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SPECTRAL THEORY OF REINHARDT MEASURES

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Let μ be a finite positive Borel measure on $\mathbf{C}^n$ $(n \geq 1)$, with compact support K, let $P^2(\mu)$ be the norm closure in $L^2(\mu)$ of the algebra of complex polynomials in $z_1, \dots, z_n$, and let $M_z = (M_{z_1}, \dots, M_{z_n})$ be the n-tuple of multiplication operators by the coordinate functions $z_1, \dots, z_n$ acting on $P^2(\mu)$. M_z is the universal model for cyclic subnormal n-tuples of operators acting on a separable Hilbert space. For $n = 1$, the spectral and algebraic properties of M_z have been the focus of extensive study (see [Con] for a survey account of the basic results in this area). One important instance, the case $d\mu(re^{i\theta}) = d\rho(r) \times \frac{d\theta}{2\pi}$ (where ρ is a positive Borel measure on $[0, +\infty)$), gives rise to the class of subnormal weighted shifts, via Berger's Theorem [Con, III.8.16]. Here, the spectral picture of M_z admits a very simple description:

(i) $\sigma(M_z)$, the spectrum of M_z, equals $D_\mu := \{\lambda \in \mathbf{C}: |\lambda| \leq \sup\{|z|: z \in K\}\}$;

(ii) The Fredholm domain of M_z is $\mathbf{C}\backslash\partial D_\mu$; and

(iii) $\operatorname{index}(M_z - \lambda) = -1$ whenever $\lambda \in \operatorname{int}(D_\mu)$.

The circular symmetry of weighted shifts, reflected in the above description, appears in several variables in the notion of *Reinhardt* set; $F \subseteq \mathbf{C}^n$ is Reinhardt if $F = \tau^{-1}(\tau(F))$, where $\tau: \mathbf{C}^n \to \mathbf{R}_+^n$ is given by $z \to (|z_1|, \dots, |z_n|)$. Correspondingly, a compactly supported positive Borel measure μ is Reinhardt if it admits a decomposition $d\mu(re^{i\theta}) = d\rho(r) \times d\theta/(2\pi)^n$, where ρ is a positive Borel measure on $\mathbf{R}_+^n$. For instance, volumetric Lebesgue measure on a complete bounded Reinhardt domain $\Omega \subseteq \mathbf{C}^n$ is a Reinhardt measure, in which case $P^2(\mu)$ is actually $A^2(\Omega)$, the Bergman space over Ω.

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The spectral and C^*-algebraic properties of M_z on $A^2(\Omega)$, for $\Omega \subseteq \mathbf{C}^n$ Reinhardt or pseudoconvex, have been extensively investigated, as have been those of M_z acting on the Hardy spaces over the Shilov boundary of bounded symmetric domains (e.g., [BC, BCK, BCZ, BdeM, Cob, CM, CS, DH, MR, P, Ra, SSU, U, V]). In this note we announce a complete description of the spectral picture of M_z in case μ is a Reinhardt measure on $\mathbf{C}^2$ whose associated weight sequences have limits at infinity in all directions (a notion to be defined later).

To describe our results, we need some notation. Let

$$V := \{(z_1, \dots, z_n) \in \mathbf{C}^n : z_1 \cdot \dots \cdot z_n = 0\}.$$

Without loss of generality, we can, and shall, assume that $K \subseteq \overline{D}^n$ and that K is not contained in V, since otherwise M_z is unitarily equivalent to the orthogonal direct sum of n-tuples of the form $(M_{(z_1, \dots, z_k)}, 0, \dots, 0)$. For μ Reinhardt, the set of *bounded point evaluations* for μ is b.p.e. $(\mu) := \{\lambda \in \mathbf{C}^n : p \to p(\lambda), p \in \mathbf{C}[z]$, extends boundedly to $P^2(\mu)\}$. The Taylor spectrum of M_z, $\sigma_T(M_z)$, is a nonempty compact subset of $\mathbf{C}^n$ defined in terms of the exactness of a cochain complex, called the Koszul complex, built from the exterior algebra on n generators and the coordinates M_{z_i}, $i = 1, \dots, n$. The Taylor spectrum enjoys most of the usual properties of the spectrum of a single Hilbert space operator, and supports an analytic functional calculus. There is also a notion of Fredholmness and of index for commuting n-tuples of operators. (For basic facts on joint spectral systems, the reader is referred to [Cu].) Finally, for a compact subset F of $\mathbf{C}^n$, we let $\widehat{F}$ denote the polynomially convex hull of F. If F is Reinhardt and $0 \in F$, then $\widehat{F} = \tau^{-1}\{\exp[\text{convex hull}(\log(\tau(F) \setminus V))]\}^-$. In particular, if $z \in F$ then the polydisk $\{w \in \mathbf{C}^n : |w_i| \le |z_i|, i = 1, \dots, n\}$ is contained in $\widehat{F}$.

Theorem 1. *Let μ be a Reinhardt measure on $\mathbf{C}^n$. Then*
(i) $\operatorname{int} \widehat{K} \subseteq$ b. p. e.$(\mu) \subseteq \widehat{K}$;
(ii) $\sigma_T(M_z^{(\mu)}) = \widehat{K}$.

To prove Theorem 1(i), we construct a dense-range operator from $P^2(\mu)$ to the Hardy space of the n-torus, $H^2(\mathbf{T}^n)$, and we then use it to pull back the Szegö kernel function from $H^2(\mathbf{T}^n)$ to $P^2(\mu)$; part (ii) requires a spectral inclusion [Cu, Theorem

7.5(ii)] together with the containment b.p.e.$(\mu) \subseteq \sigma_T(M_z)$. To discuss our calculation of the Taylor essential spectrum of M_z, we require some preparations. To begin with, the existence of bounded point evaluations in a neighborhood Ω of the origin (Theorem 1(i)) gives rise to a kernel function $k(w, z)$ such that $f(z) = \langle f, k(\cdot, z)\rangle$ for all $f \in P^2(\mu)$, $z \in \Omega$. Let $\lambda \in K\backslash V$ and let $\varepsilon := (\min_i |\lambda_i|)/3$. Use of the Cauchy-Schwarz inequality now yields the following key estimate: There exists a constant $C > 0$ such that

$$\iint_{\overline{D}(0,\varepsilon)\cap K} |f|^2\, d\mu \le C \iint_{K\backslash \overline{D}(0,\varepsilon)} |f|^2\, d\mu,$$

for every $f \in P^2(\mu)$, where $D(0, \varepsilon)$ is the open polydisk centered at the origin and of multiradius $(\varepsilon, \dots, \varepsilon)$. From this we can derive the next result.

Proposition 1. *Let μ be a Reinhardt measure on $\mathbf{C}^n$. Then M_z is (jointly) bounded below, i.e., there exists $\delta > 0$ such that $||z_1 f||^2 + \cdots + ||z_n f||^2 \ge \delta^2 ||f||^2$ for all $f \in P^2(\mu)$.*

Since M_z is bounded below, we can use the groupoid machinery introduced in [CM] to analyze $C^*(M_z)$. This is done as follows. First, observe that M_z is unitarily equivalent to an n-tuple of n-variable weighted shifts; for, if we let $e_\alpha := z^\alpha/||z^\alpha||_{L^2(\mu)} (\alpha \in \mathbf{Z}^n_+)$, it follows from the Reinhardtness of μ that $\{e_\alpha\}_{\alpha\in\mathbf{Z}^n_+}$ is an orthonormal basis for $P^2(\mu)$, and that $M_{z_i} e_\alpha = w_i(\alpha) e_\alpha$, where

$$w_i(\alpha) := ||z^{\alpha+\varepsilon_i}||/||z^\alpha||,\ \alpha \in \mathbf{Z}^n_+,\ i = 1, \dots, n,$$
$$\varepsilon_i := (0, \dots, 0, \overset{i}{1}, 0, \dots, 0).$$

Similarly, if $\beta \in \mathbf{Z}^n_+$, the powers $M_z^\beta := M_{z_1}^{\beta_1} \cdot \cdots \cdot M_{z_n}^{\beta_n}$ are associated with weight sequences $w_\beta(\cdot)$. Extend w_β to all of $\mathbf{Z}^n$ via $w_\beta(\alpha) := 0$ $(\alpha \notin \mathbf{Z}^n_+)$, and let $\mathscr{A}$ be the *closed translation-invariant subalgebra* of $l^\infty(\mathbf{Z}^n)$ generated by $\{w_\beta\}_{\beta\in\mathbf{Z}^n_+}$, not including the constants. The maximal ideal space of $\mathscr{A}$, denoted Y, is a noncompact, locally compact Hausdorff space on which $\mathbf{Z}^n$ acts by translation. The map $\varphi: \mathbf{Z}^n \to Y$ given by $\varphi(\alpha)(a) := a(\alpha)$, $\alpha \in \mathbf{Z}^n$, $a \in Y$, is injective and open, and $X := \overline{\varphi(\mathbf{Z}^n_+)} \subseteq Y$ is compact. Thus, X is a suitable *compactification* of $\mathbf{Z}^n_+$ [CM, Lemma 2.1 and Lemma 2.3]. If we let $\mathfrak{G} := Y \times \mathbf{Z}^n|_X := \{(y, \alpha) \in Y \times \mathbf{Z}^n : y \in X \text{ and } y + \alpha \in X\}$, we see that $\mathfrak{G}$ is the groupoid obtained by reducing the transformation group $Y \times \mathbf{Z}^n$ to X, which therefore becomes the unit space of $\mathfrak{G}$. A careful analysis of X leads to a detailed description of the ideal structure of $C^*(M_z)$, based on the correspondence between open invariant subsets of X and closed ideals in $C^*(M_z)$. Since X is obtained from $\mathbf{Z}_+^n$ by adding suitable limit points at infinity, we need to impose conditions on μ that guarantee a tractable identification of $X \backslash \mathbf{Z}_+^n$.

We shall say that a Reinhardt measure μ has *convergent weight sequences* if for every $i, j = 1, \dots, n$ and for every $\alpha \in \mathbf{Z}_+^n$, the sequence $\{w_i(\alpha + k\varepsilon_j)\}_{k=1}^{\infty}$ is convergent. The following theorem says that one can always assume that μ has no mass near the origin.

Theorem 2. *Let μ be a Reinhardt measure on $\mathbf{C}^n$, let $K := \operatorname{supp} \mu$, assume that μ has convergent weight sequences, let Ω be a neighborhood of $\partial \widehat{K}$, and let $\nu := \mu|_\Omega$. Then $C^*(M_z^{(\mu)})$ is $*$-isomorphic to $C^*(M_z^{(\nu)})$. Moreover, $M_z^{(\mu)}$ is a compact perturbation of $M_z^{(\nu)}$ (when each is regarded as an n-tuple of n-variable weighted shifts on $l^2(\mathbf{Z}_+^n)$). In particular, $M_z^{(\mu)}$ and $M_z^{(\nu)}$ have identical spectral pictures.*

Our description of the spectral picture of M_z relies on some special properties of the Koszul complex for M_z in case $n = 2$. Recall that

$$K(M_z) : 0 \to P^2(\mu) \xrightarrow{D^0(\mu)} P^2(\mu) \oplus P^2(\mu) \xrightarrow{D^1(\mu)} P^2(\mu) \to 0,$$

where

$$D^0(\mu) f = z_1 f \oplus z_2 f$$

and

$$D^1(\mu)(f \oplus g) = -z_2 f + z_1 g (f, g \in P^2(\mu)).$$

It follows from Proposition 1 that $D^0(\mu)$ is bounded below, and a trivial calculation then shows that $K(M_z)$ is exact at the middle stage, so that, by Theorem 1, $\operatorname{index}(M_z^{(\mu)}) = 1$ once we establish that 0 is in the Fredholm domain of M_z. In the sequel, we assume that $n = 2$.

To analyze X, we proceed as in [CM]. $\varphi(\mathbf{Z}_+^2)$ is an open invariant subset of X, whose associated ideal in $C^*(M_z)$ is the ideal of compact operators; on the other hand, we let ∞_G denote the subset of X consisting of all limit points of sequences

$\{\varphi(\mathbf{k}^{(j)})\}_{j=1}^{\infty}$, where $k_i^{(j)} \to +\infty$ for $i = 1, 2$. Clearly ∞_G is a closed invariant subset of X, and $\mathfrak{G}|_{\infty_G} = \infty_G \times \mathbf{Z}^2$. When both $\varphi(\mathbf{Z}_+^2)$ and ∞_G are removed from X, we are left with two disjoint subsets, ∞_N and ∞_E, consisting of all points in X obtained by taking limits along vertical and horizontal directions, respectively; e.g., $\infty_N := \{x \in X : x = \lim_j \varphi(\mathbf{k}^{(j)}),\ \{k_1^{(j)}\}$ is *bounded* and $k_2^{(j)} \to +\infty\}$. In the spectral and algebraic descriptions of M_z, the key role is played by ∞_G, on which we now focus our attention. Given a direction $\vec{u} \in \mathbf{R}_+^2$ we let

$$\infty_{\vec{u}} := \{x \in X : \exists \{\varphi(\mathbf{k}^{(j)})\}_{j=1}^{\infty} \text{ with } \varphi(\mathbf{k}^{(j)}) \xrightarrow{w^*} x,\ \mathbf{k}^{(j)} = p_j\vec{u} + q_j\vec{u}^{\perp},\ \text{and } q_j/p_j \to 0\}.$$

Clearly $X = \bigcup_{\vec{u}\in\mathbf{R}_+^2} \infty_{\vec{u}}$, although two directions may give rise to the same limit points, and different limit points may correspond to the same direction. Nevertheless, the sets $\infty_{\vec{u}}$ carry important information.

A Reinhardt measure μ on $\mathbf{C}^n$ is said to have *convergent weight sequences in all directions* (c.w.s.a.d.) if for every direction $\vec{u} \in \mathbf{R}_+^2$ and every sequence $\{\mathbf{k}^{(j)} = p_j\vec{u} + q_j\vec{u}^{\perp}\} \xrightarrow{w^*} x \in \infty_{\vec{u}}$ with $q_j/p_j \to 0$, the convergence of $\{q_j\}$ to some $q \in \mathbf{R}$ implies the convergence of $\{w_i(\mathbf{k}^{(j)})\}_{j=1}^{\infty}$, $i = 1, 2$. Volumetric Lebesgue measure on a complete pseudoconvex Reinhardt domain and surface measure on the boundary of such a domain are two canonical examples of such Reinhardt measures; additional examples are given by Reinhardt measures μ such that $\operatorname{supp} \mu|_{\partial\widehat{K}} = K \cap \partial\widehat{K}$. Intuitively, a measure μ has c.w.s.a.d. if it admits "balayage" to the boundary. There are, however, measures which do not have c.w.s.a.d.

Following the notation in [SSU], we let C be the closed convex hull of $\log(\tau(K\backslash V))$. Then $\partial C = \partial^0 C \cup \partial^1 C$ (the boundary of C is the union of its 0- and 1-dimensional faces).

Proposition 2. *Let μ be a Reinhardt measure on $\mathbf{C}^2$, and assume that μ has c.w.s.a.d. Then ∞_G can be identified with $\partial C\backslash(F_v \cup F_h)$, where F_v and F_h are the vertical and horizontal (open) faces of ∂C, if they exist.*

Each oblique 1-dimensional face of ∂C gives rise to a direction $\vec{u} \in \mathbf{R}_+^2$; if μ has c.w.s.a.d., the corresponding $\infty_{\vec{u}}$ is topologically equivalent to the two-point compactification of the real line, with the action of $\mathbf{Z}^2$ given by $t + (\alpha_1, \alpha_2) = t + \alpha_1 u_1 - \alpha_2 u_2$. This puts into evidence the presence of a copy of an *irrational rotation* C^*-algebra when $u_1/u_2 \notin \mathbf{Q}$, intrinsic to the proof of (iii) below.

Theorem 3. *Let μ be a Reinhardt measure on $\mathbf{C}^2$, and assume that μ has c.w.s.a.d. Then*

(i) *$M_z - \lambda$ is bounded below if and only if*

$$\lambda \notin (\exp(\partial^0 C \times \mathbf{T}^2))^-,$$

(ii) *$M_z - \lambda$ is invertible if and only if $\lambda \notin \widehat{K}$,*

(iii) *$M_z - \lambda$ is Fredholm if and only if $\lambda \notin \partial\widehat{K}$,*

(v) $\operatorname{index}(M_z - \lambda) = \begin{cases} 1 & \text{if } \lambda \in \operatorname{int}\widehat{K}, \\ 0 & \text{if } \lambda \notin \widehat{K}. \end{cases}$

Theorem 3 should be compared with [SSU, Theorem 1.3], where μ is volumetric Lebesgue measure on a complete pseudoconvex Reinhardt domain. Unlike the sheaf-theoretical methods used in [CS], [P], and [SSU] for the Bergman space case (obviously not applicable in the case of a general measure), our proof uses J. Bunce's characterization of the left spectrum [B], results from multiparameter spectral theory, and a covering lemma for the spectrum of $C^*(M_z)$ to reduce the problem to the case when $\operatorname{int}\widehat{K}$ is the *L-shaped domain* $\Omega_{\delta_1,\delta_2} := \{(z_1, z_2) \in \mathbf{C}^2 : (|z_1| < \delta_1, |z_2| < 1)$ or $(|z_1| < 1, |z_2| < \delta_2\}$ $(0 < \delta_1, \delta_2 < 1)$. For $\Omega_{\delta_1,\delta_2}$, we calculate $\sigma_{Te}(M_z)$ by explicitly exhibiting a pair of (1-variable) bilateral weighted shifts acting on $l^2(\Gamma)$ (Γ a subgroup of $\mathbf{R}$), obtained via a suitably built faithful representation of $C^*(M_z)/\mathscr{K}$ associated with the direction $\vec{u} = (-\log\delta_2, -\log\delta_1)$. Our techniques also allow us to handle certain cases of Reinhardt measures which do not have c.w.s.a.d., e.g., the example studied in [S].

Details of this work will be forthcoming.

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