treated in the book, or wondered to what parts of the book they referred, the place where each property is treated is given below. I did not intend to create an impression that the properties were not in the book.

Property i) page 399 (This is Helgason's definition of spherical functions.)

ii) page 408, Lemma 3.2

iii) page 419, Theorem 4.5 (This is a form of the general principle valid for non-compact G. As noted in the review, the much stronger form valid for compact groups, which serves as motivation for the general result, is not treated except by example in the Introduction.)

iv) page 402, Proposition 2.4

v) page 414, Theorem 37

vi) page 400, Proposition 2.2.

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Direct and inverse scattering on the line, by Richard Beals, Percy Deift, and Carlos Tomei. Mathematical Surveys and Monographs, No. 28. American Mathematical Society, Providence, 1988, xiii + 209 pp. $57.00. ISBN 0-8218-1530X

The direct and inverse scattering theory for linear ordinary differential operators has been the subject of recent renewed interest. This stems in part from the so-called inverse scattering method for solving certain nonlinear partial differential equations, which uses scattering theory to convert these special nonlinear problems into linear ones. This technique was discovered by Gardner, Greene, Kruskal, and Miura [6], who described how to solve the Korteweg–de Vries equation (KdV)

$$q_t = 6qq_x - q_{xxx}$$

using the scattering theory for the ordinary differential operator family

$$L(t) = \frac{d^2}{dx^2} + q(x, t).$$

Lax [8] observed that one could view the KdV equation as an operator equation

$$\frac{dL}{dt} = [B, L],$$

where

$$B = -4\frac{d^3}{dx^3} + 3\left(q(x, t)\frac{d}{dx} + \frac{d}{dx}q(x, t)\right).$$

This helped explain some of the structure underlying the discoveries of Gardner *et al.* since from this form one may readily deduce that the spectrum of $L(t)$ is the same for all t and that the so-called scattering data evolve linearly.

For the operators of the form given by L above, a more or less complete theory for the direct and inverse scattering problems had been worked out previously, leading to the following inverse scattering method for solving the initial value problem for the KdV equation [6]: Compute the scattering data for the initial value; let it evolve in time using the linear equation; reconstruct the potential $q(x, t)$ by inverse scattering. It was later observed by Deift and Trubowitz [4] that the earlier scattering theory had a technical gap and they developed a fully rigorous version.

Shortly after the work of Gardner, Greene, Kruskal, and Miura, it was observed by several groups that other operators led a formal approach for solving other nonlinear PDEs of interest [1, 7, 9, 10]. In these cases, the corresponding direct and inverse scattering theory was not necessarily in place.

The monograph under review follows the work of Deift and Trubowitz [4], Deift, Tomei, Trubowitz [5], Beals [2], and Beals and Coifman [3] in developing rigorous results for such direct and inverse scattering problems. The class of operators considered in this monograph is the generalization of the KdV case to higher-order operators of the form:

$$L = D^n + p_{n-2}(x)D^{n-2} + \cdots + p_0(x),$$

where

$$D = \frac{1}{i}\frac{d}{dx}$$

and it is assumed that the potentials $p_j(x)$ are smooth and decay as $|x| \to +\infty$.

The theory is quite technical, so we shall content ourselves here to give the flavor of the ideas involved. In particular certain differences between the cases n even and odd as well as the self-adjoint case will be ignored. The main underpinning in all scattering theories of this type is to relate L to the bare operator D^n since by the decay assumption L is a small perturbation of D^n for large values of $|x|$.

The constant coefficient operator D^n has exponentials as eigenfunctions; namely, $D^n(e^{izx}) = z^n e^{izx}$. Moreover, if ω_n denotes the primitive nth root of unity $e^{2\pi i/n}$, it is clear that the n solutions of $D^n u = z^n u (z \neq 0)$ are given by $e^{i\omega_n^k zx}$ with $k = 0, 1, 2, \dots, n-1$. As $z \to 0$, these solutions became linearly dependent and one must include various z derivatives or equivalently worry about the order of vanishing as $z \to 0$. The n solutions for $x \neq 0$ real are also ordered by magnitudes in the open sectors of the complex z-plane where the real parts of the exponents are distinct. This leads to a distinguished ordering in each sector of the solutions based on their sizes as $x \to +\infty$ (or $-\infty$).

All of this carries over the the more complicated operator L so that one may construct a distinguished solution of $Lu = z^n u$ which is holomorphic in z in open sectors by solving a Volterra equation. Using wedge products of solutions, the authors in fact construct two distinguished sets of n solutions using Volterra equations. As z approaches a point of the bounding ray other than 0 from either side, one finds relations between the solution families provided certain determinants are nonzero. These relations (along with added data at zeros of the determinants) comprise the scattering data. For "generic" potentials, meaning those for which the zeroes of the determinants are all simple, distinct from one another, stay away the boundary of the sectors, and behave as simply as possible as $z \to 0$, Beals, Deift, and Tomei give a complete characterization of the scattering data. They also prove that generic potentials form an open dense subset of the set of all Schwartz space potentials. The set of zeroes is called the singular set Z.

In Part II of the monograph, the inverse problem of recovering the operator L from the scattering data is considered. The procedure is roughly as follows: (1) observe that it suffices to find the first row of the matrix fundamental solution, since the other rows are the x-derivatives of this row; (2) derive an equation for this row for each $x \in \mathbf{R}$ using the "$\bar{\partial}$ method" to show that the suitably extended version of this row vector is uniquely determined by its values along the sector boundaries and at the singular set Z; (3) using a suitable matrix factorization near $z = 0$, show that the equations can be reduced to Fredholm equations of index zero for each x; (4) *for self-adjoint operators,* deduce a "vanishing lemma" which shows that the kernel is trivial, hence the inverse problem is always uniquely solvable for each x. In the general case, step (4) can fail, but it was shown by Beals [2] that there is a dense open set where the inverse problem is solvable for all x.

In the final part, the question of evolution equations is taken up. It is shown that the corresponding Lax equations associated to an nth-order operator L lead to linear evolutions of scattering data. These linear equations can have exponentially growing or decaying parts as well as pieces which oscillate. This provides a decomposition of the solution to the nonlinear evolution equation into exponentially stable, exponentially unstable, and "central" parts. Some algebraic aspects of the process of adding or removing bound states and the link with the theory of first-order matrix systems are also considered.

Direct and Inverse Scattering on the Line is a carefully written, clear, and complete monograph. The techniques used to analyze the direct and inverse problem are a beautiful blend of ideas from complex analysis, algebra, and functional analysis. Rigorous results in this subject are hard to come by, so this monograph is a welcome addition to the research literature. The authors have done a good job of laying out their arguments. Unfortunately, the publisher has not. The page layout is a bit tight, with many formulas set too close to surrounding text. This makes for a bit of eyestrain.

This book is heartily recommended for the serious students of the subject.

References

1. M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, *Method for solving the Sine-Gordon equation*, Phys. Lett. **30** (1970), 1262–1264; *The inverse scattering transform-Fourier analysis for nonlinear problems*, Stud. Appl. Math. **53** (1974), 249–315.
2. R. Beals, *The inverse problem for ordinary differential operators on the line*, Amer. J. Math. **107** (1985), 281–366.
3. R. Beals and R. R. Coifman, *Scattering and inverse scattering for first order systems I, II*, Comm. Pure Appl. Math. **38** (1985), 29–42; Inverse Problems **3** (1987), 577–593.
4. P. Deift and E. Trubowitz, *Inverse scattering on the line*, Comm. Pure Appl. Math **32** (1979), 121–251.

5. P. Deift, C. Tomei, and E. Trubowitz, *Inverse scattering and the Boussiesq equation*, Comm. Pure Appl. Math. **35** (1982), 567–628.
6. C S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Method for solving the Korteweg–de Vries equation*, Phys. Rev. Lett. **19** (1967), 1095–1097;*Korteweg–de Vries equation and generalizations VI. Methods for exact solution*, Comm. Pure Appl. Math. **27** (1974), 97–133.
7. G. L. Lamb, Jr., *Analytical descriptions of ultra-short optical pulse propagation in a resonant medium*, Rev. Mod. Phys. **43** (1971), 99–129.
8. P. D. Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure Appl. Math. **21** (1968), 467–490.
9. V. E. Zakharov, *On the stochastization of one-dimensional chains of nonlinear oscillators*, Soviet Phys. JETP **38** (1974), 108–110.
10. V. E. Zakharov and A. B. Shabat, *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media*, Soviet Phys. JETP **34** (1972), 62–69.

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Controlabilité Exacte, Perturbations et Stabilisation de Systèmes Distribués, by J.-L. Lions. Masson, Paris, 1988, Vol. I: Controlabilité Exacte, X+ 537 pp. ISBN 2-225-81477-5 Vol. II: Perturbations; xiii+272 pp. ISBN 2-225-81474-0.

The two volume work reviewed here continues Professor Lions' lengthy list of fundamental contributions to the control theory of distributed systems—systems governed by partial differential and other infinite-dimensional processes—constituting just part of the work of a long and distinguished scientific career. The main subject matter concerns HUM, the Hilbert space Uniqueness Method, as a tool for studying Hilbert spaces of controllable states for a variety of linear partial differential equations, notably the wave equation, but the work also includes a contribution to asymptotic energy decay theory for the wave equation and some studies of the controllability of "perturbed" systems of the same sort, such as the wave equation in a "perforated" medium, applying homogenization techniques, and problems involving perturbations of the