A SIMPLIFIED VERSION OF THE
ABSTRACT CAUCHY-KOWALEWSKI THEOREM
WITH WEAK SINGULARITIES

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ABSTRACT. A simplified version of the abstract Cauchy-Kowa-
lewski theorem of Nirenberg [9], Ovsjannikov [12], Nishida
[10], Baouendi and Goulaouic [3], and Asano [1] is presented.
The new version requires more specific information on the form
of the equation but recovers a stronger result in that the region
of existence is not forced to shrink at each step of an iteration
and that weak singularities are allowed along the boundary of
the region of existence.

The Cauchy–Kowalewski theorem is the basic existence theorem
for analytic solutions of partial differential equations and in its ab-
stract form [1, 3, 9, 10, 12] can be applied to equations that involve
nonlocal operators, such as the water wave equations [8], the Boltz-
mann equation in the fluid dynamic limit [11], the incompressible
fluid equations in the zero-viscosity limit [2] and the vortex sheet
equations [4–6, 13]. The proof of the abstract Cauchy–Kowalewski
theorem in [1, 3, 9, 10, 12] is of "Nash–Moser type" in that it re-
quires a loss in the size of the existence region at each step of an
iteration (but without use of Newton iteration).

The purpose of this paper is to present a new version of the
abstract Cauchy–Kowalewski theorem that does not use a "Nash–
Moser type" proof. The domain of existence does not shrink at
each step of the iteration, and in addition mild singularities are
allowed at the boundary of the existence region. The main signifi-
cance of this new version is that the simpler proof makes it more
adaptable to new applications and that more control over the re-
gion of existence allows solution of problems with singularities.

The new version of the theorem does require more information
on the structure of the equation. However these new hypothe-

ses are valid for most of the applications of the theorem. For the applications in [4–6, 8, 13] the hypotheses are easily verified; verification for the more complicated applications in [2, 11] has not yet been completed. This new version of the abstract Cauchy–Kowalewski theorem is used in [7] for construction of vortex sheet solutions with singularities.

The result is stated in a family of Banach spaces B_ρ for $0 < \rho < \rho_0$, with norm $\|\cdot\|_\rho$ such that $B_{\rho'} \subset B_\rho$ and $\|u\|_\rho < \|u\|_{\rho'}$ for $0 < \rho < \rho' < \rho_0$. Consider the problem

$$(1) \qquad\qquad u_t = A(u, t),$$

$$(2) \qquad\qquad u(0) = 0,$$

with $u = u(t)$, as a differential equation in the family of spaces B_ρ. The statement of the theorem also uses a linear operator D which maps $B_{\rho'}$ continuously into B_ρ for any $0 < \rho < \rho' < \rho_0$. In addition suppose that $0 < \beta < 1$ and define the norm

$$(3) \qquad \||u\|| = \sup_{\substack{0<\rho<\rho_0 \\ 0<t<\rho_0-\rho}} \{\|u(t)\|_\rho + (\rho_0 - \rho - t)^\beta \|Du(t)\|_\rho\}.$$

Assume that A satisfies the following conditions: There are positive constants ε and R such that

(i) $A(\cdot, t)$ maps $B_\rho \cap \{u: \|u\|_\rho \leq R, \}$ into $B_{\rho'}$ continuously in u and t, for $0 < \rho < \rho' < \rho_0 - t$.

(ii) If $u, w, Du, Dw \in B_{\rho'}$ with $\|u\|_{\rho'} \leq R$, $\|w\|_{\rho'} \leq R$,

then

$$(4) \quad \begin{aligned} &\|A(u, t) - A(w, t)\|_\rho \\ &\qquad \leq \varepsilon\{(1 + \|Du, Dw\|_\rho)\|u - w\|_\rho + \|D(u - w)\|_\rho\}, \end{aligned}$$

$$(5)$$
$$\|DA(u, t) - DA(w, t)\|_\rho$$
$$\qquad \leq \varepsilon(\rho' - \rho)^{-1}\{(1 + \|Du, Dw\|_{\rho'})\|u - w\|_{\rho'} + \|D(u - w)\|_{\rho'}\},$$

for $0 < \rho < \rho' < \rho_0 - t$.

(iii) For $0 < \rho < \rho_0 - t$,

$$(6) \qquad\qquad \|A(0, t)\|_\rho \leq \varepsilon(\rho_0 - \rho - t)^{-\beta},$$

$$(7) \qquad\qquad \|DA(0, t)\|_\rho \leq \varepsilon(\rho_0 - \rho - t)^{-\beta-1}.$$

These hypotheses have the following interpretation: The operator D will usually be $\partial/\partial z$. Then (4) says that A acts like a function of u times u_z, and (5) is the Cauchy estimate on the derivative of A. The function space B_ρ will consist of functions that are analytic and bounded in $\{z: |\operatorname{Im} z| < \rho\}$ with a norm like $\|u\|_\rho = \sup_{|\operatorname{Im} z| < \rho} |u|$ (or the related norm in (25) below). Finally (6) and (7) allow A to have a singularity on $|\operatorname{Im} z| = \rho_0 - t$. It will also be assumed that ε is small, which, by rescaling of t or u, is equivalent to existence for a short time or for small nonlinearity. The notation $\|f, g\| = \|f\| + \|g\|$ has been used to save space.

Theorem. *For any positive R and ρ_0 and any $0 < \beta < 1$ there are numbers $\varepsilon_0 > 0$ and $0 < a < R$, such that for $0 < \varepsilon \le \varepsilon_0$ and for A satisfying assumptions* (i)–(iii), *the system* (1), (2) *has a solution $u(t) \in B_\rho$ for $0 < \rho < \rho_0 - t$ with $\||\, u \,\|| \le a$. In other words there is a solution $u(t)$ in B_ρ with*

$$\|u(t)\|_\rho \le a,$$

(8)

$$\|Du(t)\|_\rho \le a(\rho_0 - \rho - t)^{-\beta}$$

for $0 \le t < \rho_0 - \rho$.

Proof. Solve (1), (2) by iteration as

(9) $$u_0 = 0,$$

(10) $$u_{n+1}(t) = \int_0^t A(u_n, s)\, ds,$$

and define

(11) $$v_{n+1} = u_{n+1} - u_n,$$

so that

(12) $$v_1(t) = u_1(t) = \int_0^t A(0, s)\, ds,$$

(13) $$v_{n+1}(t) = \int_0^t A(u_n, s) - A(u_{n-1}, s)\, ds.$$

It is easy to bound

(14) $$\||\, u_1 \,\|| = \||\, v_1 \,\|| \le a/2$$

in which

(15) $$a = 2\varepsilon \left(\frac{\rho_0^{1-\beta}}{1-\beta} + \frac{1}{\beta} \right).$$

Take ε_0 small enough that $a < R$ for $0 < \varepsilon < \varepsilon_0$.

Next assume by way of induction that $\||u_k\|| \leq a$ for $1 \leq k \leq n$ and estimate $\||v_{n+1}\||$. The estimate on $\|v_{n+1}\|_\rho$ for $0 < \rho < \rho_0 - t$ is easily found as

(16)

$$\|v_{n+1}(t)\|_\rho \leq \int_0^t \|A_n(s) - A_{n-1}(s)\|_\rho \, ds$$

$$\leq \varepsilon \int_0^t \{(1 + \|Du_{n-1}, Du_n\|_\rho)\|v_n\|_\rho + \|Dv_n\|_\rho\} \, ds$$

$$\leq \varepsilon \||v_n\|| \int_0^t 1 + (1 + 2a)(\rho_0 - \rho - s)^{-\beta} \, ds$$

$$\leq \||v_n\|| /4,$$

if ε is small enough that

(17)
$$\varepsilon \left(\rho_0 + (1 + 2a)\frac{\rho_0^{1-\beta}}{1 - \beta} \right) \leq \frac{1}{4}.$$

The estimate on $\|Dv_{n+1}\|_\rho$ is the main step. For $0 < \rho < \rho_0 - s$ define

(18)
$$\rho'(s) = \rho + (\rho_0 - \rho - s)/2$$

so that

(19)
$$\rho'(s) - \rho = \rho_0 - \rho'(s) - s = (\rho_0 - \rho - s)/2.$$

Then estimate

$$\|Dv_{n+1}(t)\|_\rho \leq \int_0^t \|D(A_n - A_{n-1})(s)\|_\rho \, ds$$

$$\leq \varepsilon \int_0^t (\rho'(s) - \rho)^{-1}$$

$$\times ((1 + \|Du_{n-1}, Du_n\|_{\rho'(s)})\|v_n\|_{\rho'(s)} + \|Dv_n\|_{\rho'(s)}) \, ds$$

(20)
$$\leq \varepsilon \||v_n\|| \int_0^t (\rho'(s) - \rho)^{-1}$$

$$\times (1 + (1 + 2a)(\rho_0 - \rho'(s) - s)^{-\beta}) \, ds$$

$$\leq 2^{1+\beta} \varepsilon \||v_n\|| \int_0^t (\rho_0 - \rho - s)^{-1}$$

$$\times (1 + (1 + 2a)(\rho_0 - \rho - s)^{-\beta}) \, ds$$

$$\leq (\rho_0 - \rho - t)^{-\beta} \||v_n\|| /4,$$

if ε is small enough that

(21)
$$\varepsilon 2^{1+\beta} \left(\rho_0^\beta (|\log \rho_0| + \beta^{-1}) + \frac{1 + 2a}{\beta} \right) \leq \frac{1}{4}.$$

Choose ε_0 small enough that (17) and (21) are true and $a < R$ for $0 < \varepsilon < \varepsilon_0$, with a defined by (15). It follows that

$$(22) \qquad \vert\!\vert\!\vert\, v_{n+1} \,\vert\!\vert\!\vert \leq \vert\!\vert\!\vert\, v_n \,\vert\!\vert\!\vert\, /2.$$

Thus $\vert\!\vert\!\vert\, v_n \,\vert\!\vert\!\vert \leq a/2^{n+1}$ and

$$(23) \qquad \vert\!\vert\!\vert\, u_{n+1} \,\vert\!\vert\!\vert \leq \vert\!\vert\!\vert\, v_1 \,\vert\!\vert\!\vert + \cdots + \vert\!\vert\!\vert\, v_{n+1} \,\vert\!\vert\!\vert < a.$$

This completes the induction step. Since the v_n are geometrically decreasing in size it follows that $u_n \to u$ in the norm $\vert\!\vert\!\vert \cdot \vert\!\vert\!\vert$, with u solving (1), (2) and with $\vert\!\vert\!\vert\, u \,\vert\!\vert\!\vert \leq a$.

An important difference between this proof and that of [1, 3, 9, 10, 12] is that in the latter blowup of u must be controlled, whereas here blowup only of Du need be handled. Since the estimates on A are linear in Du, the blowup is more easily controlled.

As an illustrative nontrivial example take $A(u) = \varepsilon(\partial_z^{1/2} u)^2 + \varepsilon(1 - e^{iz+1-t})^{-\beta}$ for u analytic and 2π periodic, in which $0 < \beta < 1$. The half derivative is a nonlocal operator defined by

$$(24) \qquad (\partial_z^{1/2} u)\hat{\ }(k) = \sqrt{|k|}\hat{u}(k).$$

A satisfies the hypotheses of the theorem in which $\rho_0 = 1$, $R = 1$, $D = \partial_z$, and B_ρ is the space of functions u that are analytic in $\{|\operatorname{Im} z| < \rho\}$ with $\|u\|_\rho < \infty$, in which

$$(25) \qquad \|u\|_\rho = \sum_k |\hat{u}(k)| e^{\rho|k|}.$$

The theorem then asserts that for small ε there is a solution u of the equation

$$(26) \qquad \begin{aligned} u_t &= \varepsilon(\partial_z^{1/2} u)^2 + \varepsilon(1 - e^{iz+1-t})^{-\beta}, \\ u(0) &= 0 \end{aligned}$$

in the domain $|\operatorname{Im} z| < 1 - t$. The solution is constructed all the way out to the curve $z = i(1 - t)$ where it is expected to have a singularity.

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