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*Controlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribués*, by J.-L. Lions. Masson, Paris, 1988, Vol. I: Controlabilité Exacte, X+ 537 pp. ISBN 2-225-81477-5 Vol. II: Perturbations; xiii+272 pp. ISBN 2-225-81474-0.

The two volume work reviewed here continues Professor Lions' lengthy list of fundamental contributions to the control theory of distributed systems—systems governed by partial differential and other infinite-dimensional processes—constituting just part of the work of a long and distinguished scientific career. The main subject matter concerns HUM, the Hilbert space Uniqueness Method, as a tool for studying Hilbert spaces of controllable states for a variety of linear partial differential equations, notably the wave equation, but the work also includes a contribution to asymptotic energy decay theory for the wave equation and some studies of the controllability of "perturbed" systems of the same sort, such as the wave equation in a "perforated" medium, applying homogenization techniques, and problems involving perturbations of the spatial region in which the process takes place. Additional applications are made to the control theory of elastic plates, thermoelastic processes, systems with memory , systems involving "transmission" of data from one region to another, simultaneous control of several systems at once, etc. In the process the author widely cites research carried out by other investigators and adds his own unique interpretation.

To give the reader some ideas of what the Hilbert space Uniqueness Method (HUM henceforth) is about, we will describe it in an abstract setting. We consider a linear system

$$\dot{x} = Ax + Bu, \qquad x \in X, \quad u \in U,$$

where the system state $x$ lies in the Hilbert space $X$ and the "control" $u$ lies in another Hilbert space $U$. A variety of control objectives may be posed but we will consider just the problem of using the control $u$ to "steer" the system, over a time interval $[0, T]$, from an initial state $x_0 \in X$ given at $t = 0$ to the zero terminal state; i.e. $x(T) = 0$. The operator $A$ is generally taken to be the generator of a strongly continuous semigroup in $X$ while $B$ may be a bounded operator from $U$ to $X$ or, in general, an "admissible" unbounded operator—as is required for the description of boundary control processes, for example, Extensive literature has developed in this area over the past thirty years or so; the problem is relatively uncomplicated in the finite-dimensional setting but becomes very challenging when various partial differential equations, such as the wave, heat, electromagnetic and other equations, become involved in the more specific description of the system.

Assuming $x(t)$, $u(t)$, $t \in [0, T]$, solve the above problem, we let $y(t)$ be a solution of the homogeneous "adjoint" system

$$\dot{y} = -A * y$$

with "endpoint" states $y(0) = y_0$, $y(TG) = y_1$. A standard computation then shows that

$$(x(T), y(T))_X - (x(0), y(0))_X = \int_0^T (u(t), B * y(t))_U \, dt,$$

or, since $x(T)$ is supposed to be zero,

$$(x_0, y_0)_X = \int_0^T (u(t), B * y(t))_U \, dt.$$

From this it can be seen that controls $u \in \mathscr{O}_T^{\perp}$, where $\mathscr{O}_T$ is the closed subspace of $L^2([0, T]; U)$ spanned by "observations" $B * y(T)$ on solutions $y$ of the adjoint system, play no role in the control process inasmuch as they simply take the zero initial state into the zero terminal state. So it is natural to restrict attention to controls $u$ which, themselves, lie in $\mathscr{O}_T$. If the observation $B * y(t)$, given on $[0, T]$, *uniquely* determines $y_0$ (a consideration which leads to the HUM designation of the method) one can assign to $y_0$ the norm $\|y_0\|_F = \|B * y\|_{L^2([0, T]; U)}$ and realize, in the by now familiar Lions fashion, a new Hilbert space $F$ which is the closure of states $y_0 \in X$ relative to $\| \ \|_F$. Then there is a representation of the dual space of $F$, call it $F'$, for which

$$F' \subset X \subset F,$$

which may be identified with the subspace of controllable states $x_0$ for the original control problem. Such a controllable state can, in fact, be "steered" to 0 during $[0, T]$ and the least norm control in $L^2([0, T]; U)$ which does the job must lie in $\mathscr{O}_T$, where it is unique, leading to a method whereby such controls can be characterized mathematically and, at least in principle, computed. Different systems generally require slight modifications of this pattern; in some cases, for example, $(x_0, y_0)_X$ is replaced by a more general form $(x_0, Ly_0)_X$, where $L$ is an appropriate operator on $X$.

Clearly, the basic structure of the method is very simple, deriving from elementary "least squares" considerations. For finite-dimensional systems the method has been around since the 1960's without any particular name being attached to it except that the associated matrix

$$W_T = \int_0^T e^{A(T-t)} BB * e^{A*(T-t)}\, dt$$

is more or less universally known as the "controllability Grammian." It seems quite unfortunate that the historical roots are not acknowledged as fully as they might be in these volumes, but it should be added that Professor Lions has noted this connection in some later accounts. A related point which makes this reviewer somewhat uncomfortable is the first volume's repeated descriptions of the HUM method in terms of hyperbolic equations, and the second volume's introduction of RHUM, the Reverse Hilbert space Uniqueness Method, without, in the reviewer's opinion, adequate notice to the reader that both HUM and RHUM are highly developed versions of concepts whose structure can readily be given a quite general description. Generally, the informal "lecture" style of the presentation is quite attractive but there are points where one does have the impression that the lectures require additional editing to form a fully cohesive whole.

In reading this work it is important to keep in mind the state of exact controllability theory for partial differential equations before HUM was introduced. Researchers in the area were then struggling with what today would be considered relatively simple problems, employing a variety of ad hoc methods (nonharmonic Fourier analysis, uniform forward/backward stabilization, clever ways of extending initial data, etc.). If Professor Lions had not introduced the HUM method it is likely we control theorists would still be doing the same things, even with the classical least squares formulation sitting right in front of us all along. Lions' work in this area gives a tremendous impetus to the whole field of exact controllability (along with related areas such as uniform stabilizability) for infinite-dimensional systems. He has shown the proper way to apply the classical least squares approach in the context of many distributed parameter control problems with significant applications. This is the essence of his contribution—and it is a vital one.

While HUM serves as the unifying theme, it is not the only subject treated in the work. Many readers, like this reviewer, will also value these books as a source of important material otherwise only available via extensive search of a rather scattered literature. Examples include the general form of Holmgren's theorem, multiplier methods for energy decay in solutions of the wave and plate equations, geometric optics methods, homogenization theory, various *a priori* estimation techniques, etc., etc. The inclusion of appendices by other researchers in the PDE control field, along with the description of a variety of open research problems, guarantees that the end effect of perusing these volumes is not that one has been sold a single methodology as an answer to the riddles of control theory but, rather, introduced to it as a useful stepping stone to a wide open universe of mathematical ideas—and introduced thereto by one of the great *virtuosi* of modern applied mathematics.

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