which make the book most welcome. The book is written in a topological mode, it is true, but it is accessible and suitable for a wider readership, being clear and careful in style, emphasizing the search for the "right" notion and "right" proof. The specialist on the other hand may still find interesting homotopy theory (fibrewise) which is new to him, in the last part of the book.

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The logarithmic integral I, by Paul Koosis. Cambridge Studies In Advanced Mathematics, vol. 12, Cambridge University Press, Cambridge, New York, New Rochelle, Melbourne, Sydney, 1988, xvi + 606 pp., $89.50. ISBN 0-521-30906-9.

One of the most important and distinctive features usually associated with the class of analytic functions is what is commonly referred to as the unique continuation property. Put in its simplest form it says: if $f(z)$ is defined and analytic on an open set Ω in the complex plane and if either

(1) $f(z) = 0$ on a set with a limit point in Ω, or
(2) $f^{(n)}(z_0) = 0$, $n = 0, 1, 2, \ldots$, at some $z_0 \in \Omega$,

then $f(z) \equiv 0$ on Ω. In short, an analytic function is completely determined by its behavior on a rather small portion of its domain of definition. Koosis's book, *The logarithmic integral*, (LI), is in large part concerned with extensions and applications of this basic fact.

By 1892—at the age of twenty-one—Émile Borel had become convinced that it must be possible to extend the uniqueness property to much larger, more general, nonanalytic classes of functions defined, for example, on sets without interior points. To some, however, it seemed highly unlikely that such a program could be carried out in any meaningful way and Poincaré had even constructed certain examples to strengthen the negative point of view. Nevertheless, Borel persisted in his conviction and at his thesis defense of 1894—at which Poincaré was the rapporteur—he continued to press his case, not yet being able to exhibit a suitable extension of the classical concept of an analytic function while retaining the distinctive feature of unique continuation.

In the years that followed, Borel sought to give more precise expression to his ideas and he eventually created the theory of monogenic functions as outlined in his book [5], published some twenty-five years later. A function f defined on an arbitrary subset E of the complex plane is monogenic at a point $x_0 \in E$ if it is differentiable at x_0 in the sense that

$$\lim_{x \to x_0, x \in E} \frac{f(x) - f(x_0)}{x - x_0}$$

exists through points of E; the set of functions monogenic at every point of E is denoted $M(E)$. What Borel discovered is this: There are sets E which have no interior, but are nevertheless sufficiently massive so that if $f \in M(E)$, then $f \in C^\infty(E)$ in the restricted sense and if either

(1′) $f = 0$ on a relatively open subset of E, or
(2′) $f^{(n)}(x_0) = 0$, $n = 0, 1, 2, \ldots$, at some $x_0 \in E$,

then $f \equiv 0$ on E. That is, the class of monogenic functions on E enjoys the uniqueness property of the analytic functions. Although Borel evidently considered these results to be of considerable importance, his work in this area never received wide recognition. It did, however, contribute indirectly to the creation and development of the theory of quasianalytic classes by A. Denjoy, T. Carleman, S. N. Bernstein, A. Beurling, and others.

A second and quite different area of study which also led in the same direction has its roots in physics. This has to do with the work of Holmgren and others on the possible uniqueness of solutions to the heat equation (cf., for example, [12, 13]). The problem was this: Is the distribution of heat $u = u(x, t)$ at time t in a thin metal rod lying along the x-axis completely determined by its initial state $u(x, 0)$? More specifically, does the initial value problem

$$\begin{aligned} &u_t - u_{xx} = 0, \qquad -\infty < x < \infty, \qquad t > 0 \\ &u(x, 0) = f(x) \end{aligned}$$

have a unique solution? As early as 1908, Holmgren had turned his attention to this and similar questions and he found that if $u(x, t)$ is any solution to the heat equation, then it is necessarily analytic in x, C^∞ in t, and when restricted to a bounded region of the xt-plane,

$$\left|\frac{\partial^n u}{\partial t^n}\right| \le AM^n(2n)! \tag{1}$$

for $n = 0, 1, 2, \ldots$, and suitable constants A and M. If $(2n)!$ could be replaced by $n!$, then u would also be analytic in t and its future would be completely determined by its past. Unfortunately, the inequalities (1) are sharp and we cannot, in general, expect a unique solution to the initial value problem as stated (cf., for example, Tychonoff [16] and also [14]).

Commenting on the situation as it existed in 1912, and on the work of Holmgren in particular, Hadamard [10] posed the following problem: What conditions on a sequence of positive real numbers A_n, $n = 0, 1, 2, \ldots$, are sufficient to ensure that if $f \in C^\infty[a, b]$ and

$$|f^{(n)}(t)| \le M^n A_n, \qquad n = 0, 1, 2, \ldots \tag{2}$$

uniformly on $[a, b]$, then f is uniquely determined by its value and the values of its derivatives at a single point? The problem stood for nearly a decade until it fell in grand fashion at the hands of Denjoy and Carleman, at which time the theory of quasianalytic classes was born (cf. [7, 8]). One consequence of their work is this: If, in addition to the requirements outlined above, $\sum_{n=1}^\infty A_n^{-1/n} = \infty$ and $f^{(n)}(c) = 0$, $n = 0, 1, 2, \ldots$, for some $c \in (a, b)$, then $f \equiv 0$. If f happens to be analytic, the inequalities (2) are automatically satisfied with $A_n = n!$ and so $\sum_{n=1}^\infty A_n^{-1/n} \ge c\sum_{n=1}^\infty \frac{1}{n} = \infty$. Thus, a genuine extension of the classical theory had now been achieved. An excellent description of the early results and subsequent development can be found in Chapter IV of *The logarithmic integral.*

In practice, however, one is sometimes forced to deal with the uniqueness question for functions defined on a set E having no underlying smoothness, where we cannot speak about the derivatives $f^{(n)}$, $n = 1, 2, \ldots$, even in the sense of Borel, and where we cannot make use of inequalities like (2). To illustrate the point let us suppose that Ω is a bounded simply connected domain in the complex plane, let dA denote two-dimensional Lebesgue measure, and let $w(z) > 0$ be a bounded continuous function defined on Ω. Consider the two spaces of functions:

(a) $A^2(\Omega, w\,dA)$, the closure of the polynomials in $L^2(w\,dA)$.

(b) $L_a^2(\Omega, wdA)$, the set of functions in $L^2(wdA)$ which are analytic in Ω.

It is easily seen that $A^2 \subseteq L_a^2$ and so we can ask if equality occurs. By duality this is equivalent to the following question: If $m \in L^2(\Omega, wdA)$ and if the integral

$$f(z) = \int_\Omega \frac{mw(\zeta)}{(\zeta - z)} dA_\zeta$$

vanishes identically in Ω_∞, the unbounded complementary component of $\overline{\Omega}$, must $f = 0$ a.e. with respect to harmonic measure on $\partial\Omega$? For purposes of this discussion the reader should envision Ω as being obtained from a Jordan domain by introducing cuts or slits in the form of simple (but nonsmooth) arcs extending from the interior to the boundary. Our task, then, is to prove that $f = 0$ on the cuts from the knowledge that $f = 0$ on $\partial\Omega_\infty$. For this it is convenient to transfer the problem to the unit disk D by means of conformal mapping. In this way we obtain a corresponding function F on D and, moreover, F has radial limits $F(\theta)$ a.e. $-d\theta$ on ∂D. Here $d\theta$ is the usual arc length measure. Under suitable conditions (arising principally from restrictions on w) it can be shown that $F(\theta)$ is nearly analytic in the sense that it has a rapidly decreasing sequence of negative Fourier coefficients. If we could prove that $F(\theta) = 0$ a.e. $-d\theta$ it would follow that $f = 0$ a.e. with respect to harmonic measure on $\partial\Omega$. Thus, we can, in principle, avoid the difficulties associated with a general boundary and work in a more familiar setting. What emerges is this: For a weight $w = w(g)$ depending only on Green's function g, the conclusion $F(\theta) = 0$ a.e., and hence $A^2 = L_a^2$, is valid whenever $y \log w(y)$ is monotonic and

$$\int_0 \log\log \frac{1}{w(y)} dy = +\infty \tag{3}$$

(cf., for example, [6]). In particular, $w(z) \to 0$ rather quickly as $z \to \partial\Omega$. The expression (3) occurs often in connection with quasianalytic classes and can be viewed as one manifestation of *The logarithmic integral*, after which Koosis's book takes its name.

Let us now turn the situation around and consider a function $F(\theta) \sim \sum_{-\infty}^{\infty} a_n e^{in\theta}$ defined and continuous or square summable on ∂D. If $a_n = 0$ for $n = 1, 2, \ldots$, then F admits an analytic extension to the interior of D and therefore $F(\theta) \equiv 0$ if any of the following occur:

(a) $F = 0$ on a subarc of ∂D;

(b) $F = 0$ on a set of positive measure on ∂D;

(c) $\int_0^{2\pi} \log|F(\theta)|d\theta = -\infty$.

However, each of these assertions remains valid under the much weaker assumption that $|a_{-n}| \to 0$ rapidly as $n \to +\infty$. If, for example, $|a_{-n}| \le e^{-\beta n}$, $\beta > 0$, then F can again be continued analytically off ∂D, this time to the annulus $e^{-\beta} < |z| < 1$, and the situation is essentially unchanged. The problem of determining the exact rate of decay required on the negative Fourier coefficients in order to ensure that the uniqueness property of the analytic functions is retained can be viewed as a natural extension of the question posed by Hadamard [10] in 1912 and was first taken up by Cartwright, Levinson, and Beurling in the late 1930s (cf., for example, [2, 3, 15]). Here is what resulted from their combined efforts: if

(i) $|a_{-n}| \le e^{-k(n)}$, $\quad n = 1, 2, \ldots$

(ii) $\sum_{n=1}^{\infty}(k(n)/n^2) = +\infty$,

where $k(x) \uparrow +\infty$ as $x \uparrow +\infty$, then $F(\theta) \equiv 0$ whenever (a) or (b) is satisfied. Actually, slightly more is required of k for part (b). In any case, if $k(n) = n/\log n$, both assertions are correct even though the corresponding $F(\theta)$ may now admit no analytic extension off ∂D.

As it stands, however, the Beurling-Levinson-Cartwright result does not suffice for the application described above. We cannot be sure in the approximation problem that $F(\theta)$ vanishes at more than a single point, as could conceivably happen if $\partial\Omega_\infty$ has harmonic measure zero. But, it can be shown that if the weight $w(z) \to 0$ fast enough as $z \to \partial\Omega$, then $\int_0^{2\pi} \log|F(\theta)|d\theta = -\infty$ in all cases and, moreover, conditions (i) and (ii) on the negative Fourier coefficients are fulfilled. Thus, we are forced to consider the uniqueness question in the more general context.

The first step is to extend $F(\theta)$ from ∂D to the entire open disk D in a suitable fashion. This is accomplished with the aid of the Legendre transform $h(y) = \sup_{x>0}(k(x) - yx)$, $\quad y > 0$, and a corresponding weight function $w(z) = w(|z|) = e^{-h(\log 1/|z|)}$ defined on D. Because $w(z) \to 0$ very rapidly as $z \to \partial D$, there is an extension, still denoted F, with the property that

$$\left|\frac{\partial F}{\partial \bar{z}}(z)\right| \le \text{const}.w(z) \tag{4}$$

almost everywhere in D. This is a consequence of a theorem of

Dyn′kin (cf., LI, pages 338–343) and, roughly speaking, is equivalent to the existence of a function $\rho \in L^\infty(D)$ such that

$$F(z) = \sum_{n=0}^{\infty} a_n z^n + \frac{1}{\pi}\int_D \frac{\rho(\zeta)w(\zeta)}{(\zeta - z)} dA_\zeta$$

has the given boundary values $F(\theta)$. Here dA refers to two-dimensional Lebesgue measure and, since $\partial F/\partial \bar{z} = -\rho w$, the required estimate (4) follows. Of course, the extended F is not itself analytic and we are back, more or less, in the situation described earlier in connection with the weighted approximation problem.

Now let $E = \{z \in D\colon |F(z)| \le w(z)\}$. Without loss of generality we may assume that E is the union of countably many disjoint smoothly bounded Jordan regions, only finitely many of which meet any compact subset of D. Setting $U = D\backslash E$ we may also assume that $\partial U \supseteq \partial D$, since otherwise $F(\theta) = 0$ on some subarc and therefore $F(\theta) \equiv 0$ by the original Cartwright-Levinson theorem. There are two possibilities: Either

(A) ∂E is *thick* near some point $\lambda_0 \in \partial D$, or
(B) ∂E is *thin* at every point of ∂D.

It is a remarkable fact that, in either case, $F(\theta) \equiv 0$ provided $\int_0^{2\pi} \log|F(\theta)| d\theta = -\infty$. This is a theorem of A. L. Vol′berg and it represents the most significant advance in the theory of quasianalytic classes since the seminal work of Cartwright, Levinson and Beurling carried out more than forty years ago (cf., [19, 20]).

There are two key ideas. One goes back to Bernstein [1], was greatly transformed and developed by Beurling [2] and is this: Although the extended F is not analytic, it can be approximated very rapidly by a sequence of functions, each of which is analytic in some annular region abutting ∂D. For each $\varepsilon < 1$ we have simply to define

$$F_\varepsilon(z) = \sum_{n=0}^{\infty} a_n z^n + \frac{1}{\pi}\int_{D_\varepsilon} \frac{\rho(\zeta)w(\zeta)}{(\zeta - z)} dA_\zeta,$$

where $D_\varepsilon = \{z\colon |z| < 1 - \varepsilon\}$. Clearly, F_ε is analytic for $1 - \varepsilon < |z| < 1$ and, moreover,

(iii) $|F(z) - F_\varepsilon(z)| \le ce^{-h(\varepsilon)}$,
(iv) $|F_\varepsilon(z)| \le K$.

The constants c and K are independent of ε and, for convenience, we have replaced $h(\log 1/(1-\varepsilon))$ by $h(\varepsilon)$. Inequalities such as these are central to the entire theory and they give precise expression to the general principle that functions which are rapidly approximable by analytic functions retain the unique continuation property.

In case (A) it is best to transform the problem to the real line $\mathbf{R}$ by means of the mapping $\chi(z) = -i\log z$. The corresponding functions, again denoted F and F_ε, are defined and periodic on $\mathbf{R}$. To prove that $F \equiv 0$ on $\mathbf{R}$ we choose an interval [A, B] containing a full period of F and we study the Fourier transform $\widehat{F}(z) = \int_A^B F(x)e^{izx}dx$. Because $\widehat{F}$ is entire and satisfies an inequality of the form $|\widehat{F}(x+iy)| \le Ke^{a|y|}$, the function $\log|\widehat{F}(z)| - ay$ is subharmonic and bounded above in the upper half-plane $y > 0$. The latter is therefore majorized by its Poisson integral and, since under the present assumptions

$$\int_1^\infty \log|\widehat{F}(t)|\frac{dt}{t^2} = -\infty, \tag{5}$$

it can be inferred that $\widehat{F} \equiv 0$ and consequently $F \equiv 0$ on $\mathbf{R}$. The integral in (5) plays a fundamental role in Koosis's book and its convergence or divergence is related to the fact that for the function k and its Legendre transform h the integrals $\int_0 \log h(y)dy$ and $\int_1^\infty (k(x)/x^2)\,dx$ converge or diverge simultaneously (LI, p. 333). Here, the second is equal to $+\infty$ by virtue of property (ii). Earlier, in the approximation problem it was the first integral that controlled the situation.

In case (B), F is replaced by a new function

$$\Phi(z) = F(z)\exp\left\{-\frac{1}{\pi}\int_U \frac{1}{F(\zeta)}\frac{\partial F}{\partial\bar{\zeta}}\frac{dA_\zeta}{(\zeta - z)}\right\}$$

and it is easily checked that

(v) Φ is analytic in U.

(vi) $c_1|F(z)| \le |\Phi(z)| \le c_2|F(z)|$ with $c_1, c_2 > 0$.

The idea of regularizing a function F in this way was first employed by Theordorescu in 1931, was rediscovered by Bers in 1951 and rediscovered again by Vol′berg in 1982. It is now commonly referred to as the Bers similarity principle (cf., [18]). Because we are in case (B), harmonic measure for U is boundedly equivalent to $d\theta$ on $\partial U \cap \partial D$ and it follows by a standard argument that $\Phi \equiv 0$ in U and, hence, $F = 0$ a.e. - $d\theta$ on ∂D.

A complete description of these and a host of similar results can be found in *The logarithmic integral*. The book contains, for example, a thorough discussion of the classical Bernstein problem for weighted polynomial approximation on the real line and its generalization to closed sets having infinite extent in both directions. Considerable space is given, therefore, to the corresponding results of Mergeljan, Akhiezer, Pollard, de Branges, Benedicks, and the author, where necessary and sufficient conditions for approximation are expressed in terms of the integral which is the subject of the book. Also included is a chapter on the moment problem together with results of Carleman and M. Riesz which involve integrals of the same form.

Nevertheless, it is my conviction that the material chosen for presentation in this review most nearly conveys the nature and scope of Koosis's book. If there is a single dominant theme, it can best be expressed in terms of the uncertainty principle which, loosely stated, is this: A function $F \in L^1(\mathbf{R})$ and its Fourier transform $\widehat{F}$ cannot be simultaneously small on a large set unless F and $\widehat{F}$ both vanish identically. More than 150 pages representing a quarter of the text are devoted, at least nominally, to making this statement precise in various situations. Here, in Chapter VII, the main uniqueness theorems and their consequences are established and the uncertainty principle plays a key role, particularly in connection with the work of Cartwright, Levinson, and Beurling. It is ironic, therefore, that the principle itself and the Fourier transform as a tool are suddenly abandoned when the more recent contributions of Vol′berg are taken up. This is due, undoubtedly, to the relative novelty of the results in question at the time that *The logarithmic integral* was written. Thus, on this topic the author follows rather closely Vol′berg's original argument as outlined in [19]. Use of the Fourier transform, however, allows a more direct approach to be taken and sharp forms of the basic uniqueness theorems are easily obtained (cf., [6] and LI, the Addendum). In this way the true relationship between the work of Beurling and Vol′berg is also brought into sharper focus. Roughly speaking, Beurling deals with case (A) described above and, building on this, Vol′berg deals with case (B).

The logarithmic integral is a well-written book and it will enable the reader to enter an area where there has been an explosion of new results. Perhaps the most striking development to date is the use of nearly or asymptotically holomorphic functions by Il'yashenko and Vol′berg in connection with an old conjecture of

Poincaré, just recently settled. In its original form Poincaré's conjecture was this: If $p(x, y)$ and $q(x, y)$ are polynomials, then there can be at most finitely many limit cycles associated with the first-order system of equations $\dot{x} = p(x, y)$, $\dot{y} = q(x, y)$. It turns out, however, that the conjecture is still valid even when p, q are only known to belong to certain quasianalytic classes, and the methods described in Koosis's book are applicable here. Additional applications can be found in connection with the existence of asymptotic values for functions in the MacLane class (cf., [11]) and possibly to the study of finely holomorphic functions (cf., [9]).

The book is excellent and I personally look forward to the appearance of volume 2 which is now in press and which contains a wealth of material on harmonic measure, capacity, extremal length, gap theorems, multiplier theorems and, in particular, the work of Beurling and Malliavin.

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Systems of equations of composite type, by A. Dzhuraev. Translated by H. Begehr and Lin Wei. Pitman Monographs and Surveys in Pure Applied Math., vol. 44, Longman Scientific and Technical, Essex; John Wiley & Sons, New York, 1989, xv+333 pp. ISBN 0-582-00321-0

The classic study of linear partial differential equations centered around the three basic types of equations: elliptic, parabolic, and hyperbolic. From physical considerations for the potential, heat, and wave equations, one was able to determine suitable boundary value problems for each category of equation or system of equations.

However, when an equation or system did not fit into one of the three types, little was known or done concerning the determination of proper boundary value problems. In the 1930s, J. Hadamard, O. Sjostrand, and others began the study of equations of *composite type* in two dimensions. These are equations that have characteristics of both elliptic and hyperbolic (or parabolic) type. For example, if one considers a first-order system of three or more equations, then the system will be of composite type if at least one of the roots of the characteristic equation is real and at