Communications in **M**athematical **A**nalysis
Volume 9, Number 1, pp. 121–127 (2010) www.commun-math-anal.org
ISSN 1938-9787

UNIQUE CONTINUATION PROPERTY FOR BOUSSINESQ-TYPE SYSTEMS

YOUCEF MAMMERI*
Institut de mathématiques de Bordeaux, CNRS UMR 5251, INRIA
Université Bordeaux 1
33405 Talence, France

(Communicated by Ronghua Pan)

Abstract

We prove that if the solution of a general family of Boussinesq systems has a compact support for all time, then that solution vanishes identically.

AMS Subject Classification: 35B60, 35Q53, 76B15.

Keywords: Boussinesq system, unique continuation property, compact support.

1 Introduction

In this paper, we study the unique continuation property for a general family of Boussinesq systems of the form [1, 2]

$$\eta_t + u_x + (u\eta)_x + au_{xxx} - b\eta_{xxt} = 0 \quad (1.1)$$
$$u_t + \eta_x + uu_x + c\eta_{xxx} - du_{xxt} = 0, \quad (1.2)$$

with $a+b = 1/2(\theta^2 - 1/3)$ and $c+d = 1/2(1-\theta^2)$, for $\theta \in [0,1]$. We wonder if there exists a solution of the Boussinesq system (1.1)-(1.2), with a compact support on a time interval other than the obvious one. Here, we propose the use of the Bourgain's method [3] based on entire function estimates. According to the Paley-Wiener theorem, the entire function is given by an analytic continuation of the Fourier transform in space of the Duhamel formula. To adapt that method, we need to prove that for all $R > 0$, there exists $k \in \mathbb{R}$ with $|k| > R$ and such that

$$|L'(k)| \geq |f(k)|, \text{ with } \lim_{|k|\to\infty} |f(k)| = \infty, \quad (1.3)$$

*E-mail address: youcef.mammeri@math.u-bordeaux1.fr

where L is the symbol of the linear evolution. In the context of the system (1.1)-(1.2), L is given, for $(1-ak^2)(1-ck^2) \geq 0$, by

$$L(k) = k\sqrt{\frac{(1-ak^2)(1-ck^2)}{(1+bk^2)(1+dk^2)}},$$

and the property (1.3) holds only if $b = d = 0$ and $ac \neq 0$.

If we denote $H^s(\mathbb{R})$ the Sobolev space of order s, our result reads as follows.

Theorem 1.1. *We set $b = d = 0$. Let us suppose $ac \neq 0$ and for all $k \in \mathbb{R}$, $(1-ak^2)(1-ck^2) \geq 0$. For $s > 4$, $(\eta, u) \in \mathcal{C}([-T,T]; H^s(\mathbb{R})) \times \mathcal{C}([-T,T]; H^s(\mathbb{R}))$ denotes the solution of the Boussinesq system (1.1)-(1.2). Let us suppose that there exists $0 < B < +\infty$ such that for all $t \in [-T,T]$*

$$\operatorname{supp}(\eta(t), u(t)) \subseteq [-B,B] \times [-B,B],$$

and for all $\xi \in \mathbb{C}$

$$\sup_{t\in[-T,T]} |\hat{\eta}(\xi,t)| = \sup_{t\in[-T,T]} |\hat{u}(\xi,t)|.$$

Then (η, u) vanishes identically.

We refer to [1, 2] for precise results of existence theory. Let us remain that the Boussinesq system describes the surface (η) and the velocity (u) of the two-way propagation of small amplitude, long wavelength, gravity waves in shallow water [1, 2, 4, 9].

The paper is organised as follows. Section 2 deals with some estimates for an entire function. In Section 3, the Bourgain method is applied to the system (1.1)-(1.2).

2 Estimates for an Entire Function

In this section, we remain the fundamental estimates introduced by Bourgain [3].

Proposition 2.1. *[3]Let $\Phi \in C_0^\infty(\mathbb{R})$ be a nonzero function. Then for all $Q > 0$ and $R > 0$, there exists $k \in \mathbb{R}$, with $|k| > R$, such that*

$$|\widehat{\Phi}(k)| \quad > \quad \mathrm{e}^{-|k|/Q}. \tag{2.1}$$

Lemma 2.2. *[3]Let $\widehat{\Phi} : \mathbb{C} \to \mathbb{C}$ be an entire function, bounded and integrable on the real axis, such that there exists $C > 0$ satisfying for all $\xi \in \mathbb{C}$*

$$|\widehat{\Phi}(\xi)| \leq C\mathrm{e}^{B|\mathrm{Im}\,\xi|}.$$

Then, for all $\xi = (k + im) \in \mathbb{C}$, with

$$|m| \leq B^{-1}\Big[1 + |\log \sup_{|k_0|\geq|k|} |\widehat{\Phi}(k_0)||\Big]^{-1}, \tag{2.2}$$

we have for k_1 and $m_1 \in \mathbb{R}$; $|m_1| \leq |m|$

$$|\widehat{\Phi}'(k - k_1 + im_1)| \leq C\,B\left(\sup_{|k_0|\geq|k|} |\widehat{\Phi}(k_0)| + \sup_{|k_0|\geq|k-k_1|} |\widehat{\Phi}(k_0)|\right)\Big[1 + |\log \sup_{|k_0|\geq|k|} |\widehat{\Phi}(k_0)||\Big].$$

We can generalize the proposition 2.1 to entire functions.

Corollary 2.3. *Let* $\Phi \in C_0^\infty(\mathbb{R})$ *be a nonzero function with* $\operatorname{supp}\Phi \subseteq [-B,B]$. *Let us suppose that, for all* $k \in \mathbb{R}$, $\widehat{\Phi}(k) \geq 0$ *decreases when we increase k. Then for all* $Q > 0$ *and* $R > 0$, *there exists* $k \in \mathbb{R}$, *with* $|k| > R$, *such that for all* $m \in \mathbb{R}$, *with* $|m| \leq B^{-1}[1 + |\log|\widehat{\Phi}(k)||]^{-1}$,

$$|\widehat{\Phi}(k+im)| \quad > \quad \frac{1}{2}\widehat{\Phi}(k) > \frac{1}{2}\mathrm{e}^{-|k|/Q}. \tag{2.3}$$

Proof. We denote Q and R the constants satisfying the proposition 2.1.
Since Φ has a compact support included in $[-B,B]$, the Paley-Wiener theorem [8] implies that $\widehat{\Phi}$ has an analytic continuation satisfying for all $(k+im) \in \mathbb{C}$,

$$|\widehat{\Phi}(k+im)| \leq C\mathrm{e}^{B|m|}.$$

Let $m \in \mathbb{R}$, we write

$$|\widehat{\Phi}(k+im)| = |\widehat{\Phi}(k+im) - \widehat{\Phi}(k) + \widehat{\Phi}(k)|,$$

and the triangle inequality gives

$$|\widehat{\Phi}(k+im)| \geq \widehat{\Phi}(k) - |\widehat{\Phi}(k+im) - \widehat{\Phi}(k)|.$$

However, we have

$$|\widehat{\Phi}(k+im) - \widehat{\Phi}(k)| \leq |m| \sup_{|m_0| \leq |m|} |\widehat{\Phi}'(k+im_0)|,$$

and it is enough to choose $m \in \mathbb{R}$ satisfying the condition (2.2), or smaller, and to apply the lemma 2.2 with $k_1 = 0$ to find

$$|\widehat{\Phi}(k+im) - \widehat{\Phi}(k)| \leq \frac{1}{2} \sup_{|k_0| \geq |k|} \widehat{\Phi}(k_0) = \frac{1}{2}\widehat{\Phi}(k).$$

Finally, we have

$$|\widehat{\Phi}(k+im)| \geq \frac{1}{2}\widehat{\Phi}(k),$$

and we conclude by applying the proposition 2.1. □

The Duhamel formula is used to prove the nonlinear case. We need estimates for an entire function taking into account the convolution.

Proposition 2.4. *[3]Let* $\Phi \in C_0^\infty(\mathbb{R})$ *be a nonzero function with* $\operatorname{supp}\Phi \subseteq [-B,B]$. *Let us suppose that, for all* $k \in \mathbb{R}$, $\widehat{\Phi}(k) \geq 0$ *decreases when we increase k. Then there exists* $C > 0$ *such that for all* $Q > 0$ *and* $R > 0$, *there exists* $k \in \mathbb{R}$, *with* $|k| > R$, *satisfying*

$$\widehat{\Phi}(k) \quad > \quad C\widehat{\Phi} * \widehat{\Phi}(k). \tag{2.4}$$

Corollary 2.5. *Let $\Phi \in \mathcal{C}_0^\infty(\mathbb{R})$ be a nonzero function with* $\operatorname{supp}\Phi \subseteq [-B,B]$. *Let us suppose that, for all $k \in \mathbb{R}$, $\widehat{\Phi}(k) \geq 0$ decreases when we increase k. Then there exists $C > 0$ such that for all $Q > 0$ and $R > 0$, there exists $k \in \mathbb{R}$, with $|k| > R$, satisfying for all $m \in \mathbb{R}$, with $|m| \leq B^{-1}[1 + |\log(|\widehat{\Phi}(k)|)|]^{-1}$,*

$$\widehat{\Phi}(k) \quad > \quad C\,|\widehat{\Phi} * \widehat{\Phi}(k+im)|. \tag{2.5}$$

Proof. Let $m \in \mathbb{R}$, the triangle inequality gives

$$|\widehat{\Phi} * \widehat{\Phi}(k+im)| \leq \widehat{\Phi} * \widehat{\Phi}(k) + |\widehat{\Phi} * \widehat{\Phi}(k+im) - \widehat{\Phi} * \widehat{\Phi}(k)|.$$

However, we have

$$\begin{aligned} |\widehat{\Phi} * \widehat{\Phi}(k+im) - \widehat{\Phi} * \widehat{\Phi}(k)| &= \left| \int_{-\infty}^{+\infty} (\widehat{\Phi}(k+im-k_1) - \widehat{\Phi}(k-k_1))\widehat{\Phi}(k_1)\, dk_1 \right| \\ &\leq |m| \int_{-\infty}^{+\infty} \sup_{|m_1| \leq |m|} |\widehat{\Phi}'(k-k_1+im_1)|\widehat{\Phi}(k_1)\, dk_1, \end{aligned}$$

and it is enough to choose $m \in \mathbb{R}$ satisfying the condition (2.2), or smaller, and to apply the lemma 2.2 to find

$$\begin{aligned} |\widehat{\Phi} * \widehat{\Phi}(k+im) - \widehat{\Phi} * \widehat{\Phi}(k)| &\leq |m| B(1 + |\log \widehat{\Phi}(k)|) \int_{-\infty}^{+\infty} (\widehat{\Phi}(k) + \widehat{\Phi}(k-k_1))\widehat{\Phi}(k_1)\, dk_1 \\ &\leq C\widehat{\Phi}(k) + C\widehat{\Phi}(k) * \widehat{\Phi}(k). \end{aligned}$$

Finally, we have

$$|\widehat{\Phi} * \widehat{\Phi}(k+im)| \leq C\widehat{\Phi}(k) + (1+C)\,\widehat{\Phi}(k) * \widehat{\Phi}(k),$$

and we conclude by applying the proposition 2.4. □

We consider the following Cauchy problem, for $(x,t) \in \mathbb{R}^2$,

$$\eta_t + u_x + (u\eta)_x + a u_{xxx} = 0 \tag{2.6}$$

$$u_t + \eta_x + u u_x + c\eta_{xxx} = 0 \tag{2.7}$$

$$(\eta(x,0), u(x,0)) = (\eta_0(x), u_0(x)). \tag{2.8}$$

Corollary 2.6. *Let $s > 4$ and $(\eta, u) \in \mathcal{C}([-T,T]; H^s(\mathbb{R})) \times \mathcal{C}([-T,T]; H^s(\mathbb{R}))$ be a nonzero solution of the Cauchy problem (2.6)-(2.7)-(2.8) with for all $t \in [-T,T]$*

$$\operatorname{supp}(\eta(t), u(t)) \subseteq [-B,B] \times [-B,B].$$

Then there exists $t_1 \in [-T,T]$ such that for all $Q > 0$ and $R > 0$, there exists $k \in \mathbb{R}$, with $|k| > R$, satisfying

$$|\hat{\eta}(k,t_1)| + |\hat{u}(k,t_1)| = \sup_{|k_0| \geq |k|} \sup_{t \in [-T,T]} (|\hat{\eta}(k_0,t)| + |\hat{u}(k_0,t)|),$$

and for all $m \in \mathbb{R}$, with $|m| \leq B^{-1}[1 + |\log(|\hat{\eta}(k,t_1)| + |\hat{u}(k,t_1)|)|]^{-1}$,

$$|\hat{\eta}(k+im,t_1)| + |\hat{u}(k+im,t_1)| > \frac{1}{2}(|\hat{\eta}(k,t_1)| + |\hat{u}(k,t_1)|) > \frac{1}{2}e^{-|k|/Q} \tag{2.9}$$

$$C(|\hat{\eta}| * |\hat{u}|(k+im,t_1) + |\hat{u}| * |\hat{u}|(k+im,t_1)) \leq |\hat{\eta}(k,t_1)| + |\hat{u}(k,t_1)|. \tag{2.10}$$

Proof. Corollaries 2.3 and 2.5 are applied with the function Φ defined by, for $k \in \mathbb{R}$

$$\widehat{\Phi}(k) = \sup_{|k_0| \geq |k|} \sup_{t \in [-T,T]} (|\hat{\eta}(k,t)| + |\hat{u}(k,t)|).$$

□

3 Proof of the main result

Let $s > 4$ and $(\eta_0, u_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$. We suppose for all $t \in [-T,T]$, the solution $(\eta, u)(t)$ of the Boussinesq system has a compact support included in $[-B,B] \times [-B,B]$. The Paley-Wiener theorem implies that $(\hat{\eta}(t), \hat{u}(t))$ has an analytic continuation satisfying there exists $C > 0$ such that for all $(k + im) \in \mathbb{C}$ and $t \in [-T,T]$,

$$|\hat{\eta}(k+im,t)| + |\hat{u}(k+im,t)| \leq C\mathrm{e}^{B|m|}. \tag{3.1}$$

Let $t_2 \in [-T,T]$, with $t_2 > t_1$. The Duhamel formula is written, for $\delta t := t_2 - t_1$, $\xi := k + im$ and $\xi\sqrt{|(1-a\xi^2)(1-c\xi^2)|} =: K + iM$,

$$\begin{aligned}
\begin{pmatrix} \hat{\eta} \\ \hat{u} \end{pmatrix}(\xi, t_2) &= \begin{pmatrix} \cos((K+iM)\,\delta t) & -i\sin((K+iM)\,\delta t) \\ -i\sin((K+iM)\,\delta t) & \cos((K+iM)\,\delta t) \end{pmatrix} \begin{pmatrix} \hat{\eta} \\ \hat{u} \end{pmatrix}(\xi, t_1) \\
&\quad - i\xi \int_{t_1}^{t_2} \begin{pmatrix} \cos((K+iM)\,(t_2-\tau)) & -i\sin((K+iM)\,(t_2-\tau)) \\ -i\sin((K+iM)\,(t_2-\tau)) & \cos((K+iM)\,(t_2-\tau)) \end{pmatrix} \begin{pmatrix} \widehat{\eta u} \\ \widehat{u^2}/2 \end{pmatrix}(\xi,\tau)\,d\tau \\
&= \mathrm{I}_1 - \mathrm{I}_2.
\end{aligned}$$

We notice

$$\begin{aligned}
K + iM &= (k+im)\big((1+(a+c)m^2-(a+c)k^2-6ack^2m^2+acm^4+ack^4)^2 \\
&\quad + (-2(a+c)km - 4ackm^3 + 4ack^3m)^2\big)^{1/4}.
\end{aligned} \tag{3.2}$$

On the other hand, if constants Q and R are large enough, then there exists $k \in \mathbb{R}$, with $|k| > R$, and $t_1 \in [-T,T]$ satisfying

$$|k| \quad > \quad B(1 + |\log(|\hat{\eta}(k,t_1)| + |\hat{u}(k,t_1)|)\,|).$$

Indeed, the inequality (2.9) can be rewritten, for all $Q > 0$ and $R > 0$, there exists $k \in \mathbb{R}$ satisfying

$$|k| > \frac{1}{2}\,(R - Q\log(|\hat{\eta}(k,t_1)| + |\hat{u}(k,t_1)|)\,),$$

and it is enough to take R and $Q > 2B$, according to the decrease of the Fourier transform. Even if we take Q and R larger, we can choose m such that

$$\frac{1}{|k|} < |m| \leq B^{-1}\Big[1 + |\log(|\hat{\eta}(k,t_1)| + |\hat{u}(k,t_1)|)\,|\,\Big]^{-1}.$$

Lemma 3.1. *Let us suppose, for all* $\xi \in \mathbb{C}$, $|\hat{\eta}(\xi,t_1)| = |\hat{u}(\xi,t_1)|$. *Then*

$$\begin{aligned}|I_1| &= e^{M\delta t}|\hat{\eta}(k,t_1)|\\ |I_2| &\leq C\frac{|k|+|m|}{M}(e^{M\delta t}-e^{-M\delta t})|\hat{\eta}(k,t_1)|.\end{aligned}$$

Proof. We have for I_1

$$\begin{aligned}|\hat{\eta}(\xi,t_2)|+|\hat{u}(\xi,t_2)| &= |\hat{\eta}(\xi,t_1)|\,|\cos((K+iM)\,\delta t)-i\sin((K+iM)\,\delta t)|\\ &= |\hat{\eta}(\xi,t_1)||e^{-i(K+iM)\delta t}| = |\hat{\eta}(\xi,t_1)|e^{M\delta t}.\end{aligned}$$

For I_2, the triangle inequality implies

$$\begin{aligned}|I_2| &\leq |\xi|\int_{t_1}^{t_2}\left(|\widehat{\eta u}(\xi,\tau)|+\frac{|\widehat{u^2}(\xi,\tau)|}{2}\right)(|\cos((K+iM)(t_2-\tau))|+|\sin((K+iM)(t_2-\tau))|)\\ &\leq C(|k|+|m|)\,(|\widehat{\eta}|*|\widehat{u}|(\xi,t_1)+|\widehat{u}|*|\widehat{u}|(\xi,t_1))\int_{t_1}^{t_2}e^{-M(t_2-\tau)}+e^{M(t_2-\tau)}\,d\tau\\ &\leq C(|k|+|m|)\,(|\widehat{\eta}|*|\widehat{u}|(\xi,t_1)+|\widehat{u}|*|\widehat{u}|(\xi,t_1))\,\frac{e^{M\delta t}-e^{-M\delta t}}{M}.\end{aligned}$$

The inequality (2.10) gives the result.

□

Finally, it remains

$$|\hat{\eta}(\xi,t_2)|+|\hat{u}(\xi,t_2)| \geq \left(C\frac{|k|+|m|}{M}e^{-M\delta t}+e^{M\delta t}(1-C\frac{|k|+|m|}{M})\right)|\hat{\eta}(k,t_1)|.$$

Even if we take Q and R larger again, we can choose m, with $m\delta t > 0$, such that $M > C(|k|+|m|)$. It gets then

$$|\hat{\eta}(\xi,t_2)|+|\hat{u}(\xi,t_2)| \geq Ce^{C|\delta t||k|}|\hat{\eta}(k,t_1)|.$$

We deduce from inequalities (2.9) and (3.1) that there exist $\widetilde{C} > 0$ and $k \in \mathbb{R}$, depending on Q and R such that

$$e^{|k|(C|\delta t|-1/Q)} \leq \widetilde{C},$$

which is impossible if we take $Q > 1/(C|\delta t|)$ and R large. The contribution $t_2 < t_1$ is dealt with similarly.

References

[1] J.L. Bona, M. Chen, J.-C. Saut, Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. I. Derivation and linear theory. *J. Nonlinear Sci.* **12** (2002), no. 4, pp 283–318.

[2] J.L. Bona, M. Chen, J.-C. Saut, Boussinesq equations and other systems for small-amplitude long waves in nonlinear dispersive media. II. The nonlinear theory. *Nonlinearity* **17** (2004), no. 3, pp 925–952.

[3] J. Bourgain, On the compactness of the support of solutions of dispersive equations, *Internat. Math. Res. Notices* **9** (1997), pp 437–447.

[4] J. V. Boussinesq, Théorie générale des mouvements qui sont propagés dans un canal rectngulaire horizontal, *C.R. Acad. Sci. Paris* **73** (1871), pp 256–260.

[5] T. Carleman, Sur les systèmes linéaires aux dérivées partielles du premier ordre à deux variables., *C. R. Acad. Sci.*, **197** (1933), pp 471–474.

[6] B. B. Kadomtsev, V. I. Petviashvili, Model equation for long waves in nonlinear dispersive systems, *Sov. Phys. Dokady.* **15** (1970), pp 891–907.

[7] J.-C. Saut, B. Scheurer, Unique continuation for some evolution equations, *J. Differential Equations*, **66** (1987), no. 1, pp 118–139.

[8] L. Schwarz, *Théorie des Distributuions*, Hermann, Paris (1966).

[9] G. B. Whitham, *Linear and Nonlinear Waves*, Wiley, New York (1999).