

THE SELBERG ZETA FUNCTION AND SCATTERING POLES FOR KLEINIAN GROUPS

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In this note we present a polynomial bound on the distribution of poles of the scattering operator for the Laplacian on certain hyperbolic manifolds M^n of infinite volume. The motivation is to understand more fully the geometry of the poles of the scattering operator. The proof uses the relationship between poles of the scattering operator and zeros of the Selberg zeta function for geodesic flow on M^n .

Recall that the classical Selberg zeta function $Z(s)$ [30] is a meromorphic function which describes the lengths $\ell(\gamma)$ of closed geodesics γ on a compact surface S :

$$(1) \quad Z(s) = \prod_{\gamma} \prod_{m=1}^{\infty} \{1 - \exp(-(s+m)\ell(\gamma))\}$$

where the product over γ runs over primitive closed geodesics. Crude estimates on the distribution of lengths show that $Z(s)$ is analytic for $\Re(s) > 1$; an application of Selberg's trace formula shows that $Z(s)$ extends to a meromorphic function on \mathbb{C} with trivial zeros at the integers $1, 0, -1, \dots$ together with spectral zeros at the numbers s_k where $s_k(1-s_k)$ is an eigenvalue of the Laplacian on S [11, 20, 30].

Here we will be concerned with the Selberg zeta functions, introduced by Patterson [25], for certain noncompact hyperbolic manifolds M^n of infinite hyperbolic volume. These zeta functions are defined by an infinite product similar to (1) (identical when $n = 2$); they share important features with the classical zeta function. In particular, their analytic structure is closely tied to the spectrum of the Laplacian on M^n . However, the Laplacian

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on M^n has at most finitely many discrete eigenvalues and absolutely continuous spectrum of infinite multiplicity [13, 14, 15]; thus the dominant contribution to the spectrum is described by the scattering operator $S(s)$ associated to the absolutely continuous spectrum. $S(s)$ is an operator-valued meromorphic function that relates the asymptotic behavior of incoming and outgoing generalized eigenfunctions of the Laplacian (see [1, 16, 17, 18, 19, 26, 28]). The poles of $S(s)$ play a role similar to the role played by eigenvalues of the Laplacian for a compact surface; the zeta function has spectral zeros at poles of the scattering operator together with an infinite sequence of trivial zeros or poles which appear to carry topological information [25, 27, 29].

In contrast to the eigenvalues of the Laplacian, the poles of the scattering operator are eigenvalues of a nonselfadjoint eigenvalue problem, so that very little is known about their nature and distribution. Recent studies of the scattering operator for acoustical scattering have derived polynomial bounds on the distribution function

$$(2) \quad N(r) = \#\{k \in \mathbf{C}: |k| \leq r \text{ and } k \text{ is a pole} \\ \text{of the scattering operator}\}$$

(see Melrose [21, 22]), Zworski [33, 34, 35] derives exact leading asymptotics for $N(r)$ in the case of Schrödinger scattering by a compactly supported potential.

Here we prove a polynomial bound for the distribution of poles of the scattering operator $S(s)$. We know of no previous result of this type for the Laplacian on a hyperbolic manifold. Using this bound, one should be able to prove a trace formula involving the poles of the scattering operator and obtain finer information on the distribution of poles (cf. [21, 22, 3] for trace formulas in acoustical scattering and [10, 12] for their application to analysis of poles of the scattering operator in certain acoustical problems).

To state our result more precisely, we first recall some notions of hyperbolic geometry. Model hyperbolic space \mathbf{H}^n as the unit ball $\mathbf{B}^n \subset \mathbf{R}^n$ with the Poincaré metric

$$ds^2 = 4|dx|^2 / (1 - |x|^2)^2,$$

where $|dx|^2$ is the Euclidean metric on \mathbf{R}^n . Then $S^{n-1} \subset \mathbf{R}^n$ is the boundary at infinity for \mathbf{H}^n , and the isometries of \mathbf{H}^n are

Möbius transformations leaving \mathbf{B}^n invariant. A discrete group Γ of isometries of \mathbf{H}^n is called *geometrically finite* if it admits a fundamental domain bounded by finitely many geodesic hyperplanes. The *limit set* Λ_Γ is the set of accumulation points in S^{n-1} of Γ -orbits of points in \mathbf{H}^n ; it is a closed, Γ -invariant set. The convex hull of the limit set, $CH(\Lambda_\Gamma)$, is the smallest hyperbolically convex subset of \mathbf{H}^n whose boundary (viewed as a subset of \mathbf{R}^n) contains Λ_Γ . $CH(\Lambda_\Gamma)$ is Γ -invariant; the quotient $FN = CH(\Lambda_\Gamma)/\Gamma$ is called the Fenchel-Nielsen region. Closed geodesics on M^n lift to geodesics in \mathbf{H}^n with both endpoints in the limit set; such geodesics are contained in $CH(\Lambda_\Gamma)$, so that closed geodesics of M^n lie in FN . A geometrically finite discrete group Γ is called *convex co-compact* if FN is a compact subset of \mathbf{H}^n/Γ .

We consider quotients $M^n = \mathbf{H}^n/\Gamma$ where

- (*) Γ is geometrically finite, convex co-compact,
and torsion-free, and $\text{vol}(\mathbf{H}^n/\Gamma) = \infty$.

The no-torsion condition guarantees that M^n is a Riemannian manifold. The convex co-compactness implies that M^n has no cusps. Since FN is compact, the closed geodesics of M^n lie in a compact set. Since closed geodesics are dense in the nonwandering set by Anosov's closing lemma [2], the nonwandering set for geodesic flow on the unit tangent bundle of M^n is also compact.

The nontrivial geometry of M^n , as well as its nontrivial dynamics, is concentrated in FN . Let FN_1 be the set of points at distance ≤ 1 from FN . Then FN_1 is a C^1 submanifold with boundary and $M^n - FN_1$ is the union of finitely many connected components diffeomorphic to cylinders $(0, 1) \times N_i$ where the N_i are compact manifolds. The geometric boundary of M is $\partial_\infty M^n = \bigcup_i N_i$ and can also be realized as Ω_Γ/Γ where $\Omega_\Gamma = S^{n-1} - \Lambda_\Gamma$ is the domain of discontinuity of Γ acting in S^{n-1} .

The Laplacian on M^n has at most finitely many eigenvalues in $[0, (n-1)^2/4)$ (owing to the infinite volume of M^n [13]), and purely absolutely continuous spectrum on $[(n-1)^2/4, \infty)$ [14]. The associated scattering operator $S(s)$ is a meromorphic pseudodifferential operator-valued function acting on line bundles over $\partial_\infty M^n$ (see Agmon [1], Froese, Hislop, and Perry [7], Mandouvalos [16, 17, 18, 19], and Perry [26, 27, 28] for further discussion). The relation between the scattering operator $S(s)$ and the zeta

function $Z(s)$ is

Theorem 0.1 ([25, 27]). *Let M^n satisfy (*). Then for $\Re(s) = (n - 1)/2$, $s \neq (n - 1)/2$, the identity*

$$\frac{Z'(s)}{Z(s)} + \frac{Z'(n - 1 - s)}{Z(n - 1 - s)} = \text{Tr}(S(s)^{-1}S'(s))$$

holds.

Here the prime denotes differentiation with respect to s , and Tr denotes a suitably renormalized trace (see [27]). In [27] it is shown that the right-hand side has a meromorphic continuation to \mathbb{C} and that each of the nontrivial poles of the scattering operator is a pole of the right-hand side with positive integral residue equal to the dimension of the zero eigenspace of a Fredholm operator that determines the poles of $S(s)$.

On the other hand, the dynamical methods of Ruelle [31] and Fried [8] show that

Theorem 0.2. *Let M^n satisfy (*). Then the zeta function $Z(s)$ for M^n is the quotient of entire functions of order $\leq n$.*

Although [8] and [31] analyze zeta functions for geodesic flow on compact manifolds rather than noncompact manifolds, their results remain true for Axiom A flows on noncompact manifolds with analytic stable and unstable foliations and a compact non-wandering set. It is relatively easy to check that these assumptions are satisfied for geodesic flow on the unit tangent bundle of the manifolds M^n considered here.

Together with a trivial estimate on the distribution of zeros of entire functions of order n , Theorems 0.1 and 0.2 imply:

Theorem 0.3. *Let $N(r)$ be the counting function (2) for poles of $S(s)$, i.e.,*

$$N(r) = \#\{s \in \mathbb{C}: |s| \leq r \text{ and } s \text{ is a pole of } S(s)\}.$$

Then for any $\epsilon > 0$ there is a $C_\epsilon > 0$ such that

$$N(r) \leq C_\epsilon(1 + r)^{n+\epsilon}.$$

This result is almost optimal, as the following examples due to Charles Epstein [6] show. Let Γ_{n-1} be a discrete group of isometries of \mathbb{H}^{n-1} so chosen that $M^{n-1} = \mathbb{H}^{n-1}/\Gamma_{n-1}$ is a compact Riemannian manifold. There is a natural embedding of $\text{Isom}(\mathbb{H}^{n-1})$ in $\text{Isom}(\mathbb{H}^n)$ such that, if Γ_n is the image of Γ_{n-1} under this embedding, \mathbb{H}^n/Γ_n is isometric to $\mathbb{R} \times M^{n-1}$ with hyperbolic metric

$ds^2 = d\tau^2 + \cosh^2(\tau)d\omega^2$, where $d\omega^2$ is the hyperbolic metric on M^{n-1} . The zeta function $Z_{\Gamma^n}(s)$ is given in terms of the zeta function $Z_{\Gamma^{n-1}}(s)$ by the formula

$$(3) \quad Z_{\Gamma^n}(s) = \prod_{m=0}^{\infty} Z_{\Gamma^{n-1}}(s+m).$$

$Z_{\Gamma^{n-1}}(s)$ has spectral zeros due to eigenvalues of the Laplacian on M^{n-1} and topological zeros or poles depending on the dimension and topology of M^{n-1} (see [5, 9, 11, 30]). The spectral zeros of $Z_{\Gamma^{n-1}}(s)$ are those s_k on the line $\Re(s) = (n-2)/2$ for which $s_k(n-2-s_k)$ is an eigenvalue of the Laplacian on M^{n-1} . Let $s_k = (n-2)/2 + it_k$. The t_k obey the estimate (Weyl's law for M^{n-1} [4])

$$N(t_k: |t_k| \leq r) \sim r^{n-1}.$$

Since the product (3) has zeros $s_{k,m} = -m + s_k$ for m a nonnegative integer, it follows that the counting function for these spectral zeros of the product (3) obeys $N(r) \sim r^n$. It can be shown by explicit calculation, using the explicit form of the metric to separate variables, that all of these zeros occur as poles of the scattering operator for M^n .

Details of the proof and the examples described above will appear in [29].

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