are equivariantly diffeomorphic. This is proved using a version of Reidemeister torsion defined for a very large class of spaces, finishing off a project started by M. Rothenberg 20 years ago.

Equivariant topology is still a very active area of research. In recent years the main development has been applying controlled methods. This goes beyond the scope of the present book, but combining controlled techniques with those presented here seems a promising area of research. It is difficult to explain controlled methods briefly. Basically, they are methods making it possible to make modifications arbitrarily close to a stratum, but in a "small" way, so the stratum fits as before. Another development, which is mentioned in the book, is relating topologically and analytically defined invariants. The basic object of study is a smooth manifold with a finite group G acting. Using the de Rham complex, it is now possible to define torsion invariant analytically, and relate these invariants to topologically defined invariants. The author has been a very active participant in the development of this area.

One of the virtues of this book is that it is carefully written with few mistakes. A reviewer nevertheless has a certain obligation to find at least one misprint: The signs in the matrices on page 223 are incorrect. The correct signs are obtained by reading the formulae instead.

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Space mappings with bounded distortion, by Yu. G. Reshetnyak. Translations of Mathematical Monographs, vol. 73, American Mathematical Society, Providence, RI, 1989, 362 pp., $129.00. ISBN 0-8218-4526-8

Quasiconformal (qc) and quasiregular (qr) mappings in euclidean n-space generalize the notions of a plane conformal mapping and of an analytic function of one complex variable, respectively. The systematic study of qc space mappings was begun by F. W. Gehring and J. Väisälä in the early 1960s, whereas the pioneering work on qr space mappings, due to Yu. G. Reshetnyak, appeared a few years later, in 1966–69. During the 1980s these mappings, particularly qc mappings, appeared in a number of applications in geometric function theory, geometry, topology, functional analysis, group theory, etc. and have thus become well known even outside the circle of specialists (cf. the surveys in [BM, G2, I, L, LF1, LF3, MA, RI2, S, V, VU1, pages ix–xv] as well as the works [DS, FH, GV, GR, MO, T, TV]). The main architect of the theory of qr mappings is Reshetnyak, who is well known also for his contributions to geometry, partial differential equations (PDEs), and nonlinear potential theory [R2].

Historical remarks. Planar qc and qr maps were introduced by H. Grötzsch in 1928 and applied by him in the study of several function-theoretic problems during the following three decades. Far-reaching results about these mappings were found by O. Teichmüller in the late 1930s through mid 1940s as well as by L. V. Ahlfors in the 1950s through 1960s. Although space qc mappings had been mentioned by M. A. Lavrent'ev already in 1938, the modern era in the theory began around 1960 when some results by E. D. Callender, C. Loewner, B. V. Shabat, F. W. Gehring, J. Väisälä, and Yu. G. Reshetnyak appeared almost simultaneously. By the mid 1960s the basic properties of space qc mappings had been established by Gehring and Väisälä. Exploiting some results of PDE theory associated with the names of J. Moser, F. John and L. Nirenberg and J. Serrin, Reshetnyak proceeded in a series of paper published in Sibirskiĭ Matematicheskiĭ Zhurnal to create the theory of general qr maps in 1966–69. He introduced these mappings under the name "mappings with bounded distortion." The shorter, nowadays more common name qr mapping was introduced by O. Martio, S. Rickman, and J. Väisälä in their joint work where they developed another approach to qr maps. Their papers appeared in Annales Academiae Scientarium Fennicae in 1969–72. V. A. Zorich [Z] made an important contribution to the theory of these maps by solving a global injectivity problem of M. A. Lavrent'ev. Later contributions to the theory of qr maps include results by many authors [BI1, BI2, GLM, I, IK, IM, MS, SE, VU1, VU2, AVV1]; see also the bibliographies of [VU1] and of the book under review.

Geometric function theory in n-space. At least part of the motivation for the investigation of qc and qr mappings comes from geometric function theory in the broad sense of the term. The right setup for the theory of qr mappings consists of a mixture of real analysis (including Sobolev spaces), PDEs and variational calculus, and geometric parts of complex analysis. Let us review briefly a few general properties of analytic functions which must be borne in mind when we want to find a fruitful extension of classical function theory.

The first property is a topological feature of a nonconstant analytic function f. Such a function f maps open sets onto open sets and is discrete ($f^{-1}(y)$ consists of isolated points). The second property is a bound for the local (or semiglobal) modulus of continuity provided by the Schwarz lemma (cf. also Cauchy's integral formula). The third property is conformal invariance: An analytic function f and its composition $h_1 \circ f \circ h_2$ with conformal mappings h_1 and h_2 are essentially identical. This third property has become especially popular in geometric function theory since the publication of the celebrated paper of Ahlfors and Beurling [AB]. As a geometer, F. Klein was a distinguished and successful advocate of invariance properties more than a century ago.

Next we are going to give the definition of a qr map. It is one of the deep and remarkable features of this definition, due to Reshetnyak, that the class of qr maps has the above-mentioned three properties. A continuous mapping f of a domain G in R^n into R^n is called qr if f belongs to the local Sobolev space $W^1_{n,\mathrm{loc}}(G)$ and if the inequality

$$|f'(x)|^n \leq K J_f(x)\,; \; |f'(x)| = \sup\{|f'(x)h| : |h| = 1\}$$

holds a.e. in G for some constant $K \geq 1$, where J_f is the Jacobian determinant of $f'(x)$. Mappings satisfying this inequality with a specific $K \geq 1$ are called $K-qr$. For $n = 2$ it is nontrivial (but true) that $1 - qr$ maps coincide with analytic functions.

We have already alluded to the fact due to Reshetnyak that qr maps have the above-mentioned three properties. Next, several additional remarks about the metric and analytic character of qr mappings are in order. Such maps are differentiable a.e. and satisfy Lusin's condition (N). The local modulus of continuity of qr and qc mappings is not Lipschitz continuous but only Hölder continuous. Consequently many odd phenomena can occur under qr and qc maps; e.g., the Hausdorff dimension is not invariant under these mappings. Generic qr maps are not smooth everywhere and it is essential to allow some degree of nonsmoothness to get interesting examples of qr maps.

Main problems: integrability and stability. There are several directions in which the theory of qc and qr mappings have made progress and numerous contexts where these mappings have played a role. May it suffice to mention BMO functions [RR], Royden algebras [LF2], automorphic functions [MS], nonlinear PDEs [GLM], value distribution theory [RI1, RI2], conformal invariants [VU1], topological results [T, DS, TV], and dynamical systems [S] as well as individual recent results [IM, FH, GR]. From this multitude of problems we wish to mention here two which are simple to state and are crucial for the book under review. (1) Find the largest $p > n$ such that $|f'| \in L^p_{\text{loc}}(G)$ for every $K - qr$ mapping $f\colon G \to R^n$ (L^p-integrability problem). (2) Let $f\colon R^n \to R^n$ be a $K - qr$ mapping, $n \geq 3$, normalized by $f(0) = 0$, $f(e_1) = e_1$, where $e_1 = (1, 0, \dots, 0) \in R^n$, and let I denote the class of all isometries of R^n keeping 0 fixed. Find a concrete number $\varepsilon(K, n)$ such that $\varepsilon(K, n) \to 0$ as $K \to 1$ and

$$(*) \qquad \inf_{A \in I} \sup\{|f(x) - Ax| \colon |x| \leq 1\} \leq \varepsilon(K, n)$$

(a special case of the stability problem). The first problem has been extensively investigated by a number of authors quoted in the book under review, in particular, by B. Bojarski [B], F. W. Gehring [G1], and T. Iwaniec [I, IK]. For each $K > 1$, a $K - qc$ radial stretching delivers an upper bound $p(n, K)$ for the largest exponent in Problem (1). The open question here is whether $|f'| \in L^{p(n,K)}_{\text{loc}}$ for each $K - qc$ mapping f. This question is rather important since "reverse Hölder-inequalities" [G1] have found many applications, e.g., in the variational calculus. See also [M].

Problem (2) is motivated by a far-reaching generalization of Liouville's theorem, due to Gehring and Reshetnyak, which states that a $1 - qr$ mapping of a domain G in R^n, $n \geq 3$, is of the form $h|G$ where h is a Möbius transformation. This means that h is an element in the group generated by reflections in spheres and hyperplanes. A simplified proof of this result is due to Bojarski and Iwaniec [BI2]. Furthermore, Problem (2) arises also in the study of elasticity properties of materials where the case of a bi-Lipschitz mapping f is considered [J1, J2, K]. The author has written four books of which the book under review is the first one and the second book [R1] is devoted entirely to his results related to this problem—called stability theory by the author. The main results of stability theory are also presented in the book under review. The basic open question here is to find explicit, concrete estimates, e.g., in Problem (2). The methods of stability theory rely on normal family properties of qr mappings which show the existence of a certain type of estimates: The actual majorant functions remain unknown. Several contributions to this problem have been made by the author's collegues and former students whose work is quoted or outlined in the book. In particular, V. I. Semenov wrote several papers during the 1980s with deep results on this subject (cf. e.g., [SE]). Semenov's work leads in some cases to explicit constants expressed in terms of special functions such as complete elliptic integrals. From the viewpoint of qc maps, the theory of these special functions has been studied in [AVV2]. Since no concrete number $\varepsilon(K, n)$ is known in the above problem $(*)$ with $\varepsilon(K, n) \to 0$ as $K \to 1$, one can try to solve simpler problems. A large class of problems is concerned with estimating a functional such as $m_1(f(x), f(y))$ in terms of $K, n, m_2(x, y)$ where f is $K - qr$ and m_1, m_2 are some prescribed metrics. An example is the qr version of the Schwarz lemma where an inequality of this type is obtained with concrete constants and a sharp bound as $K \to 1$ [AVV1]. Further, from the Schwarz lemma one can derive a concrete asymptotically sharp bound as $K \to 1$ for

$$\sup\{|f(x)|: |x| \leq 1, f \in QC_K\}$$

where QC_K is the family of all $K - qc$ mappings $f: R^n \to R^n$ with $f(0) = 0$ and $f(e_1) = e_1$ [VU2]. Results of this kind may be regarded as first steps towards the quantitative solution of $(*)$. Many more problems of the same nature can be obtained by the following general procedure: Take any theorem dealing with qc or qr space maps. Find a version of the theorem with concrete constants and asymptotically sharp behavior as $K \to 1$. This large class of problems will offer challenging tasks for many years to come.

Reshetnyak's book. The book under review is a considerably extended and revised translation of the original 1982 Russian version. The book provides a balanced, well-organized, and self-contained introduction to the fundamental properties of qr mappings. The topics included cover material from Sobolev space theory, differential forms, conformal capacity, and PDEs. One of the highlights of the book is a proof of the fact that a nonconstant qr mapping is discrete open. The new material in the translation also covers important theorems from stability theory due to the author and his colleagues as well as some related results. Due to the revision and expansion of the text the translation is more up-to-date than the original text. The author states that the choice of material reflects his own research interests (pages xi, 246). The topics chosen are fundamental for the theory of qr mappings, and therefore the book fills a gap in mathematical literature. At its time of publication, 1989, the book under review is the second book in English about space qr maps. The first book (1988) of the reviewer [VU1] exploits conformal invariants as instruments for investigating qr maps and emphasizes topics different from those in Reshetnyak's book. Accordingly these two books are largely nonoverlapping and complement each other. It should be pointed out, however, that even these two books together do not cover all the existing results and hence there is room for new books on space qc and qr mappings in the monograph literature.

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Distance-Regular Graphs, by A. E. Brouwer, A. M. Cohen, and A. Neumaier. Springer-Verlag, New York, 1989, 485 pp., $84.50. ISBN 3-540-50619-5

The title of the present book refers to a class of graphs whose regularity properties go back to the platonic solids of antiquity. Examples of such graphs are provided by the vertices and the edges of the cube, or of the icosahedron. On the other hand, the graphs have deep connections to many topics of the present-day theory and applications of groups, geometries, codes, and designs. Thus the book represents a large part of discrete mathematics. The geometry of the graphs is phrased in terms of distances; it has a direct translation into algebra. Many mathematical disciplines, ranging from functional analysis to computation, contribute to the understanding of our graphs. Now let us first give some definitions. The