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An Algebraic Spectrum Condition

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Abstract. It is shown that a translationally invariant algebra $\mathfrak{A}$ of local observables (see [1]) admits a representation in a Hilbert space having a vacuum state. Furthermore an algebraic criterion is given which is necessary and sufficient for the existence of at least one representation of $\mathfrak{A}$ in which the usual spectral condition for the energy-momentum operators holds.

I. Introduction

A local ring system [1] is defined by giving a B^*- algebra* on which a space-time translation x operates as a *-automorphism

$$A \to A_x , \quad A \in \mathfrak{A} , \tag{1}$$

and by assigning to each space-time domain $\mathcal{O}$ a Banach *-subalgebra $\mathfrak{A}(\mathcal{O})$ of $\mathfrak{A}$ such that

$$\mathfrak{A}(\mathcal{O}_1) \text{ commutes with } \mathfrak{A}(\mathcal{O}_2) \text{ if } \mathcal{O}_1 \text{ is spacelike to } \mathcal{O}_2 , \tag{2a}$$

$$[\mathfrak{A}(\mathcal{O})]_x = \mathfrak{A}(\mathcal{O} + x) , \tag{2b}$$

$$\text{if } \cup \mathcal{O}_n = \mathcal{O}, \cup \mathfrak{A}(\mathcal{O}_n) \text{ generates } \mathfrak{A}(\mathcal{O}); \overline{\cup \mathfrak{A}(\mathcal{O})} = \mathfrak{A}. \tag{2c}$$

The important work of Borchers, Haag and Schroer [2] showed that if there exists a Hilbert space representation of a local ring system in which the transformations (1) are induced by a unitary representation of the translation group

$$A_x = U(x)\, A\, U(x)^{-1} , \tag{3}$$

then there also exists such a representation with a vacuum state:

$$U(x)\, \Psi_0 = \Psi_0 . \tag{3'}$$

This representation satisfies the spectrum condition

$$P^2 > 0, \quad P_0 > 0 \qquad (\text{where } \exp[i P_\mu x^\mu] = U(x)) \tag{4}$$

if the initial one did [3].

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* Terminology of C. E. Rickart, *General Theory of Banach Algebras*. Naimark [4] uses the term "completely regular Banach ring".

A priori, it could happen that no representation has property (3). In the following we show that this is not the case, and give an algebraic condition for the existence of a representation with property (4).

II. Existence of the vacuum

We will not make full use of the structure (2); all of what follows holds for any B^* algebra $\mathfrak{A}$ having a locally compact Abelian group of automorphisms $\mathfrak{G}$; the role of the future light cone in (4) is more generally assumed by an arbitrary closed subset of the character group $\overline{\mathfrak{G}}$. For simplicity of language, however, we shall continue to refer only to the translation group.

Proposition I. $\mathfrak{A}$ *possesses an invariant state f (positive linear functional of norm 1):*

$$f(A) = f(A_x) \quad \text{for all} \quad A, x\,. \tag{5}$$

By the Gelfand-Segal construction* this functional gives rise to a representation

$$\mathfrak{A} \to \mathfrak{A}^{(f)}$$

with a cyclic vector $\Psi_0^{(f)}$ which satisfies

$$f(A) = (\Psi_0^{(f)}, A^{(f)} \Psi_0^{(f)})\,.$$

Equation (5) implies in a standard way that $\mathfrak{A}^{(f)}$ and $\Psi_0^{(f)}$ satisfy eq. (3) and (3′).

Proof: Proposition I is a straightforward consequence of the Markov-Kakutani theorem**: Let X be a locally convex topological vector space, and K a compact convex subset of X; if a commuting set of linear continuous transformations of X into itself leave K invariant, they have a common invariant element in K.

The state space $\mathfrak{P}(\mathfrak{A})$ of $\mathfrak{A}$ is a convex compact subset of the locally convex linear topological space $\mathfrak{X}$, the dual of the real Banach space $\mathfrak{A}_H$ of the Hermitian elements in $\mathfrak{A}$, furnished with the weak topology. A neighbourhood basis for this topology is defined as follows: given

$$A_1, \ldots, A_n \in \mathfrak{A}, \varepsilon > 0, \quad f_0 \in \mathfrak{X}\,,$$

$$U(f_0; A_1, \ldots, A_n; \varepsilon) = \{f : f \in \mathfrak{X}, |f(A_i) - f_0(A_i)| < \varepsilon, i = 1, \ldots, n\}\,.$$

The linear transformations of $\mathfrak{X}$ given by

$$T_x f = f_x\,, \quad f_x(A) = f(A_x)$$

are obviously weakly continuous, leave $\mathfrak{P}(\mathfrak{A})$ invariant, and commute with one another. Under these conditions, the Markov-Kakutani theorem says that there exists in $\mathfrak{P}(\mathfrak{A})$ an invariant element:

$$f \in \mathfrak{P}(\mathfrak{A}), \quad T_x f = f \quad \text{for all} \quad x\,.$$

Thus there is always a vacuum representation.

* Ref. [4], § 17, n. 3.

** Ref. [5], Part I, V. 10.6.

III. Spectral conditions

Let us consider functions $z(x)$ which are summable and whose Fourier transforms have their support outside of the future light cone:

$$\mathrm{supp}\,\tilde{z} \cap \overline{\mathscr{V}^+} = \emptyset\,, \qquad z \in \mathscr{L}^1(d^4x)\,. \tag{6}$$

Such functions exist in any locally compact Abelian group, and can be chosen to have a Fourier transform vanishing outside of a given open set (in the character group) and equal to 1 on a given compact set contained in it*.

If z satisfies (6) and $A \in \mathfrak{A}$ we define

$$A_z = \int A_x z(x)\, d^4x\,,$$

which is again an element of $\mathfrak{A}$ since z is summable and $\|A_x\| = \|A\|$**.

Let us define

$$\mathfrak{B} = \{A_z : A \in \mathfrak{A}\,,\quad z \text{ satisfies } (6)\}\,. \tag{7}$$

The minimal left ideal containing $\mathfrak{B}$, i.e. the linear span of $\mathfrak{A}\mathfrak{B}$ will be denoted by $\mathfrak{J}_l$.

Proposition II. *$\mathfrak{A}$ possesses a representation $\mathfrak{A} \to \mathfrak{A}^{(f)}$, with $f \in \mathfrak{P}(\mathfrak{A})$ and $f = f_x$, such that the representation of the translation group*

$$U^{(f)}(x)\, A^{(f)} \Psi_0^{(f)} = A_x^{(f)} \Psi_0^{(f)}$$

satisfies the spectrum condition (4), *i.e.*

$$U^{(f)}(x) = \int_{\overline{\mathscr{V}^+}} e^{ipx}\, dE^{(f)}(p)\,,$$

if and only if $\mathfrak{J}_l \neq \mathfrak{A}$.

If $\mathfrak{A}$ has the local ring structure (2), f can be chosen such that $\mathfrak{A}^{(f)}$ is irreducible and $\Psi_0^{(f)}$ is the only vacuum state in the representation space $\mathfrak{H}^{(f)}$.

Proof: Suppose that there exists a state f with the desired properties; then, if $B = A_z \in \mathfrak{B}$, we have

$$\begin{aligned} B^{(f)} \Psi_0^{(f)} &= A_z^{(f)} \Psi_0^{(f)} \\ &= \int z(x)\, A_x^{(f)}\, d^4x\, \Psi_0^{(f)} \\ &= \int z(x)\, U^{(f)}(x)\, d^4x\, A^{(f)} \Psi_0^{(f)} \\ &= E^{(f)}(\tilde{z})\, A^{(f)} \Psi_0^{(f)} = 0\,, \end{aligned}$$

all the steps being allowed since every representation of a Banach algebra $\mathfrak{A} \to \mathfrak{A}^{(f)}$ is continuous:

$$\|A^{(f)}\| \leqq \|A\|\,^{***}\,.$$

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* Ref. [4], § 31, n. 5.

** Because a *-isomorphism of a B^*-ring into a B^*-ring is isometric: see ref. [4], § 24, Th. 3.

*** Ref. [4], § 17, Th. 1.

Thus we get

$$f(B^* B) = \|B^{(f)} \Psi_0^{(f)}\|^2 = 0 . \tag{8}$$

The same argument says that equation (8) holds for $B \in \mathfrak{J}_l$ and by continuity for B in the closure $\overline{\mathfrak{J}}_l$. Thus $\overline{\mathfrak{J}}_l$ is contained in the left kernel of f and therefore, by Schwartz's inequality $f(B) = 0$ for $B \in \overline{\mathfrak{J}}_l$. Since f is a state, i.e. a non-zero functional by hypothesis, $\overline{\mathfrak{J}}_l \neq \mathfrak{A}$, or, what amounts to the same thing*,

$$\mathfrak{J}_l \neq \mathfrak{A} .$$

Conversely, suppose that $\mathfrak{J}_l \neq \mathfrak{A}$; then* $\overline{\mathfrak{J}}_l \neq \mathfrak{A}$; the set

$$\mathfrak{P}_s(\mathfrak{A}) = \{f : f \in \mathfrak{P}(\mathfrak{A}), f(A^* A) = 0 \quad \text{if} \quad A \in \overline{\mathfrak{J}}_l\}$$

is nonvoid; it is evidently convex and closed in the weak topology, since in this topology

$$f \to f_0 \quad \text{means} \quad f(A) \to f_0(A) \quad \text{for each } A .$$

Thus it is a compact subset of $\mathfrak{X}$. Now $\mathfrak{J}_l$ (and thus $\overline{\mathfrak{J}}_l$) is invariant under translations,

$$A \in \mathfrak{J}_l \Rightarrow A_x \in \mathfrak{J}_l ,$$

so that $\mathfrak{P}_s(\mathfrak{A})$ is invariant under the transformations T_x:

$$f \in \mathfrak{P}_s(\mathfrak{A}) \Rightarrow T_x f \in \mathfrak{P}_s(\mathfrak{A}) .$$

The same trivial application of the Markov-Kakutani theorem used in the proof of Proposition I guarantees that $\mathfrak{P}_s(\mathfrak{A})$ has an invariant element f:

$$T_x f = f \qquad \text{for all } x .$$

The construction used above then shows:

$$E^{(f)}(\tilde{z}) = 0 \quad \text{for all } z \text{ satisfying (6)} , \tag{9}$$

i.e. $U^{(f)}(x)$ satisfies the spectrum condition (4).

If $\mathfrak{A}$ possesses a local ring structure (2), theorems of REEH and SCHLIEDER [6] and BORCHERS [7] (which can be simply translated into local ring language [8]) allow us to decompose f

$$f(A) = \int_{\mathfrak{P}_s(\mathfrak{A})} f'(A)\, d\mu(f')$$

in such a way that each f' satisfies the spectrum condition (9) and, in the representation space $\mathfrak{H}^{(f')}$, $\Psi_0^{(f')}$ is the only vacuum state vector, and $\mathfrak{A}^{(f')}$ is irreducible.

Proposition III. *If* $\mathfrak{A}$ *is simple and has the local ring structure* (2), *and if* $\mathfrak{J}_l \neq \mathfrak{A}$, *then*

$$\overline{\mathfrak{J}}_l \cap \mathfrak{A}(\mathcal{O}) = \{0\} \tag{10}$$

for each bounded region $\mathcal{O}$.

Proof: According to Proposition II there exists a representation $\mathfrak{A}^{(f)}$ in which the spectrum condition holds and in which there is a cyclic vacuum. In this representation the Reeh-Schlieder theorem is valid, and

* Ref. [4], § 8, IV, n. 3.

so the vacuum is a separating vector for $\mathfrak{A}(\mathcal{O})$ if $\mathcal{O}$ is any bounded region. As shown before $B \in \overline{\mathfrak{J}}_l$ implies $B^{(f)} \Psi_0^{(f)} = 0$. Thus $B \in \overline{\mathfrak{J}}_l \cap \mathfrak{A}(\mathcal{O})$ implies $B^{(f)} = 0$. If $\mathfrak{A}$ is simple this means $B = 0$.

IV. Remarks

This work is incomplete in two respects. First, it deserves further investigation to see whether the "spectral condition"

$$\mathfrak{J}_l \neq \mathfrak{A}$$

is automatically satisfied in the local ring case. For instance, it could happen that equation (9) follows from the axioms (2).

This would be desirable since it would throw a bridge between locality and the spectrum condition, principles which are strictly related in classical physics.

Second, nothing is said in this approach about the separability of the Hilbert spaces underlying the representations in question.

Finally, if $\mathfrak{A}$ is not simple, it might happen that some representation obtained by Proposition II is trivial; for instance, this occurs if $\mathfrak{J}_l$ is a two-sided ideal in $\mathfrak{A}$. In this case each element $A \in \mathfrak{A}/\mathfrak{J}_l$ has its spectrum in $\overline{\mathscr{V}^+}$, whereas the spectrum of A^* is also contained in $\overline{\mathscr{V}^+}$; hence the spectrum of A is contained in $\overline{\mathscr{V}^+} \cap \overline{\mathscr{V}^-} = \{0\}$. Thus each element in $\mathfrak{A}/\mathfrak{J}_l$ is invariant, and by local commutativity $\mathfrak{A}/\mathfrak{J}_l$ is Abelian*.

To avoid this one should require:

$\mathfrak{J}_l$ is not contained in any proper two-sided ideal, i.e.

$$\text{linear span } \mathfrak{A}\mathfrak{B}\mathfrak{A} = \mathfrak{A}\,.$$

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* This remark was suggested by H. J. Borchers.