\quad + \frac{1}{g^{44}} \{(g^{i4} \delta_j{}^\alpha + g^{i\alpha} \delta_j{}^4 - g^{\alpha 4} \delta_j{}^i) W_\alpha{}^4 + (2g^{i4} \delta_j{}^4 - g^{44} \delta_j{}^i) W_4{}^4\} \\ &= A_j{}^{i\alpha\beta} W^*_{\alpha\beta} + \frac{1}{g^{44}} [g^{i4} \delta_j{}^k + g^{ik} \delta_j{}^4 - g^{k4} \delta_j{}^i] W_k{}^4 , \end{aligned}$$

which coincides with (8), § 2.

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