Communications in **M**athematical **A**nalysis
Volume 17, Number 2, pp. 311–337 (2014) www.math-res-pub.org/cma
ISSN 1938-9787

Parabolic Singular Integrals on Ahlfors Regular Sets

Jorge Rivera-Noriega*
Facultad de Ciencias
Universidad Autónoma del Estado de Morelos
Av. Universidad 1001 Col. Chamilpa
Cuernavaca Mor. CP 62209

(Communicated by Vladimir Rabinovich)

Abstract

We present a survey of recent developments on a parabolic version of uniform rectifiability and parabolic singular integrals. In particular we describe some ideas to prove the equivalence between the parabolic uniform rectifiability of a set E and the $L^2(E)$ boundedness of a class of Calderón-Zygmund integrals of parabolic type. We also describe a result on compactness of certain parabolic singular integrals, as well as some related open problems and conjectures.

AMS Subject Classification: 28A75, 42B20, 42B25, 35K08

Keywords: Parabolic uniform rectifiability, parabolic singular integrals, big pieces of parabolic Lipschitz graphs, parabolic Corona type decompositions, compact parabolic singular integrals

The topics contained in this survey article may be viewed as extensions to more general settings of Calderón-Zygmund's theory of singular integrals. In the first part we briefly describe some basic facts on singular integral operators and uniform rectifiability. We refer the reader to [6] and to the introductory remarks in [16, 17] for more details. A recent enlightening introduction to the topic is in the unpublished manuscript [13]. The second part contains adaptations of definitions and the description of some recent results in the parabolic setting.

Needless to say, this work does not aim to be an exhaustive account of the theory, but rather a description that allows one to get to the definitions and results in the parabolic setting, including some of its motivations.

In particular, we do not include the discussion of the Cauchy integral over a 1-dimensional set, in which there is some special features and a better understanding of certain aspects (see e.g. [47]).

*E-mail address: rnoriega@uaem.mx

Part I
Brief review of the standard case

The topics in this introductory part are rather old, and we apologize in advance for reviewing what it could be a well known set of ideas and topics for some readers. The point of presenting them in this section is for purposes of motivating the parabolic analogues of definitions and ideas. We also mention a couple of recent fundamental results and pose a couple of important conjectures in the area.

1 Basic definitions and results

1.1 Singular Integrals.

As is well known, the basic example of a singular integral operator is the *Hilbert transform* defined by the principal value integral:

$$Hf(x) = \frac{1}{\pi}\,\mathrm{pv}\int \frac{f(y)}{x-y}\,dy = \frac{1}{\pi}\lim_{\epsilon\to 0}\int_{|x-y|>\epsilon} \frac{f(y)}{x-y}\,dy. \tag{1.1}$$

To obtain n-dimensional extension of this operator one considers *Riesz transforms*

$$R_j f(x) = \lim_{\epsilon\to 0}\int_{|x-y|>\epsilon} \frac{x_j-y_j}{|x-y|^{n+1}}\,f(y)\,dy, \quad 1\le j\le n.$$

Both operators may be viewed as convolution principal value type operators of the form

$$\mathcal{T} f(x) = \mathrm{pv}\int K(x-y)f(y)dy \tag{1.2}$$

with an odd kernel K. Moreover, the kernel satisfies the following estimates

$$|K(x)| \le \frac{C}{|x|^n} \tag{1.3}$$

$$|K(x)-K(y)| \le \frac{C\,|x-y|^{\alpha}}{|x|^{n+\alpha}} \quad \text{for } |x|>2|x-y|. \tag{1.4}$$

In order to pose the Main Topic of this note, we would like to consider a generalization of previously defined objects, defined this time on the graph of a Lipschitz function.

Let $\Gamma = \{(x,A(x)) : x\in\mathbb{R}\}$ be the graph of a function $A:\mathbb{R}\to\mathbb{R}$ satisfying a *Lipschitz condition* of the form $|A(x)-A(y)|\le M|x-y|$, for a uniform constant $M>0$. Consider the *Cauchy integral over* Γ

$$Cf(z) = \int_\Gamma \frac{f(w)}{z-w}\,dw \quad \text{for } z\in\Gamma. \tag{1.5}$$

Using graph coordinates we have $z = x+iA(x)$, $w = y+iA(y)$, $dw = (1+iA'(y))\,dy$, hence

$$C_A\tilde{f}(x) = \int_{\mathbb{R}} \frac{1}{x-y+i(A(x)-A(y))}\tilde{f}(y)\,dy$$

where $\tilde{f}(y) = f(y + iA(y))(1 + iA'(y))$.

Notice that this operator is not of convolution type, yet the resemblance of (1.5) with (1.2) and the fact that (1.5) seems to be an appropriate generalization of the Hilbert transform (1.1) make us want to consider operators such as (1.5).

In fact, a problem originally posed by A. P. Calderón, motivated by a variety of questions related to partial differential equations (see [2]), was to decide whether Cauchy integral (1.5) is $L^p(\Gamma)$ bounded, $1 < p < \infty$. After a partial answer was provided by Calderón himself [2], this problem was settled with more generality in [7] (see also [3, 39] and references therein).

From this, and given the well known connection between Lipschitz functions and the so called *rectifibility* of a set, (see e.g. [46]), the problem of extending these results to a class of rectifiable sets seemed the natural next step in this theory. This is the idea pursued and then reported mainly in the fundamental works [16, 17]. And our aim is to describe some variations and recent developments in the parabolic setting.

In a short phrase one can make a rough description of the motivation of this type of problems as follows:

> *To find sets $E \subset \mathbb{R}^n$ of integer dimension $d < n$ on which one can define Cauchy integral and its generalizations, and where the L^2 boundedness of these operators hold.*

In the rest of this section we will quickly describe some details of this statement.

1.2 Ahlfors-David regular sets

Let us start recalling the definition of Hausdorff measure and dimension. More detailed accounts can be found for instance in [46]. Given a set $E \subset \mathbb{R}^n$, $n > 2$, $\epsilon > 0$ and $s > 0$ define

$$H^s_\epsilon(E) = \inf \sum_{A \in \mathcal{A}} [\operatorname{diam} A]^s$$

where the infimum is taken over countable covers of E with sets of diameter $< \epsilon$. The *s-dimensional Hausdorff measure* of E is given by

$$H^s(E) = \lim_{\epsilon \to 0} H^s_\epsilon(E).$$

Fixing $E \subset \mathbb{R}^n$ there exists a unique $s_0 \in [0, \infty]$ satisfying

$$H^s(E) = \infty \quad \text{if } s < s_0, \qquad H^s(E) = 0 \quad \text{if } s > s_0.$$

Hausdorff dimension is then defined as this critical value. This value coincides with our intuition when s is a positive integer. Moreover, a basic result is that in $\mathbb{R}^n$ the Lebesgue measure and the n dimensional Hausdorff measure coincide. Intuitively, s-dimensional Hausdorff measure "recongnizes" and measures correctly every s-dimensional set; it takes an infinite value on sets of dimension less than s and it vanishes on sets of dimension greater to s.

Consider now a closed set $E \subset \mathbb{R}^n$ with Hausdorff dimension $d \in \mathbb{Z}$, $0 < d \leq n$, and suppose that d dimensional Hausdorff measure H^d is locally finite when restricted to E. We also adopt the notation dy when integrating with respect to $dH^d|_E$, the restriction of H^d to E. Also, for a set $A \subseteq \mathbb{R}^n$ we write $|A|$ to denote $H^d(A)$.

Define singular integral operators described as a *principal value integral* of the form

$$Tf(x) = \mathrm{pv} \int_E K(x-y) f(y) dy \tag{1.6}$$

where $K \in C^\infty(\mathbb{R}^n \setminus \{\vec{0}\})$ is odd and it satisfies

$$|\nabla^j K(x)| \leq \frac{C(j)}{|x|^{d+j}} \quad j = 0, 1, 2, \ldots.$$

Call $\mathcal{K}_d(\mathbb{R}^n)$ the family of all such kernels. Note that the kernels given by

$$K_j f(x) = \frac{x_j}{|x|^{d+1}} \quad j = 1, 2, \ldots, n. \tag{1.7}$$

are all in $\mathcal{K}_d(\mathbb{R}^n)$. And if $d = n-1$, which means we have the *codimension 1 case*, then all of the kernels associated to the Riesz transforms are in $\mathcal{K}_d(\mathbb{R}^n)$.

Fundamental Question. *Find geometric and analytic conditions on E so that the following estimate holds*

$$\sup_{\epsilon>0} \int_E \left| \int_{E\cap\{|x-y|>\epsilon\}} K(x-y) f(y)\, dy \right|^2 dx \leq C(K) \int_E |f|^2 dx \quad \textit{for every } f \in L^2(E). \tag{1.8}$$

The estimate (1.8) provides a way to obtain $L^2(E)$ boundedness of the operator T without proving the existence of the limit defining the principal value integral in (1.6). To be more precise, consider the *truncated operators*

$$T_\epsilon f(x) = \int_{E\cap\{|x-y|>\epsilon\}} K(x-y) f(y)\, dy \quad \text{for } \epsilon > 0, \tag{1.9}$$

deifined, say, for $f \in C(E)$. In (1.8) it is implicitly required that $T_\epsilon f$ can be uniformly bounded on $\epsilon > 0$ so as to be extended to $L^2(E)$.

The set $E \subset \mathbb{R}^n$ is *Ahlfors-David regular* if it is closed and it satisfies

$$\frac{1}{C} R^d \leq |E \cap B_R(x)| \leq C R^d \tag{1.10}$$

for every $x \in E$, $R > 0$, and with a constant C independent of x and R.

As a *preliminary definition*, we say that E is *uniformly rectifiable* if it is Ahlfors-David regular and (1.8) holds *for every* $K \in \mathcal{K}_d(\mathbb{R}^n)$.

The next conjecture is one of the main questions in this area. It has been solved only if E has codimension 1 by F. Nazarov, X. Tolsa and A. Volberg [48]. The general case remains an open question.

Conjecture 1. *An Ahlfors-David regular set E is uniformly rectifiable if and only if the estimate* (1.8) *hods for all of the kernels* (1.7).

Now we carry on with the description of some features of Ahlfors-David sets. First we recall (see [12, Part III Proposition 1.4]) that the right hand side of (1.10) is actually a consequence of the fact that (1.8) holds for the kernels in (1.7).

The left hand side of (1.10) is not only a non-degeneracy condition, but it also implies the doubling property of H^d on E. Recall that a measure μ defined on Borel sets of E is a *doubling measure on E* if it is finite on compact sets relative to E, and there exists $K_1 > 0$ such that for $x \in E$ and every $r > 0$ one has $\mu(B_{2r}(x)) \leq K_1 \mu(B_r(x))$.

This way, Ahlfors-David regular sets equipped with Hausdorff measure H^d and euclidean distance form a *space of homogeneous type* in the sense of [8]. As is well known, (see e.g. [56]) Calderón-Zygmund theory for singular integrals is applicable on this generality.

In the rest of the section we will describe some conditions which turn out to be equivalent to the uniform rectifiability.

1.3 Big pieces of Lipschitz graphs and images

We say that $\mathcal{G} \subset \mathbb{R}^n$ is a *graph* if there exists a d dimensional hyperplane P, an $(n-d)$-dimensional hyperplane $P^\perp$, which is orthogonal to P, and a function $A : P \to P^\perp$ such that $\mathcal{G} = \{p + A(p) : p \in P\}$. In this case we write $\mathcal{G} = \mathcal{G}(A)$.

An Ahlfors-David regular set E contains *big pieces of Lipschitz graphs*, and we will write $E \in BPLG$, if there exist constants $C_1 > 1$ and $\theta > 0$ such that for every $x \in E$ and every $0 < r < R$ there exists a graph $\mathcal{G}(A)$, with $\|\nabla A\|_\infty \leq C_1$, such that

$$|E \cap B_r(x) \cap \mathcal{G}| \geq \theta r^d. \tag{1.11}$$

Sometimes the notation $E \in BPLG(C_1, \theta)$ is adopted to show explicitly the dependance on the constants involved.

The notion of Big Pieces of Lipschitz Graphs was a first attempt to provide an answer to the Fundamental Question posed above, and in fact if $E \in BPLG$ then E is uniformly rectifiable [11, Part III Section 3]. A particular example of a set E with property BPLG is addressed in [14]. On the other hand, according to [17, p. 16], there is an example of a set which is uniformly rectifiable but it does not satisfy the BPLG property. This is an unpublished result that appears as an exercise in [13].

Despite this negative result, there is a recent characterization of uniform rectifiability using a variant of the "big pieces of Lipschitz graphs" property that we now describe.

An Ahlfors-David regular set $E \subset \mathbb{R}^n$ is in the class $(BP)^2LG$, and we say that *E contains big pieces of big pieces of Lipschitz graphs*, if there exist constants $C_1 > 0$, $\theta > 0$ and $\alpha > 0$ such that for any ball B centered on E, there exists an Ahlfors-David regular set $F \subset \mathbb{R}^n$ such that

(a) $|E \cap F \cap B| \geq \alpha |E \cap B|$,

(b) F is an Ahlfors-David regular set with constant less than or equal to C_1,

(c) $F \in BPLG(C_1, \theta)$.

Having recalled this definition, we note that in [1] it has been recently established that an Ahlfors-David regular set E is uniformly rectifiable if and only if E is in the class $(BP)^2LG$.

In the search of more general conditions that may be equivalent to uniform rectifiability, the following definition was introduced. This is often adopted as the condition that defines uniform rectifiability for its geometric appearance.

A set $E \subset \mathbb{R}^n$ is in the class *BPBI* if E is Ahlfors-David regular and there exist constants $C_1 > 1$ and $\theta > 0$ such that for every ball centered on any point in E one can find a compact set $A \subset \mathbb{R}^d$ and a bilipschitz function $\psi : A \to \mathbb{R}^n$, i.e.

$$C_1^{-1}|x-y| \leq |\psi(x)-\psi(y)| \leq C_1|x-y|, \quad \text{es decir que } \psi \text{ es } \textit{bilipschitz}; \tag{1.12}$$

and such that $|E \cap \psi(A) \cap B| \geq \theta |E \cap B|$. In this case one says that *E has big pieces of bilipschitz images* (thus the acronym *BPBI*).

1.4 Two more equivalent conditions

We now recall yet another geometric condition that is equivalent to uniform rectifiability. First recall that a measure μ defined on Borel sets of $E \times (0,\infty)$ is *a Carleson measure* if $\mu(T(B')) \leq C|B'|$, where $B' = B \cap E$ is a *surface ball with center on E* and $T(B')$ denotes the *Carleson region associated to* B', defined as $T(B') = B' \times (0, \operatorname{rad} B')$ (here $\operatorname{rad} B'$ denotes the radius of B').

Let E be a d-dimensional Ahlfors-David regular set, and let $1 \leq q < \infty$, $x \in E$ and $t > 0$ be given. Define

$$\beta_q(x,t) = \inf_P \left\{ \frac{1}{t^d} \int_{E\cap\{|x-y|<t\}} \left[\frac{d(y,P)}{t} \right]^q dy \right\}^{1/q}$$

where the infimum is taken over d-dimensional planes P in $\mathbb{R}^n$. Define also

$$\beta_\infty(x,t) = \inf_P \left[\sup \left\{ \frac{d(y,P)}{t} : y \in E, \quad |x-y| < t \right\} \right]$$

where the same kind of infimum is taken. The idea of using functionals as β_q, $1 \leq q \leq \infty$, in connection with *rectifiability* appears in [38].

Theorem 1.1. *[16] Let E be an Ahlfors-David regular set. Then E is uniformly rectifiable if and only if* $|\beta_2(x,t)|^2 dxdt/t$ *defines a Carleson measure on* $E \times (0,\infty)$.

For the next definitions, recall that $E \subset \mathbb{R}^n$ is an Ahlfors-David regular set with Hausdorff dimension $d \in \mathbb{Z}$, $0 < d \leq n$. It is possible (cf. [12, Appendix 1]) to construct a family of partitions Δ_j of E, for each $j \in \mathbb{Z}$, that will make the job of a *dyadic decomposition* of E, in the sense that the following conditions hold:

(CD1) If $j \leq k$, $Q \in \Delta_j$, $Q' \in \Delta_k$, then $Q \cap Q' = \emptyset$, or else $Q \subseteq Q'$;

(CD2) If $Q \in \Delta_j$ then there exists a constant $C > 1$ such that $C^{-1}2^j \leq \operatorname{diam} Q \leq C2^j$ and $C^{-1}2^{jd} \leq |Q| \leq C2^{jd}$.

Denote by $\Delta = \bigcup_j \Delta_j$ the union of all the partitions, and call the elements of Δ *dyadic cubes of E*. This way, if Q is a dyadic cube and $\lambda > 1$ we can define $\lambda Q = \{x \in E : d(x,Q) \leq (\lambda - 1) \operatorname{diam} Q\}$.

Given $Q \in \Delta$, say $Q \in \Delta_j$, the *descendants of* Q are the cubes $Q' \in \Delta_{j-1}$ such that $Q' \subset Q$. Assuming that $Q \subset \tilde{Q}$ for certain $\tilde{Q} \in \Delta_{j-1}$, the *siblings of* Q are all the descendants of $\tilde{Q}$. The family of all the siblings of Q is denoted by $\varsigma(Q)$.

A family $\mathcal{F}$ of elements of Δ is *coherent* if every $S \in \mathcal{F}$ has a maximal element $Q(S) \in \Delta$ that contains every element of S and it satisfies:

- $Q' \in S$ provided that $Q \subset Q' \subset Q(S)$ and $Q \in S$;
- if $Q \in S$ then, either all of their children of are in S, or neither of them is.

A family of dyadic cubes $\mathcal{A} \subset \Delta$ satisfies a *Carleson packing condition* if there exists $C > 0$ such that for every $Q \in \Delta$

$$\sum_{\substack{Q' \in \mathcal{A} \\ Q' \subset Q}} |Q'| \leq C|Q|. \tag{1.13}$$

Let E be an Ahlfors-David regular set. We say that *E admits a Corona type decomposition* if for every $\eta > 0$ there exists $C = C(\eta) > 0$ such that one can partition Δ in two families of sets $\mathcal{G}$ and $\mathcal{B}$ with the following properties:

(Cor1) $\mathcal{B}$ satisfies a Carleson packing condition with constant C;

(Cor2) $\mathcal{G}$ can be partitioned in a family $\mathcal{F}$ of subsets of S of elements in $\mathcal{G}$, in such a way that $\mathcal{F}$ is a coherent family;

(Cor3) The maximal cubes denoted by $Q(S)$, for $S \in \mathcal{F}$, satisfy a Carleson packing condition;

(Cor4) Given $S \in \mathcal{F}$ there exists a d-dimensional Lipschitz graph Γ with constant less than η, and such that for $Q \in S$ one has

$$d(x,\Gamma) \leq \eta \operatorname{diam} Q \quad \text{for } x \in E \text{ satisfying } d(x,Q) \leq \operatorname{diam} Q$$

Following [16] we refer to the $S \in \mathcal{F}$ as the *stopping-time regions*, since they are usually constructed through algorithms using stopping time arguments. The triple $(\mathcal{B},\mathcal{G},\mathcal{F})$ satisfying (Cor1)-(Cor4) is called a *coronization of E* (see [17, p.55]).

Theorem 1.2. *[16] Let E be an Ahlfors-David regular set. Then E is uniformly rectifiable if and only if E admits a Corona type decomposition.*

Part II

The parabolic case

The problems of parabolic type that motivate the development of the following results arise from questions related to Dirichlet type problems associated to the heat equation. This kind of questions go back at least to work of E. Fabes and collaborators, where layer potentials and parabolic singular integrals were considered (see e.g. [19, 24, 23, 27, 25, 22, 26]), and subsequent developments using caloric measure (see e.g. [30, 20, 21, 28, 53, 29, 54]).

The first idea is to equip the Euclidean space $\mathbb{R}^{n+1}$ with a *parabolic homogeneity*, in which the *parabolic dimension* is $n+2$. (A more thorough explanation is in [4, 5]). This *non-isotropic* homogeneity of $\mathbb{R}^{n+1}$ is reflected in the fact that any *parabolic cylinder of radius* $r > 0$ should be defined as $B_r \times (0, r^2)$, where B_r is a Euclidean n-dimensional ball of radius $r > 0$. This in particular implies that the *measure* of a parabolic cylinder of radius $r > 0$ is of the order of r^{n+2}.

On the other hand, the basic theory described in the first part considers d-dimensional sets, with $1 \le d < n$, embedded into $\mathbb{R}^n$. For our Euclidean space $\mathbb{R}^{n+1}$ endowed with the parabolic homogeneity we have only considered surfaces in $\mathbb{R}^{n+1}$ of codimension 1. Another component of this parabolic theory is that the surfaces separate $\mathbb{R}^{n+1}$ in exactly two components, and this restriction is never mentioned in the standard theory described in the first part.

The notion of parabolic uniformly rectifiable sets have been introduced in [35, 36] in order to generalize results from [41]. However, in those works the parabolic version of the Fundamental Question of Part 1 was never addressed. We will emphasize the fact that the appropriate substitute for Lipschitz functions have a particular regularity that allows one to have the result on boundedness of parabolic singular integrals from [44, 31, 32, 33, 34].

The main result described in this second part (see Theorem 2.1) appeared in [51], and it establishes the equivalence of the uniform rectifiability in the parabolic sense with the L^2-boundedness of certain parabolic singular integrals over parabolic Ahlfors-David regular sets. A result of compactness of parabolic singular integrals (Theorem 5.1, see [52]) is briefly described in the last section.

2 Basic definitions

2.1 Uniformly rectifiable sets in the parabolic sense

For $(X,t) \in \mathbb{R}^n \times \mathbb{R}$ we denote by $C_r(X,t)$ the *cylinder of radius* $r > 0$ *centered at* (X,t) defined by $C_r(X,t) = \{(Y,s) \in \mathbb{R}^{n+1} : |X-Y| < r, |t-s| < r^2\}$. In some instances we denote points in $\mathbb{R}^{n+1}$ as $(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$, to stress that the variable x_0 is dependent of (x,t).

The *parabolic distance* between points (X,t) and $(Y,s) \in \mathbb{R}^{n+1}$ is defined as $d(X,t;Y,s) \equiv \|X-Y, t-s\| = |X-Y| + |t-s|^{1/2}$. The *parabolic norm* in $\mathbb{R}^{n+1}$ is given by $\|X,t\| = d(X,t;\vec{0})$. We can also talk about *parabolic distance* between sets E and F through

$$d(E,F) = \inf\{d(X,t;Y,s) : (X,t) \in E, (Y,s) \in F\}.$$

In this case E and F are either both in $\mathbb{R}^{n+1}$ or both in $\mathbb{R}^n$, and in either case they contain the variable t.

The *symmetric parabolic distance* between sets $K_1, K_2 \subset \mathbb{R}^{n+1}$ as described before is given by

$$D(K_1;K_2) = \sup_{(Y,s)\in K_1} d(Y,s;K_2) + \sup_{(Y,s)\in K_2} d(Y,s;K_1). \tag{2.1}$$

Given a Borel set $F \subset \mathbb{R}^{n+1}$, define the *surface measure of* F through the product measure $\sigma(F) = \int_F d\sigma_t dt$, where dt denotes 1-dimensional Lebesgue measure and σ_t is the $(n-1)$-dimensional Hausdorff measure of the *time slice* $F_t \equiv F \cap (\mathbb{R}^n \times \{t\})$.

Let us start with the description of the parabolic version of uniform rectifiability. We say that a closed set $E \subset \mathbb{R}^{n+1}$ satisfies an (M,R) *Ahlfors condition*, for certain constants $M \geq 1$ and $R > 0$, if for every $0 < \rho \leq R$ and $(X,t) \in E$ the following holds:

$$\rho^{n+1} M^{-1} \leq \sigma(C_\rho(X,t) \cap E)) \leq M\rho^{n+1}. \tag{2.2}$$

If $E \subset \mathbb{R}^{n+1}$ satisfies an (M,R) Ahlfors condition, we denote by $\Delta_r(X,t)$ the *surface cube* $C_r(X,t) \cap E$. To shorten notations we say that $E \subset \mathbb{R}^{n+1}$ is a *parabolic hypersurface* if it satisfies an (M,R) Ahlfors condition and $\mathbb{R}^{n+1} \setminus E$ consists of exactly two connected components. These components will be denoted by $\Omega_1 \equiv \Omega_1(E)$ and $\Omega_2 \equiv \Omega_2(E)$.

Fixing $E \subset \mathbb{R}^{n+1}$, let $d(X,t) = d(X,t;E)$. We call the n-dimensional planes containing a line parallel to the t axis the *t-planes*. Define

$$\gamma(Z,\tau;r) = \inf_P \left[\frac{1}{r^{n+3}} \int\limits_{E \cap C_r(Z,\tau)} d((Y,s),P)^2 d\sigma(Y,s) \right], \tag{2.3}$$

where the infimum is taken over all t-planes P. Also define the measure

$$d\nu(Z,\tau;r) = \gamma(Z,\tau;r)\, d\sigma(Z,\tau) \frac{dr}{r}. \tag{2.4}$$

We say that $E \subset \mathbb{R}^{n+1}$ is *uniformly rectifiable in the parabolic sense* (URPS), if E satisfies an (M,R) Ahlfors condiition, and for every $(X,t) \in E$ and $C_\rho(X,t) \subset C_R(X,t)$ the following Carleson measure condition holds:

$$\nu([C_\rho(X,t) \cap E] \times (0,\rho)) \leq C\rho^{n+1} \tag{2.5}$$

for certain uniform constant $C > 0$. The smallest constant for which (2.5) holds is denoted by $||\nu||_+$, and is referred to as the *Carleson norm of* ν.

2.2 Parabolic Lipschitz graphs

Consider a function $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$. We say that ψ is a *Lip(1,1/2) function with constant* $A_1 > 0$ if for every $(x,t),(y,s) \in \mathbb{R}^n$ it holds $|\psi(x,t) - \psi(y,s)| \leq A_1 ||x-y, t-s||$. The function ψ is a *parabolic Lipschitz function with constant* $A > 0$ if the following conditions holds:

- ψ satisfies a Lipschitz condition in the variable x

$$|\psi(x,t) - \psi(y,t)| \leq A|x-y| \tag{2.6}$$

  uniformly in $t \in \mathbb{R}$.

- For every interval $I \subseteq \mathbb{R}$ and every $x \in \mathbb{R}^n$

$$\frac{1}{|I|} \int_I \int_I \frac{|\psi(x,t) - \psi(x,s)|^2}{|s-t|^2}\, dt ds \leq A < \infty. \tag{2.7}$$

The regularity condition addressed in (2.7) can be recalled as a *BMO-Sobolev scale* in the t-variable, by results in [57]. It roughly states that a half order derivative of $\psi(x,t)$ with respect to t variable is in BMO. See more details in [33].

On the other hand, it is well known (see e.g. [31]) that every parabolic Lipschitz function is a Lip(1,1/2) function.

The use of this type of graphs goes back to works of J. L. Lewis and collaborators [44, 45]. The main motivation of exploring this type of regularity is to adapt to solutions of the heat equation on time-varying domains the results about solvability of L^p Dirichlet problems for Laplace equation on Lipschitz domains (see e.g. [9, 10, 59]).

More precisely, according to J. L. Lewis (see [33, p. 349]), R. Hunt posed the problem of finding (optimal) conditions on the variable t, that guarantee the solvability of Dirichlet-type problems associated to the heat equation on domains whose boundary is given locally by graphs of functions $\psi(x,t)$ satisfying a Lipschitz condition as in (2.6).

After examples of [40] it was known that the Lip(1,1/2) condition did not suffice to answer Hunt's question, and so the regularity in the t variable remained as an open problem.

Later seminal work of S. Hofmann, along with collaborations with J. L. Lewis [31, 32, 33, 34], settled Hunt's question from the viewpoint of parabolic singular integrals as well as L^2 solvability for heat equation on non-cylindrical domains. In those fundamental works it is also matured the notion of appropriate parabolic singular integrals over parabolic Lipschitz graphs.

As in the standard case, we need a geometric condition for sets with big pieces of parabolic Lipschitz graphs. First we say that a *basic parabolic Lipschitz domain with function* ψ is a domain of the form

$$\Omega(\psi) = \{(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : x_0 > \psi(x,t)\},$$

where ψ is a parabolic Lipschitz function.

We say that an (M,R) Ahlfors regular set E *contains big pieces of parabolic Lipschitz graphs*, or that E has BPPLG, if there exists a constant B_1 with the following property: Given $(Y,s) \in E$ and $0 < r_0 < R$ there exists, after a possible rotation in space variables, a basic parabolic Lipschitz domain $D = \Omega(\psi)$, with parabolic Lipschitz function ψ, such that

$$\sigma(E \cap C_{r_0}(Y,s) \cap \partial D) \geq B_1 r_0^{n+1} \tag{2.8}$$

For shortness sake we write $E \in BPPLG(B_1)$.

2.3 Parabolic Singular Integrals and statement of the Main Theorem

As in the standard case we want to consider integral operators of the form

$$Tf(\mathbf{X}) = \int_E K(\mathbf{X} - \mathbf{Y}) f(\mathbf{Y}) d\sigma(\mathbf{Y}), \tag{2.9}$$

where $\mathbf{X} = (X,t)$, $\mathbf{Y} = (Y,s)$, and where the kernel $K(X,t)$ is an odd function in the X variable, and satisfies the following properties:

- $|K(X,t)| \leq C_1/\|X,t\|^{n+1}$;

- $|\nabla_X K(X,t)| \le C_2/\|X,t\|^{n+2}$;
- $|\nabla_X^2 K(X,t)|, |\partial_t K(X,t)| \le C_2/\|X,t\|^{n+3}$.

Here $|\nabla_X^2 K|$ denotes the Euclidian magnitude of the vector whose entries are the second order derivatives of K with respect to X. If all of these conditions hold we say that K is a *good parabolic kernel*.

Incidentally, we will use the notation $\mathbf{X} = (X,t)$ for points in $\mathbb{R}^{n+1}$ whenever the time variable is not essential in the argumentation. This will become handy when describing some ideas for the proof of the main theorem.

The choice of T guarantees its L^2 boundedness over parabolic Lipschitz graphs, by well known techniques following [32] (see [50, Theorem 2.1]).

From the experience on the standard case, when dealing with a singular integral operator over an (M,R)-Ahlfors regular set E, we say that T is *bounded over* $L^2(E,d\sigma)$ if the *maximal operator associated to* T

$$T^* f(\mathbf{X}) = \sup_{\epsilon>0} \int_{E\cap\{\|X-Y,t-s\|>\epsilon\}} K(\mathbf{X}-\mathbf{Y}) f(\mathbf{Y}) d\sigma(\mathbf{Y}), \tag{2.10}$$

originally defined for $C_0(E)$ functions, can be extended as an $L^2(E,d\sigma)$ bounded operator.

We are now ready to state the main theorem of this survey note.

Theorem 2.1. *Suppose that $E \subset \mathbb{R}^{n+1}$ is a parabolic hypersurface. Then E is URPS if and only if any operator T as in* (2.9)*, associated to a good parabolic kernel is $L^2(E,d\sigma)$ bounded.*

3 Some ideas of the proof of Theorem 2.1: Construction of parabolic Corona-type decompositions

Before starting the proof, we make a notational remark. In several of the estimates in the bulk of the next sections we may use a generic constant that may change from line to line, and still is denoted with the same letter.

Thus, we adopt the nowadays standard notation $A \lesssim B$ to mean that there is a constant k that may depend on the geometric features of the set introduced in previous sections, or the dimension n, such that $A \le kB$. In any case, the dependance on other parameters does not interfere the essence of the estimates. Likewise, $A \approx B$ means $A \lesssim B$ and $B \lesssim A$ hold simultaneously.

As in the standard case (see [16]), the proof uses Corona type decompositions. We describe what this means in the parabolic setting. Notice first that the definition of *dyadic decompositions* are directly adaptable, bearing in mind the change of homogeneity and dimension described in the previous section. In particular if $E \subset \mathbb{R}^{n+1}$ satisfies an (M,R) *Ahlfors condition* then the Hausdorff dimension of E is actually $d = n+1$. Also, in (CD2) of the definition of dyadic cubes the *diameter of a surface cube* $\Delta \equiv \Delta_r(Q,s) = C_r(Q,s) \cap E$ is defined to be $\operatorname{diam}\Delta = r$, and $|\Delta|$ must be replaced by $\sigma(\Delta)$.

On the other hand, if E coincides with a t-plane, then we can choose dyadic cubes which, after a translation and rotation if necessary, take the form

$$Q_{2^j}(X,t) = \{(Z,\tau) : |Z-X| < 2^j, \quad |\tau - t| < 2^{2j}\}, \quad \text{for certain } (X,t) \in \mathbb{R}^{n+1}.$$

We will use the notation $Q_r(p)$ for any cube contained on a t-plane, centered at p with radius $r > 0$.

Now in order to define the *parabolic Corona type decomposition* we use the same properties (Cor1), (Cor2) and (Cor3) previously introduced, and the only property that must be changed is (Cor4). Keeping the notations introduced before, (Cor4) must be replaced by

(Cor4P) For each $S \in \mathcal{F}$ there exists a parabolic Lipschitz graph Γ with constant η such that for every $Q \in S$ the estimate $d(\mathbf{X};\Gamma) \le \eta\,\mathrm{diam}(Q)$ holds provided that $\mathbf{X} \in E$ satisfies $d(\mathbf{X}, Q) \le \mathrm{diam}(Q)$.

For a parabolic Corona decomposition we will still refer to the properties defining it as (Cor1), (Cor2), (Cor3) and (Cor4P).

The main theorem in [49] can now be stated as follows:

Theorem 3.1. *Suppose that $E \subset \mathbb{R}^{n+1}$ is a parabolic hypersurface and consider the following conditions:*

(a) E es URPS.

(b) Any singular integral operator T as in (2.9)*, associated to a good parabolic kernel is $L^2(E, d\sigma)$ bounded.*

Then, assuming any of these conditions, E admits a parabolic Corona type decomposition.

The proof of this theorem requires some adaptations of some of the original arguments of [16, Chapters 3-9 and 12-14].

Before discussing ideas of the aforementioned adaptations notice that the proof of Theorem 2.1 is now reduced to prove reciprocals of both parts (a) and (b) of Theorem 3.1. This is actually the essential contents of [51].

3.1 Ideas to prove Theorem 3.1

In this subsection we only describe the adaptations from the standard case in [16] to our parabolic setting, as implemented in [49]. The details of the following argumentation can be checked in [49].

3.1.1 Description of the proof of (a)

In order to obtain a parabolic Corona decomposition from the parabolic uniform rectifiability of E we follow several steps and the lines of [16].

Step 1. *Construction of $\mathcal{F}$, $\mathcal{B}$ and the stopping-time regions.*

Let $0 < \varepsilon < \delta$ be two small positive numbers so that ε/δ is also small. Let k be a large constant to be determined. Denote by $\mathcal{G} \equiv \mathcal{G}(\varepsilon)$ the set of cubes $Q \in \Delta$ for which there is a t-plane P_Q such that $d(\mathbf{X}, P_Q) \le \varepsilon\,\mathrm{diam}\,Q$ for all $\mathbf{X} \in kQ$. Define now $\mathcal{B} = \Delta \setminus \mathcal{G}$ so that we already have a decomposition $\Delta = \mathcal{B} \cup \mathcal{G}$.

Now define

$$\gamma_\infty(Z,\tau;r) = \inf_P \sup \left\{ \frac{d(Y,s;P)}{r} : (Y,s) \in \Delta_r(Z,\tau) \right\}, \tag{3.1}$$

where the infimum is taken over all t-planes P, and recall that in [35, p. 359] it is proved that

$$\gamma_\infty(Z,\tau;r)^{n+3} \leq C(n)\gamma(Z,\tau;2r) \tag{3.2}$$

Then it can be verified that (2.5) in the definition of parabolic uniform rectifiability implies

$$\sum_{\substack{\gamma_\infty(Q)>\epsilon \\ Q\subseteq R}} \sigma(Q) \leq C(\epsilon)\sigma(R), \tag{3.3}$$

where R is any surface cube of E. Here in analogy with (3.1), for a dyadic cube Q we define

$$\gamma_\infty(Q) = \inf_P \sup\left\{\frac{d(Y,s;P)}{\operatorname{diam} Q} : (Y,s) \in 2Q\right\}.$$

Estimate (3.3) can be proved with a well known technique of associating dyadic cubes on E with certain rectangles in $E \times (0,\infty)$. In turn, (3.3) implies the Carleson packing condition for $\mathcal{B}$ in (Cor1) above.

Now we organize the sets in $\mathcal{G}$ and define the family $\mathcal{F}$ of regions S. The construction of these regions in [49] is different from the original one in [16], so we provide some details.

For the next construction we localize our work in a fixed surface cube $R_0 \subset E$. Let $\Delta(R_0) = \bigcup_{\Delta_j \subset R_0} \Delta_j$ and denote by $\mathcal{G}(R_0)$ the family of subcubes of R_0 in $\mathcal{G}$. Let Q_0 be a maximal cube in $\mathcal{G}(R_0)$ and let π_0 denote the projection of points in $\mathbb{R}^{n+1}$ on P_0, the t-plane that is associated to Q_0 according to the definition of $\mathcal{G}$. Define for $Q \subset Q_0$ the set $P_0(Q) = \pi_0^{-1}(\pi_0(Q))$, which is a cylindrical region perpendicular to P_0 and with level surfaces given by $\pi_0(Q)$.

Now we construct a family $K(Q_0)$ of cubes as follows:

- $Q_0 \in K(Q_0)$
- $R \in \Delta(R_0)$ is added to $K(Q_0)$ if all of the following holds:
  - R is a descendant of Q, for certain $Q \in K(Q_0)$;
  - every sibiling of R, including R itself, is an element of $\mathcal{G}(R_0)$;
  - $d(P_0(R) \cap P_R; P_0(kR) \cap P_0) \leq \delta \operatorname{diam} R$.

This last inequality refers indirectly to the angle between P_0 and P_Q, as originally considered in [16]. However we will not use especifically the angle between these planes, but rather we measure how close the portions of planes $P_0(R) \cap P_R$ and $P_0(kR) \cap P_0$ are from each other.

Now define

$$S(Q_0) = \left(\bigcup_{Q\in K(Q_0)} Q\right) \cup \left(\bigcup_{\substack{R\in\varsigma(Q) \\ Q\in K(Q_0)}} R\right)$$

and repeat the procedure inductively, choosing at each stage a maximal cube in $\mathcal{G}(R_0)$ not contained in any of the previously constructed $S(Q)$. Let $\mathcal{F}$ denote the collection of all the regions so obtained, and let us introduce the notation $Q_0 = Q(S)$ if and only if $S = S(Q_0)$.

Observe that if $S \in \mathcal{F}$ then the only options for $\mathbf{X} \in S$ is that either $\mathbf{X}$ belongs to a minimal cube, or there exists an infinite sequence of elements in S such that $\mathbf{X}$ belongs to all of the terms of the sequence.

Step 2. *Construction of the approximating graphs and verification of its properties.*

According to (Cor4P), we must associate to each $S \in \mathcal{F}$ the graph of a parabolic Lipschitz function, and so in the next construction we fix $S \in \mathcal{F}$ and let $P = P_{Q(S)}$. Let $P^{\perp}$ denote the normal vector to P pointing towards the region Ω_1 and let π and $\pi^{\perp}$ denote the canonical projections from $\mathbb{R}^{n+1}$ onto P and $P^{\perp}$, respectively. Note that $\pi^{\perp}$ might be negative by the chosen orientation of $P^{\perp}$.

The proof of the next lemma provides the essential construction for this step. After stating it we will only sketch ideas for the construction.

Lemma 3.2. *There exists a parabolic Lipschitz function $\psi : P \longrightarrow P^{\perp}$ with character of the order of δ, and such that for every $\mathbf{X} \in Q(S)$ one has*

$$d\,(\mathbf{X}; (\pi(\mathbf{X}), \psi(\pi(\mathbf{X})))) \lesssim \varepsilon d(\mathbf{X}). \tag{3.4}$$

Given $\mathbf{X} \in \mathbb{R}^{n+1}$ define

$$d(\mathbf{X}) = \inf_{Q \in S} \left[d(\mathbf{X}, Q) + \operatorname{diam} Q \right] \tag{3.5}$$

The beginning of the construction of the function ψ is over $\mathcal{Z} = \{\mathbf{Z} \in E : d(\mathbf{Z}) = 0\}$. For $\mathbf{Z} \in \mathcal{Z}$ define

$$\psi(\pi(\mathbf{Z})) = \pi^{\perp}(\mathbf{Z}).$$

This function turns out to be of Lip(1,1/2) type.

The issue is now twofold: to define ψ outside of $\pi(\mathcal{Z})$ and to prove that ψ is actually a parabolic Lipschitz function.

For the extension of ψ off of $\pi(\mathcal{Z})$, one can use use a Whitney-type extension adapted for the parabolic setting. To prove the regularity that parabolic Lipschitz functions require, one first observe that $\{\mathbf{P} + \psi(\mathbf{P}) : \mathbf{P} \in \pi(\mathcal{Z})\} \subset E$, and since E is URPS then one can take advantage of the Carleson measure estimate (2.5) as in [35, 36] to conclude that this extension yields a parabolic Lipschitz function. The technical details are in [49].

Step 3. *Carleson packing condition for the stopping-time regions.*

We only describe the ideas that differ from the ones in [16], and in particular, since the construction of our stopping time regions S is slightly different, the subsequent argumentation is changed accordingly.

The property (Cor3) in the definition of the parabolic Corona decomposition is proved through the following lemma.

Lemma 3.3. *For every $R \in \Delta$*

$$\sum_{\substack{S \in \mathcal{F} \\ Q(S) \subseteq R}} \sigma(Q(S)) \le C\sigma(R). \tag{3.6}$$

Fixing $S = S(Q_0)$, let $m(S)$ denote the set of minimal cubes of S and let $\mathcal{U}$ denote the union of all the cubes in $m(S)$. We separate $m(S)$ into two families, according to the options that any minimal cube R has: either $R \in K(S)$ and at least one descendant of R is in $\mathcal{B}(R_0)$; or $R \in S \setminus K(S)$.

Let $m_1(S)$ denote the set of minimal cubes with at least one descendant in $\mathcal{B}(R_0)$, and let $m_2(S)$ denote the set of minimal cubes $R \in S \setminus K(S)$ with $d(P_R \cap P_0(R); P_{Q_0} \cap P_0(kR)) > \delta \operatorname{diam} R$ (recall definitions in page 323). Also we set

$$\mathcal{U}_i = \mathcal{U}_i(S) \equiv \bigcup_{Q \in m_i(S)} Q, \quad i = 1, 2.$$

Accordingly we have three types of regions:

$$\mathcal{I} = \{S \in \mathcal{F} : \sigma(Q(S) \setminus \mathcal{U}) \geq \theta\sigma(Q(S))\}, \quad \mathcal{II} = \{S \in \mathcal{F} : \sigma(\mathcal{U}_1) \geq \theta\sigma(Q(S))\}$$
$$\text{and} \quad \mathcal{III} = \{S \in \mathcal{F} : \sigma(\mathcal{U}_2) \geq \theta\sigma(Q(S))\}.$$

Condition (3.6) for the class $\mathcal{I}$ is essentially due to the disjointneess of the different S in $\mathcal{F}$. For the class $\mathcal{II}$ one can use the Carleson packing condition for $\mathcal{B}(R_0)$ to obtain (3.6). Indeed,

$$\sum_{\substack{S \in \mathcal{II} \\ Q(S) \subseteq R}} \sigma(Q(S)) \leq \frac{1}{\theta} \sum_{\substack{S \in \mathcal{II} \\ Q(S) \subseteq R}} \sum_{Q \in m_1(S)} \sigma(Q) \leq \frac{1}{\theta} \sum_{\substack{S \in \mathcal{II} \\ Q(S) \subseteq R}} \sum_{Q \in m_1(S)} \sigma(B_Q) \lesssim \sigma(R),$$

where B_Q denotes one of the descendants of Q in the class $\mathcal{B}(R_0)$, and in the last inequality we use (CD1).

In order to prove (3.6) for class $\mathcal{III}$ we use the approximating graphs. Again, the details can be found in [49].

3.1.2 Description of the proof of (b)

Let $\psi(X, t)$ be a smooth function on $(X, t) \in \mathbb{R}^n \times \mathbb{R}$ which is odd in the X variable, and with compact support. In order to define a kernel to which we can apply the hypotesis, consider the set Ω of sequences $\omega = \{\omega_j\}$, with $w_j \in \{-1, 1\}$, endowed with the product topology. The measure Π on Ω assigns equal probability to the values ± 1. Consider projections $\epsilon_j : \Omega \longrightarrow \{-1, 1\}$ given by $\epsilon_j(\omega) = \omega_j$. Recall that it is well known that

$$\sum_{j=-m}^{m} \left| \int_E \psi_j(X - Y, t - s) f(Y, s) d\sigma(Y, s) \right|^2 =$$
$$= \int_\Omega \left| \sum_{j=-m}^{m} \int_E \epsilon_j(\omega) \psi_j(X - Y, t - s) f(Y, s) d\sigma(Y, s) \right|^2 d\Pi(\omega),$$

where $\psi_j(X, t) = 2^{-j(n+1)} \psi(2^{-j} X, 2^{-2j} t)$. Define

$$K_m(X, t; \omega) = \sum_{j=-m}^{m} \epsilon_j(\omega) \psi_j(X, t).$$

Observe that this kernel is still odd in the X variable and it satisfies the other assumptions for good kernels. By our assumptions

$$\int_E \left| \int_E K_m(X-Y,t-s;\omega) f(Y,s) d\sigma(Y,s) \right|^2 d\sigma(X,t) \leq C(m,\omega) \int_E |f(X,t)|^2 d\sigma(Y,s)$$

with a constant $C(m,\omega)$ that depends on m and ω. After the "completeness argument" of [16, p.22] one gets

$$\sum_{j=-m}^{m} \left| \int_E \psi_j(X-Y,t-s) f(Y,s) d\sigma(Y,s) \right|^2 \leq C \int_E |f(X,t)|^2 d\sigma(Y,s), \tag{3.7}$$

this time with a constant independent of m and ω. Applying (3.7) to characteristic functions of balls one can prove that

$$\left(\sum_{j=-\infty}^{\infty} \left| \int_E \psi_j(X-Y,t-s) d\sigma(Y,s) \right|^2 \right) d\sigma(X,t) d\delta_{2^k}(u)$$

is a Carleson measure over $E\times(0,\infty)$.

Using this we now construct a collection of cubes satisfying a Carleson measure type property. For $\tau>0$ small let $R(\tau)$ denote the set of cubes $Q\in\Delta$ with the property that there exist $\mathbf{X},\mathbf{Y}\in 2Q$ such that $d(2\mathbf{X}-\mathbf{Y};E)\geq\tau\operatorname{diam} Q$.

Claim 3.1. *With the definitions and notations above one has*

$$\sum_{\substack{Q\in R(\tau)\\ Q\subset R}} \sigma(Q) \leq C(\tau)\sigma(R).$$

The proof of this claim can be easily adapted from [16, p. 24].

Let $\varepsilon>0$ be given and define $\mathcal{G}(\varepsilon)$ as the family of cubes $Q\in\Delta$ for which there is a plane P_Q satisfying the following properties:

$$d(\mathbf{X};P_Q)\leq\varepsilon\operatorname{diam} Q \quad \text{for all } \mathbf{X}\in 2Q; \tag{3.8}$$

$$\text{if } \mathbf{Y}\in P_Q \text{ and } d(\mathbf{Y};Q), \quad \text{then } d(\mathbf{Y};E)\leq\varepsilon\operatorname{diam} Q. \tag{3.9}$$

Claim 3.2. *Set $\mathcal{B}(\varepsilon)=\Delta\setminus\mathcal{G}(\varepsilon)$. Then*

$$\sum_{\substack{Q\in\mathcal{B}(\varepsilon)\\ Q\subset R}} \sigma(Q) \leq C(\varepsilon)\sigma(R) \tag{3.10}$$

for all $R\in\Delta$ and all $\varepsilon>0$.

The proof can be adapted this time from [16, p. 28-32]. The point is now that the estimates in the two claims above will lead us to a parabolic *generalized* Corona decomposition, defined in the following

Proposition 3.4. *Suppose that E admits a generalized parabolic Corona decomposition, in which (Cor4) above is substituted by:*

(GCor4) For each $S \in \mathcal{F}$ there exists a set $E_S \in BPPLG(C(\eta))$ such that for every $Q \in S$ the estimate $d(\mathbf{X}; E_S) \leq \eta \operatorname{diam}(Q)$ holds whenever $\mathbf{X} \in E$ and $d(\mathbf{X}, Q) \leq \operatorname{diam}(Q)$.

Then E admits a parabolic Corona decomposition. The constant $C(\eta) > 0$ may depend on η but not on S.

To establish this result, we use fundamental results of [17, I Chapter 3] on coronizations of Ahlfors-David regular sets. The proposition is actually an adaptation of [17, Theorem I.3.42]. At this point of the proof we follow ideas in [17, II Chapter 2] and [18].

4 Some ideas to prove Theorem 2.1: Consequences of a parabolic Corona-type decomposition

We start describing a reciprocal to part (b) of Theorem 3.1. Since E satisfies (M, R) Ahlfors condition, then it is a space of homogeneous type in the sense of Coifman-Weiss. In particular, arguments related to the so called $T1$ theorem can be applied, bearing in mind the parabolic homogeneity and dimension (see e.g. [15, 6] and the discussion on parabolic singular integrals in [32, p. 209]).

Given $\Delta_r(\mathbf{Z}) = C_r(\mathbf{Z}) \cap E$ a surface cube on E, for certain $\mathbf{Z} \in E$. Let $\Lambda(\mathbf{Z}, r)$ denote the space of functions $\varphi \in C_0^\infty(\Delta_r(\mathbf{Z}))$ for which

- $|\varphi(\mathbf{X}) - \varphi(\mathbf{Y})| \leq \|\mathbf{X} - \mathbf{Y}\|/r$
- If ℓ denotes an $(n-1)$-dimensional multi-index, say $\ell = (\ell_1, \ldots, \ell_{n-1})$, we set $|\ell| = \ell_1 + \cdots + \ell_{n-1}$, and as usual $(\partial/\partial x)^\ell = (\partial/\partial^{\ell_1} x_1, \ldots, \partial/\partial^{\ell_{n-1}} x_{n-1})$. Then, for $k \in \{0, 1, 2\}$

$$\sup_{0 \leq |\ell| + k \leq 2} r^{2k + |\ell|} \left\| \left(\frac{\partial}{\partial x}\right)^\ell \left(\frac{\partial}{\partial t}\right)^k \varphi \right\|_\infty \leq 1.$$

If any $Q \in \Delta$ is such that $\operatorname{diam} Q = r$, we can define $\Lambda(Q)$ similarly.

Now an argument of Littlewood-Paley type, adapted to parabolic setting implies that in order to prove $L^2(E, d\sigma)$ boundednes of T, it suffices to prove *the restricted boundedness* of T (see [56, p. 294]). This means that for $0 < r < R$ and for every $\varphi \in \Lambda(\mathbf{Z}, r)$ the following holds:

$$\int_{\Delta_r(\mathbf{Z})} |T\varphi|^2 d\sigma \leq C\sigma(\Delta_r). \tag{4.1}$$

An equivalent way to describe this is

$$\int_Q |T\varphi|^2 d\sigma \leq C\sigma(Q) \quad \text{for every } Q \in \Delta \quad \text{and every } \varphi \in \Lambda(Q). \tag{4.2}$$

Moving on, in order to define a parabolic version of *Tolsa's functionals* and hence get the right estimates, we need some more technical definitions.

Let $\mathcal{M}$ be the collection of t-flat measures, that is, $\mu \in \mathcal{M}$ if $\mu \equiv c\sigma|_P$ for certain constant $c > 0$ and certain t-plane P. Let $\mathbf{Z}_Q$ denote the center of the cube Q and define C_Q as the cylinder centered at $\mathbf{Z}_Q$ with radius $3 \operatorname{diam} Q$, i.e. $C_Q = C_{3 \operatorname{diam} Q}(\mathbf{Z}_Q)$. Denote by σ_E the measure σ restricted to E.

Given a Borel measure ν over $\mathbb{R}^{n+1}$ define the *parabolic Tolsa functional* for $Q \in \Delta$ as

$$\alpha_\nu(Q) = \frac{1}{(\operatorname{diam} Q)^{n+2}} \inf_{\mu \in \mathcal{M}} \operatorname{dist}_{C_Q}(\nu, \mu), \tag{4.3}$$

where the *distance on C_Q between ν and μ* is defined as

$$\operatorname{dist}_{C_Q}(\nu,\mu) = \sup\left\{\left|\int g d\nu - \int g d\mu\right| : |\nabla g| \le 1, \operatorname{supp} g \subset C_Q\right\}.$$

Here $|\nabla g|$ denotes the magnitude of the $(n+1)$-dimensional vector that consists of all the first order derivatives of g (including that with respect to t). If $\nu \equiv \sigma_E$ then we simply write $\alpha(Q)$ instead of $\alpha_{\sigma_E}(Q)$.

A fundamental feature of this functional is the fact that it measures how flat the measure ν is, by measuring the *closeness* between ν and the family of t-flat measures. In contrast, γ identifies whenever σ_E is supported on a t-plane, even though σ_E could be far from being a t-flat measure. In this sense the next result confirms this claim (see [58, 51]).

Proposition 4.1. *For every $Q \in \Delta$ one has $\gamma(Q) \lesssim \alpha(Q)$.*

On the other hand, $\alpha(Q)$ will prove useful to verify both the *geometric property* of URPS, and the *analytic property* of L^2 boundedness of operators associated to good parabolic kernels.

Next we introduce some more definitions. Give a non-negative measurable function $f : E \to \mathbb{R}$ and $Q \in \Delta$ define

$$\alpha_f(Q) = \frac{1}{(\operatorname{diam} Q)^{n+2}} \inf_{\mu \in \mathcal{M}} \operatorname{dist}_{C_Q}(f\, d\sigma, \mu). \tag{4.4}$$

The next propositions provide the essential steps to finish the proof of Teorema 2.1.

Proposition 4.2. *Let μ a measure which is absolutely continuous with respect to σ_E, i.e. $\mu \ll \sigma_E$, with $d\mu/d\sigma_E = f \in C_0^\infty(E)$ a non-negative function. If E admits a parabolic Corona-type decomposition, the there exists a constant $c_2 > 0$ such that for any $Q \in \Delta$*

$$\sum_{\substack{Q' \in \Delta \\ Q' \subset Q}} \left[\alpha_\mu(Q')\right]^2 \mu(Q') \le c_2 \mu(Q). \tag{4.5}$$

Proposition 4.3. *If a singular integral operator T as in* (2.9) *is associated to a good parabolic kernel K, then for every non-negative function $f \in C_0^\infty$ supported on E the following holds:*

$$\|T^* f\|_2 \lesssim \sum_{Q \in \Delta} \left[\alpha_f(Q)\right]^2 \int f\, d\sigma + \int f\, d\sigma. \tag{4.6}$$

The proof in [58, Theorem 1.3] can be adapted to the parabolic setting with few direct changes. One of the new ingredients is the following *Cotlar-type inequality*:

$$\|T^* f\|_2 \lesssim \limsup_{m \to \infty} \|T_{(m)} f\|_2 + \int f\, d\sigma, \quad \text{donde} \quad T_{(m)} f = \sum_{j=-m}^{m} T_j f, \tag{4.7}$$

where T_j is a localization of T that we now describe.

Let $\psi : \mathbb{R}^{n+1} \to \mathbb{R}$ be a non-negative function in $C^\infty(Q_{1/4}(\vec{0}))$ satsifying $\psi \geq 0$ over $C_{1/8}(\vec{0})$. Define $\psi_j(X,t) = \psi(2^j X, 4^j t)$ and $\varphi_j = \psi_j - \psi_{j+1}$. Observe that φ_j is a non-negative function suported on $C_{2^{-j-2}}(\vec{0}) \setminus C_{2^{-j-4}}(\vec{0})$, and that

$$\sum_{j\in\mathbb{Z}} \varphi_j = 1.$$

For each $j \in \mathbb{Z}$ define $K_j(X,t) = \varphi_j(X,t)K(X,t)$ and

$$T_j f(X,t) = \int K_j(X-Y, t-s) f(Y,s)\, dY ds.$$

The inequality (4.7) reduces the proof of (4.6) to proving that

$$\|T_{(m)} f\|_2 \lesssim \sum_{Q\in\Delta} \left[\alpha_f(Q)\right]^2 \int_Q f\, d\sigma$$

which is the core of the argumentation in [58, Sections 5 and 6]. In turn, these arguments are again adaptable to parabolic setting with few changes.

We point out that the only regularity required for (4.6) to hold is precisely that regularity required by our good parabolic kernels, as it can be verified by following carefully the arguments in [58], right after his Lemma 6.1.

Applying (4.5) and (4.6) to $\varphi \in \Lambda(Q)$ we obtain

$$\|T^*\varphi\|_2 \lesssim \sigma(Q)^{1/2} \left(\int_Q |\varphi|^2 d\sigma \right)^{1/2} \lesssim \sigma(Q)^{1/2},$$

which as observed in (4.2), it suffices to conclude that T is $L^2(E, d\sigma)$ bounded.

Another important remark is that Propositions 4.1 and 4.2 imply also the reciprocal of part (a) of Theorem 3.1. This time it is enough to note that teh Carleson measure condition defining the URPS property can be stated in terms of cubes, as we now recall.

For $Q \in \Delta$ define

$$\gamma(Q) = \inf_{P\in\mathcal{P}} \frac{1}{(\operatorname{diam} Q)^{n+3}} \int_Q d(Y,s;P)^2 d\sigma(Y,s).$$

Then condition (2.5) is equivalent to

$$\sum_{\substack{Q'\in\Delta \\ Q'\subset Q}} \gamma(Q')\sigma(Q') \leq C\sigma(Q). \tag{4.8}$$

And this estimate can be obtained from Propositions 4.1 and 4.2.

Fundamental Remark. *An interesting project is to explore parabolic versions of some (as many as possible) of the equivalent conditions defining the uniformly rectifiable sets from the main theorem of [16]. A partial result is related to the so called ω regular surfaces, and is briefly discussed in [44]. This could be a starting point of these investigations.*

5 Compactness of certain Parabolic Singular Integrals

In this section we describe a result that establishes the compactness on L^p spaces of a class of Parabolic Singular Integrals, contained in [52].

Partially motivated by one of the results in [36], and the technical developments in [37, Section 4.5], in [52] we explore the compactness of a class of parabolic singular integrals including the boundary double layer potentials associated to the heat equation over sets with appropriate regularity resembling the parabolic chord arc property with vanishing constant.

5.1 Parabolic vanishing chord arc condition

It turns out that adding some extra conditions to a parabolic uniformly rectifiable set E yields some good analogues of well known results of potential theory. This was established in [41, 42, 43] for the standard case and in [35, 36] for the parabolic version.

We say that a set $E \subset \mathbb{R}^{n+1}$ *separates* $\mathbb{R}^{n+1}$, if there is $0 < \delta_0 < 1/10$ such that the following is true: for any $(Q,s) \in E$ and $r > 0$ there exists a t-plane $\widehat{P} \equiv \widehat{P}(Q,s;r)$ containing (Q,s), with unit normal $\hat{n} \equiv \hat{n}(Q,s;r)$, and such that

$$\begin{aligned}
&\Big\{(X,t) + \rho\hat{n} \in C_r(Q,s) : (X,t) \in \widehat{P}, \quad \rho > \delta_0 r\Big\} \subset \Omega_1, \\
&\Big\{(X,t) - \rho\hat{n} \in C_r(Q,s) : (X,t) \in \widehat{P}, \quad \rho > \delta_0 r\Big\} \subset \Omega_2.
\end{aligned} \tag{5.1}$$

We also say that E has the δ_0 *separation property*. The constant δ_0 may be referred to as the *Reifenberg constant of* E.

We now make an important remark on the definition of the δ_0 separation property. It turns out that the δ_0 separation property (5.1) implies the existence of a normal vector pointing towards the exterior on every cross section $\Omega(s)$.

This is a consequence of the fact that the δ_0 "parabolic" separation property described in (5.1) implies the corresponding separation property of $\Omega(s)$ in the usual sense (as defined for instance in [37, p. 2688]). And this in turn implies that $\Omega(s)$ has locally finite perimeter, since the topological boundary of Ω coincides with the so called *measure theoretic boundary* of Ω.

Now, in analogy with the elliptic case, and following [35, p. 357], we say that E is a *parabolic regular set* if it has a δ_0 separation property and is uniformly rectifiable in the parabolic sense.

The set E is a *parabolic chord-arc set with vanishing constant* if it is a parabolic regular set and

$$\lim_{r\to 0}\left[\sup_{\substack{(X,t)\in E\\ 0<\rho\le r}} \frac{\nu([C_\rho(X,t)\cap E]\times(0,\rho))}{\rho^{n+1}}\right] = 0. \tag{5.2}$$

5.2 The class of Parabolic Singular Integrals

Recall first that the *fundamental solution to the heat equation* is given by

$$W(X,t) = c_n t^{-n/2} \exp\left(\frac{|X|^2}{4t}\right)\chi_{(0,\infty)}(t) \quad \text{for } (X,t) \in \mathbb{R}^{n+1}.$$

for a suitable absolute constant $c_n > 0$. Recall also that in [44, 33] (see also references therein) certain L^p-Dirichlet problem is solved over a basic parabolic Lipschitz domain $\Omega(\psi)$.

Recall that *basic parabolic Lipschitz domain* is a domain of the form

$$\Omega(\psi) = \{(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : x_0 > \psi(x,t)\}$$

for some parabolic Lipschitz function ψ.

Given $f \in C(\partial\Omega)$, one defines the *boundary double layer potential* of f as the principal value type integral

$$\mathcal{D}f(P,t) = \text{pv} \int_{-\infty}^{t} \int_{\Omega(s)} \langle N(Q,s), \nabla W(P-Q, t-s)\rangle \, d\sigma_s(Q) ds, \quad (P,t) \in \partial\Omega,$$

where $N(Q,s)$ denotes the outer unit normal vector to Ω_s at the point (Q,s). When writing this operator in graph coordinates one is lead to consider bilinear operators like Calderón commutators on which certain regularity in the t variable made it a far from trivial object to study. Details are in [32] an references therein.

Note that the boundary double layer potential can be written as a *principal value-type* operator of the form

$$\mathcal{D}f(P,t) = \text{pv} \int_{-\infty}^{t} \int_{\Omega(s)} \langle N(Q,s), P-Q\rangle \mathcal{K}(P-Q, t-s) d\sigma_s ds, \quad P \in \partial\Omega(t),\ Q \in \partial\Omega(s), \tag{5.3}$$

where $\mathcal{K}(X,t) = \tilde{c}_n t^{-n/2-1} \exp\left(|X|^2/4t\right) \chi_{(0,\infty)}(t)$.

We generalize (5.3) on parabolic hypersurfaces E as follows. Define first the *truncation* of the kernel $\mathcal{K} \in C^\infty(\mathbb{R}^{n+1} \setminus \{\vec{0}\})$ by

$$\mathcal{K}_\epsilon(X,t) = \begin{cases} \mathcal{K}(X,t) & \text{if } |X| > \epsilon,\ t > \epsilon^2 \\ 0 & \text{otherwise.} \end{cases} \tag{5.4}$$

Whenever this is well defined, the corresponding truncated operator is given by

$$\mathcal{T}_\epsilon f(X,t) = \int_{-\infty}^{t} \int_{E_s} \langle N(Q,s), X-Q\rangle \mathcal{K}_\epsilon(X-Q, t-s) d\sigma(Q,s), \tag{5.5}$$

where E is a compact parabolic hypersurface, E_s is the cross section $E \cap \{(X,s) : X \in \mathbb{R}^n\}$, and $N(Q,s)$ denotes the unit exterior normal vector in the sense explained in the previous section.

Denote by $\mathcal{T}^*$ the *maximal operator associated to* $\mathcal{T}$ defined by

$$\mathcal{T}^* f(X,t) = \sup_{\epsilon>0} |\mathcal{T}_\epsilon f(X,t)|, \quad (X,t) \in E. \tag{5.6}$$

We say that $\mathcal{T}^*$, originally defined on $C_0(E)$, is *bounded on* $L^p(E, d\sigma)$, $1 \le p < \infty$, if it can be extended as a bounded operator on $L^p(E, d\sigma)$. As usual, we write $\|\mathcal{T}^*\|_{L^p(E,d\sigma)}$ to denote the minimal constant $C > 0$ for which the inequality $\|\mathcal{T}^* f\|_{L^p(E,d\sigma)} \le C\|f\|_{L^p(E,d\sigma)}$ holds.

Notice that the kernel $\mathcal{K}(X,t)$ is even in the X variable, and satisfies

$$|\mathcal{K}(X,t)| \leq \frac{C_1}{\|X,t\|^{n+2}}, \quad |\nabla_X \mathcal{K}(X,t)| \leq \frac{C_1}{\|X,t\|^{n+3}}, \tag{5.7}$$

$$|\nabla_X^2 \mathcal{K}(X,t)|,\ |\partial_t \mathcal{K}(X,t)| \leq \frac{C_1}{\|X,t\|^{n+3}} \tag{5.8}$$

with an absolute constant $C_1 > 0$.

After all these notions are introduced we are ready to state our main theorem.

Theorem 5.1. *Let E be a parabolic chord-arc set with vanishing constant, and let $\mathcal{T}^*$ be the maximal singular integral operator as defined in* (5.6) *over E. Then $\mathcal{T}^* : L^p(E, d\sigma) \to L^p(E, d\sigma)$ is a compact operator for every $1 < p < \infty$.*

Notice that this theorem says nothing about the existence of principal values of $\mathcal{T}_\epsilon$ on parabolic regular sets. In fact, the existence of this type of principal value operators seems to be a challenging problem.

The proof of Theorem 5.1 in [52] is given through a couple of reductions that we now describe.

5.3 A local parabolic Semmes decomposition and a local separation property

We first recall a result essentially established in [35], in which it is obtained a parabolic version of local Semmes decomposition. This could be viewed as a *big pieces of graphs* property in which a more precise quantitative information is provided at certain (small) scales. The original ideas in the standard case are in [55].

In the following arguments we are given a parabolic hypersurface E that induces two regions Ω_1 and Ω_2.

Let $0 < r_0 < R$. Given $0 < \delta < 1/10$ and $(X,t) \in E$, we say that the surface cube $\Delta_{r_0}(X,t) \equiv E \cap C_{r_0}(X,t)$ *contains very big pieces of parabolic Lipschitz graphs with constant δ*, or that $\Delta_{r_0}(X,t)$ has VBPPLG(δ), if there exists, after a possible rotation in space variables, a basic parabolic Lipschitz domain $D = \Omega(\psi)$, and a constant $B_2 = B_2(M)$ such that the constant of ψ is controlled by $B_2\delta$ and the following holds:

- $\sigma(\Delta_{r_0}(X,t) \setminus \partial D) + \sigma(\partial D \cap C_{r_0}(X,t) \setminus E) \leq e^{-1/(B_2\delta)} r_0^{n+1}$;
- $\Delta_{2r_0}(X,t) = \mathcal{G} \cup \mathcal{B}$, with $\mathcal{G} \subset \partial D$ and $\sigma(\mathcal{B}) \leq e^{-1/(B_2\delta)} r_0^{n+1}$;
- if $(Y,s) \equiv (y_0, y, s) \in \mathcal{B}$ and $\pi : \mathbb{R}^{n+1} \to \{\vec{0}\} \times \mathbb{R}^n$ denotes the canonical projection $\pi(x_0, x, t) = (x,t)$, then $|y_0 - \psi(y,s)| \leq B_2\delta \cdot d(\pi(Y,s); \pi(\mathcal{G}))$.

The local decomposition described in this VBPPLG property is also referred to as a *local parabolic Semmes decomposition relative to $\Delta_{r_0}(X,t)$ with constant δ.*

Lemma 5.2. *Let $E \subset \mathbb{R}^{n+1}$ be a parabolic chord arc set with vanishing constant. Then there exists a small constant $0 < \hat{\delta} < \delta_0$ such that for any $0 < \delta < \hat{\delta}$ there is $0 < r_\delta < R$ such that E has a local parabolic Semmes decomposition relative to $\Delta_\rho(X_0, t_0)$ with constant strictly smaller than δ, for every $0 < \rho < r_\delta$, and every $(X,t) \in E$.*

Bearing this in mind, it is observed in [35, p. 360] that, if we denote by $\widehat{P} = \widehat{P}(Y,s;\rho)$ the plane through (Y,s) and parallel to the t-plane that minimizes $\gamma_\infty(Y,s;\rho)$, then the following *local separation property* holds: for any $(Y,s) \in E$, $0 < \rho < r$

$$\{(Z,\tau) + s\widehat{n} \in C_\rho(Y,s) : (Z,\tau) \in \widehat{P}, \quad s > \delta_1\rho\} \subset \Omega_1 \tag{5.9}$$

$$\{(Z,\tau) - s\widehat{n} \in C_\rho(Q,s) : (Z,\tau) \in \widehat{P}, \quad s > \delta_1\rho\} \subset \Omega_2. \tag{5.10}$$

Here $\widehat{n} = \widehat{n}(Y,s;r)$ denotes the unit normal to $\widehat{P}$ pointing towards Ω_1, and $\delta_1 \approx \delta^{\kappa/(n+3)} < \delta_0/10$.

The proof of Theorem 5.1 now proceeds using the local Semmes decomposition in Lemma 5.2 and the local separation condition in estimates (5.9), (5.10). One of the differences of arguments in [52] with respect to those in [37] is that, as observed above, the parabolic chord arc condition has no apparent condition on the normal unit vector of E, but rather the smallness of a Carleson measure norm from the *parabolic uniform rectifiability* property of E. The arguments are adaptations of the so called *good-λ method* of G. David [11] and ideas in [37]. The details are in [52].

Acknowledgments

This work contains an extended version of a Plenary Conference delivered at the *International Workshop: Analysis, Operator Theory and Physical Mathematics* held on February of 2014 at Ixtapa, México. The author received partial support from CONACYT-México *Ciencia Básica* Grant 103900 and from a *Redes de CA-PROMEP* Grant.

References

[1] J. Azzam and R. Schul, Hard Sard: Quantitative implicit function and extension theorems for Lipschitz maps. *Geom. Funct. Anal.* **22** (2012), pp 1062-1123.

[2] A. P. Calderón, Cauchy integrals on Lipschitz curves and related operators. *Proc. Nat. Acad. Sci. U.S.A.* **74** (1977), pp 1324-1327.

[3] A. P Calderón and C. P. Calderón, A representation formula and its applications to singular integrals. *Indiana Univ. Math. J.* **49** (2000), pp 1-5.

[4] A. P. Calderón and A. Torchinsky, Parabolic maximal functions associated with a distribution I. *Adv. in Math.* **16** (1975), pp 1-64.

[5] A. P. Calderón and A. Torchinsky, Parabolic maximal functions associated with a distribution II. *Adv. in Math.* **24** (1977), pp 101-171.

[6] F. M. Christ, *Lectures on singular integral operators*, CBMS Regional Conference Series in Mathematics no. 77, Amer. Math. Soc., Providence 1990.

[7] R. R. Coifman, A. McIntosh, and Y. Meyer, L'Integrale de Cauchy definit une opérateur bourné sur les courbes lipschitziennes. *Ann. of Math.* **116** (1982), pp 361-387.

[8] R. R. Coifman and G. Weiss, *Analyse harmonique noncommutative sur certain espaces homegenes*, Lecture Notes in Mathematics Vol. 242, Springer-Verlag, Berlin 1971.

[9] B. E. J. Dahlberg, On estimates of harmonic measure. *Arch. Rat. Mech. Anal.* **65** (1977), pp 272-288.

[10] B. E. J. Dahlberg, On the Poisson integral for Lipschitz and C^1 domains. *Studia Math.* **66** (1979), pp 13-24.

[11] G. David, Opérateurs intégraux singuliers sur certains courbes du plan complexe. *Ann. Sci. Ec. Norm. Sup. (4)* **17** (1984), pp 157-189.

[12] G. David, *Wavelets and Singular Integrals on curves and surfaces*, Lecture Notes in Mathematics Vol. 1465, Springer-Verlag, Berlin Heidelberg 1991.

[13] G. David, *Uniformly rectifiable sets*, Unpublished Lecture Notes from a Mini-Course at Park City Summer School, 2012.

[14] G. David and D. Jerison, Lipschitz approximation to hypersufaces, harmonic measure, and singular integrals. *Indiana Univ. Math. J.* **39** (1990), pp 831-845.

[15] G. David and J. L. Journé, A boundedness criterion for generalized Calderón-Zygmund operators. *Ann. Math.* **120** (1984), pp 371-397.

[16] G. David and S. Semmes, *Singular integrals and rectifiable sets in $\mathbb{R}^n$: Au-delà des graphes lipschitziens*, Astérisque Vol 191, Sociètè Mathèmatique de France, Paris 1991.

[17] G. David and S. Semmes, *Analysis of and on uniformly rectifiable sets*, Mathematical Surveys and Monographs Vol. 38, American Mathematical Society, Providence 1993.

[18] G. David and S. Semmes, Quantitative rectifiability and Lipschitz mappings. *Trans. Amer. Math. Soc.* **337** (1993), pp 855-889.

[19] E. B. Fabes, Singular integrals and partial differential equations of parabolic type. *Studia Math.* **28** (1966/1967), pp 81-131.

[20] E. B. Fabes, N. Garofalo, and S. Salsa, Comparison theorems for temperatures in non-cylindrical domains. *Atti Acc. Lincei Rendi Fis., serie VIII* **77** (1984), pp 1-12.

[21] E. B. Fabes, N. Garofalo, and S. Salsa, A backward Harnack inequality and Fatou theorem for non-negative solutions of parabolic equations, *Illinois J. Math.* **30** (1986), pp 536-565.

[22] E. B. Fabes, M. Jodeit, and N. M. Rivière, Potential techniques for boundary value problems in C^1 domains. *Acta Math.* **141** (1978), pp 165-186.

[23] E. B. Fabes and N. M. Rivière, Parabolic partial differential equations with uniformly continuous coefficients. *Bull. Amer. Math. Soc.* **72** (1966), pp 116–117.

[24] E. B. Fabes and N. M. Rivière, Singular integrals with mixed homogeneity. *Studia Math.* **27** (1966), pp 19-38.

[25] E. B. Fabes and N. M. Rivière, Symbolic calculus of kernels with mixed homogeneity. In *Singular Integrals (Providence) (A. P. Calderón, ed.)*, Proc. Symp. Pure Math. Vol. 10, Amer. Math. Soc., 1967, pp 106-127.

[26] E. B. Fabes and N. M. Rivière, Dirichlet and Neumann problems for the heat equation in C^1 cylinders. In *Symposia in Pure Mathematics (Providence)* Vol. 2, Amer. Math. Soc., 1979, pp 179-196.

[27] E. B. Fabes and C. Sadosky, Pointwise convergence for parabolic singular integrals. *Studia Math.* **26** (1966), pp 225-232.

[28] E. B. Fabes and M. V. Safonov, Behavior near the boundary of positive solutions of second order parabolic equations. In *Proceedings of the International Congress on Harmonic Analysis at El Escorial 1996*, CRC Press, Boca Ratón, Florida 1998, pp 871-882.

[29] E. B. Fabes, M. V. Safonov, and Y. Yuan, Behavior near the boundary of positive solutions of second order parabolic equations II. textitTrans. Amer. Math. Soc. **351** (1999), pp 4947-4961.

[30] E. B. Fabes and S. Salsa, Estimates of caloric measure and the initial-Dirichlet problem for the heat equation in Lipschitz cylinders. *Trans. Amer. Math. Soc.* **279** (1983), pp 635-650.

[31] S. Hofmann, A characterization of commutators of parabolic singular integrals. In *Proceedings of the International Conference of Fourier Analysis and Partial Differential Equations. Miraflores de la Sierra 1992*, CRC Press, Boca Ratón, Florida 1995, pp 195-209.

[32] . Hofmann, Parabolic Singular Integrals of Calderón-type, rough operators, and caloric Layer Potentials. *Duke Math. J.* **90** (1997), pp 209-259.

[33] S. Hofmann and J. L. Lewis, L^2 solvability and representation by caloric layer potentials in time varying domains. *Ann. Math.* **144** (1996), pp 349-420.

[34] S. Hofmann and J. L. Lewis, The L^p-Neumann problem for the heat equation in non-cylindrical domains. *J. Funct. Anal.* **220** (2005), pp 1-54.

[35] S. Hofmann, J. L. Lewis, and K. Nyström, Existence of big pieces of graphs for parabolic problems. *Ann. Acad. Sci. Fenn. Math.* **28** (2003), pp 355-384.

[36] S. Hofmann, J. L. Lewis, and K. Nyström, Caloric measure in parabolic flat domains. *Duke Math. J.* **122** (2004), pp 281-346.

[37] S. Hofmann, M. Mitrea, and M. Taylor, Singular integrals and elliptic boundary problems on regular Semmes-Kenig-Toro domains. *Int. Math. Res. Notices* **14** (2010), pp 2567-2865.

[38] P. Jones, Rectifiable sets and the travelling salesman problem. *Invent. Math.* **102** (1990), pp 1-15.

[39] J. L. Journé, *Calderón-Zygmund operators, Pseudo-Differential operators and the Cauchy integral of Calderón*, Lecture Notes in Mathematics Vol. 994, Springer, New York, 1983.

[40] R. Kaufman and J-M Wu, Singularity of parabolic measures. *Comp. Math.* **40** (1980), pp 243-250.

[41] C. E. Kenig and T. Toro, Harmonic measure on locally flat domains. *Ann. Math.* **87** (1997), pp 501-551.

[42] C. E. Kenig and T. Toro, Free boundary regularity for harmonic measures and Poisson kernels. *Ann. Math.* **150** (1999), pp 369-454.

[43] C. E. Kenig and T. Toro, Poisson kernel characterization of Refifenberg flat chord arc domains. *Ann. Scient. Ec. Norm. Sup (4)* **36** (2003), pp 323-401.

[44] J. L. Lewis and M. A. M. Murray, *The method of layer potentials for the heat equation in time-varying domains*, Memoirs of the Amer. Math. Soc. Vol. 114 No.545, American Mathematical Society, Providence 1995.

[45] J. L. Lewis and J. Silver, Parabolic measure and the Dirichlet problem for the heat equation in two dimensions. *Indiana Univ. Math. J.* **37** (1988), pp 801-839.

[46] P. Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics vol 44, Cambridge University Press, Cambridge 1995.

[47] P. Mattila, M. S. Melnikov, and J. Verdera, The Cauchy integral, analytic capacity, and uniform rectifiability. *Ann. Math.* **144** (1996), pp 127-136.

[48] F. Nazarov, X. Tolsa, and A. Volberg, *On the uniform rectifiability of AD-regular measures with bounded Riesz transform operator: the case of codimension 1*, Unpublished manuscript available at http://arxiv.org/pdf/1212.5229.pdf.

[49] J. Rivera-Noriega, A parabolic version of Corona decompositions. *Illinois J. Math.* **53** (2009), pp 533-559.

[50] J. Rivera-Noriega, Two results over sets with big pieces of parabolic Lipschitz graphs. *Houston J. Math.* **36** (2010), pp 619-635.

[51] J. Rivera-Noriega, Parabolic singular integrals and uniformly rectifiable sets in the parabolic sense. *J. Geom. Anal.* **23** (2013), pp 1140-1157.

[52] J. Rivera-Noriega, Compact Singular Integrals over Parabolic Ahlfors Regular Sets, Submitted, 2014.

[53] M. V. Safonov, Estimates near the boundary for solutions of second order parabolic equations. In *Proceedings of the International Congress of Mathematicians (Berlin)*, Vol. I, *Doc. Math*, Extra Vol. I, 1998, pp 637-647.

[54] M. V. Safonov and Y. Yuan, Doubling properties for second order parabolic equations. *Ann. Math.* **150** (1999), pp 313-327.

[55] S. Semmes, Chord-arc surfaces with small constant I. *Adv. Math.* **85** (1991), pp 198-223.

[56] E. M. Stein, *Harmonic analysis: Real variable methods, orthogonality, and oscillatory integrals*, Princeton Univ. Press, Princeton 1993.

[57] R. S. Strichartz, Bounded mean oscillation and Sobolev spaces. *Indiana Univ. Math. J.* **29** (1980), pp 539-558.

[58] X. Tolsa, Uniform rectifiability, Calderón-Zygmund operators with odd kernel, and quasiorthogonality. *Proc. London. Math. Soc.* **98** (2009), pp 393-426.

[59] G. Verchota, Layer Potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains. *J. Funct. Anal.* **59** (1984), pp 572-611.