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The theory of fixed point classes, by Kiang Tsai-Han. Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, and Science Press, Beijing, 1980, xi+174 pp. ISBN 3-540-10819-X and 0-3887-10819-X

Every pampered Western mathematician should read at least the epilogue to this book. The book itself is a readable and helpful account of an important topic in genuine mathematics.

From a technical point of view, I am the wrong person to review this book. When I wrote a little book on fixed points, I avoided the methods of algebraic topology as far as I could and did the things that could be done using basic methods of metric space topology and elementary functional analysis. I took on the job of reviewing Kiang's book because of the publisher's blurb: "... it will serve as a good introduction to algebraic topology and geometry." I thought it would be an easy way to learn some more about these things. (I never read much of Brown's book on the Lefschetz fixed-point theorem because of its formidable-looking machinery.) I also hoped to find the answer to a question that a student asked the author: "How did Lefschetz get the idea to formulate his theorem and give his proof?"

So what did I get out of it? I learned a lot about covering spaces and homotopies (much of which I had seen before, some time; Kiang gives an appendix reminding the reader of the basic properties). I enjoyed the way these tools are used. I learned all I want to know about Nielsen fixed-point theory. But I still don't know how Lefschetz got his ideas, and I still can't read Brown's book. There is a gap in the technical machinery used, and in the level of exposition. Brown is not user-friendly, but Kiang is, for someone with my background.

What is the book about? A compact connected polyhedron X has a "universal covering space" $\widehat{X}$ which we can imagine as lying over it (in much the same way that a Riemann surface lies over the part of the plane where an analytic function behaves itself). Unless X is simply connected, several or many points of $\widehat{X}$ lie over each point of X. There is a natural projection p (imagine it as straight down from $\widehat{X}$ onto X). Any mapping T of X can be "lifted" to a mapping $\widehat{T}$ of $\widehat{X}$, such that $p\widehat{T} = Tp$. The book's arguments (following Nielsen, who started the theory) turn on the relation between fixed points of T and fixed points of its liftings $\widehat{T}$. A given T has various liftings, which fall into equivalence classes under certain automorphisms of $\widehat{X}$ (those which don't alter the set of points lying over any $x \in X$). These are the "lifting classes" of T. Different lifting classes correspond to disjoint components of the fixed-point set of T. These components are the "fixed-point classes." (Another way to describe them is given by the fact that two fixed points can be brought together by a homotopy (deformation) of T, if and only if they belong to the same fixed-point class.)

Some fixed-point classes can be made to disappear by a homotopy process. Forget them! The others are the *essential* fixed-point classes. The Nielsen number $N(T)$ is the number of essential fixed-point classes of T. The Nielsen fixed-point theorem says that $N(T)$ is at most the number of fixed points of T. A converse says that some mapping of the same "homotopy type" as T has only $N(T)$ fixed points.

So $N(T)$ gives a lower bound (and in a sense, the best possible) for the number of fixed points. Very nice, but how does one calculate $N(T)$? That's more difficult, and is the main part of the book (the part that I haven't yet got to reading).

This book is remarkably similar to Jiang Boju's book *Nielsen Fixed Point Theory* (Contemporary Mathematics, vol. 14, Amer. Math. Soc.). If your librarian can't find it, try Boju Jiang. The similarity is explained by the fact that Jiang was Kiang's pupil, that Jiang obtained the most important results (on computation of $N(T)$) and that he was an active member of the team which helped to write Kiang's book. If you regard Jiang's book as an updated softcover version of Kiang's (and equally readable), you won't be too far wrong, though Jiang gives a briefer treatment of some matters. The present book is a translation (by the author) of a revision of the 1979 Chinese original; it is a historical accident that it appears after Jiang's book. Nevertheless, I am glad that it has appeared in English.

Every reviewer should make some unfair criticisms, so here goes. Even in 174 pages, I would have liked to see some remarks about how the theory can be made applicable. In particular I would like some answers to the questions:

(a) If the Nielsen theory tells that a mapping has *several* fixed points, does it give any way to locate them? There are various algorithms to locate fixed points (or rather, to locate ε-fixed points) but they don't seem to cover this question. (However, I do not think that computation is an important aspect of this theory; its significance is conceptual, in its explanation of what happens to fixed points under deformations of the mapping.)

(b) How does the theory extend to sets that are not compact polyhedra? (That is to say, to the sets of interest in any conceivable application.) Jiang remarks that "... there seems to be no essential difficulty in extending further to compact absolute neighborhood retracts (ANR) or even to compact maps on noncompact ANRs... ". (He suggests using methods given in Brown's book but I don't see anything about noncompact ANRs there.) Compact mappings of noncompact sets are common in analysis, so this would be an important extension.

Why should I stop there? Having asked for the moon, why not the stars as well? I would like someone to replace the ANR condition with something that is easier to check (perhaps something like "locally contractible"). And while he/she/they are about it, they can rewrite about 50 years of elegant high-powered work (of which Brown's book can serve as an example) replacing "ANR" with "locally contractible" (or whatever) all along the line.

As a test: Their work should cover the theorems of Cellina and Fryszkowski (see Studia Mathematica, 1984) on fixed points of compact mappings of some function spaces (certain nonconvex subsets of L^1). These spaces are easily seen to be contractible and locally contractible but there seems to be no known general theorem which would cover this case. (Incidentally Fryszkowski's paper has a slight defect: He should assume that his measure space has no atoms.)

I have great hopes for this book. Perhaps it could even lead some other Far Eastern fixed-point theorists—who specialize in complicated useless absurd variations of Banach's fixed-point theorem—to turn their industry and ingenuity to more genuine problems. (At present they seem to be trying to give fixed-point theory a bad name.)

Yevtushenko wrote a poem about Siberian willows—their ability to survive the frost of winter. Kiang and his school must be congratulated on their vigorous survival. I hope that they continue to flourish.

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