These refine and strengthen the now classical versions of PL immersion theory, equivariant immersion theory and smoothing theory. This extra information enables one to exploit local geometric properties such as D. Stone's notion of curvature of PL immersions. The construction of the PL Grassmannian and associated universal PL bundle is more complicated than in the smooth category, but it is quite natural. It abstracts the notion of link in a combinatorial manifold and has one j-cell for each j-dimensional abstract link. The book is written for experts and assumes a thorough knowledge of PL topology, bundles and smoothing theory.

Levitt has successfully reintroduced local geometry into the PL category. It remains to see if sufficiently simple local formulas for characteristic classes or sufficiently interesting global results involving curvature etc., can be obtained to justify the conceptual complications introduced by using the PL category.

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Predicative arithmetic, by Edward Nelson. Mathematical Notes, vol. 32, Princeton University Press, Princeton, 1986, vii+189 pp., $21.00 (paperback). ISBN 0-691-08455-6

This book presents a formalist account of the foundations of arithmetic and "to one who takes a formalist view of mathematics", Nelson reminds us in his penultimate chapter, "the subject matter of mathematics is the expressions themselves together with the rules for manipulating them—nothing more." This view is expressed even more forcefully in the final sentence of the book: "I hope that mathematics shorn of semantical content will prove useful as we expore new terrain." Now these views are not, of course, new or even particularly extreme but the reader who has reached this point in the book will have realised just how much of conventional mathematical reasoning, and even reasoning usually accepted as totally finitary, Nelson regards as containing unjustifiable semantic elements. Let me, therefore, now turn to the beginning of the book and present some examples of arguments that Nelson finds problematic.

Many logicians would argue that finitary mathematical statements are adequately captured by formulas of the predicate calculus in the language containing a constant symbol for zero (0) and function symbols for the successor, addition and multiplication of natural numbers ($S, +, \cdot,$ respectively), and that finitary arguments are adequately modelled by formal proofs (using classical logic) in the system of first-order Peano Arithmetic (PA). (Statements and arguments about finite objects other than numbers can be coded into this system, but our concern here is with the natural numbers themselves.) The axioms of PA consist of the three successor axioms:

1. $Sx \neq 0$;
2. $Sx = Sy \to x = y$;
3. $x \neq 0 \to \exists y\ Sy = x$;

the recursive defining equations for addition and multiplication:

4. $x + 0 = x$;
5. $x + Sy = S(x + y)$;
6. $x \cdot 0 = 0$;
7. $x \cdot Sy = (x \cdot y) + x$;

together with, for each formula $\phi(x)$ of the language described above, the axiom of induction for ϕ:

$$I(\phi): \qquad [\phi(0) \wedge \forall x(\phi(x) \to \phi(Sx))] \to \forall x \phi(x).$$

Now by Gödel's second incompleteness theorem the consistency of PA cannot be proved within PA (and hence, if the comments above are correct it has no finitary proof at all) but this consistency hardly seems a controversial issue. After all, if we regard a natural number as being something that we eventually reach in constructing the sequence 0, SO, SSO, SSSO, $\cdots$ then surely this description carries with it the fact that the induction axioms are simply true. Certainly there seems to be no appeal here to any non-formal notions such as a completed infinite set (a view reinforced perhaps by the fact that Peano Arithmetic is equivalent (or rather, bi-interpretable with) the system obtained from Zermelo-Frankael set theory by replacing the axiom of infinity by its negation). Nelson disagrees. He argues that since we have specified a certain *predicative* construction of the natural number sequence (and it does seem impossible to formulate a finitary justification of the principle of induction without using some notion of 'constructing') then the only instances of the induction scheme for which the justification above is valid are the corresponding predicative ones. That is for those axioms $I(\phi)$ where the property $\phi(x)$ can be verified or refuted by reference to only those numbers that have been constructed at the time x appears (or at worst when $t(x)$ appears, where t is some pre-given term of the language). For example, if $\phi(x)$ is the usual formula expressing 'x is either even or odd' then the simple inductive proof of the sentence $\forall x\phi(x)$ is clearly predicatively correct since we always have available the number y such that $2 \cdot y = x$ or $2 \cdot y + 1 = x$ by the time x is constructed. However, the same is not true for the formula, $\Delta(x)$ say, expressing "there exists a number divisible by every number between 1 and x". For while the hypotheses of the induction axiom $I(\Delta)$ are predicatively verifiable the conclusion, $\forall x\Delta(x)$, is clearly impredicative in the above sense. Thus we really cannot justify this instance of induction (and others like it where the formulas involved contain essential occurrences of *unrestricted* existential quantification) by predicative means, and this invalidates the attempted justification of PA given above. Nelson describes the situation thus: "... numbers are symbolic constructions; a construction does not exist until it is made; when something new is made, it is something new and not a selection from a pre-existing collection."

Before describing Nelson's program for developing arithmetic predicatively perhaps a few comments are in order on how impredicativity is normally dealt with by mathematical logicians. Proof theorists have extensively studied the following situation. Suppose we are given a proof from some set of axioms, T say, of a statement of the form $\forall x \exists y R(x, y)$ where the relation R is bounded (i.e. definable in the language of arithmetic without the use of unrestricted quantifiers). Then can we extract from this proof a (description of a) computable function $f: IN \to IN$ such that $\forall n \in IN R(n, f(n))$ holds and, if so, how complex (usually measured in terms of the various hierarchies of computable functions) must f be in worst cases? It turns out that an investigation of this question can lead to a *predicative* description of another theory T^* such that any statement of the above form is derivable from T^* if and only if it is derivable from T. This is particularly interesting when T is a theory of infinitary objects (e.g. if T is some theory of analysis, or of a set theory with an axiom of infinity) in which case the discovery of the corresponding theory T^* can be viewed as carrying out Hilbert's program, in so far as this is possible, for T. More relevant in the present context, however, is the fact that the construction of T^* from T is now very well understood for most of the natural subsystems of PA. For example, a theorem of Parsons asserts that if T is Σ_1-induction (i.e. we only allow the induction axioms $I(\phi)$ for ϕ of the form $\exists y R(x, y)$ with R bounded) then the corresponding class of functions is the class of *primitive* recursive functions and the corresponding theory T^* is Primitive Recursive Arithmetic. Of course this result has little meaning for Nelson since he neither regards all primitive recursive functions as predicatively defined nor accepts Σ_1-induction (the axiom $I(\Delta)$ mentioned above is an axiom of Σ_1-induction) but I think he is being a little unfair when he asserts that it "appears to be universally taken for granted by mathematicians... that the impredicativity inherent in the induction principle is harmless... ", for while they may think it harmless at least the degree of impredicativity has been thoroughly and quantitatively investigated.

Having briefly described the background to Nelson's ideas I shall now turn to the more technical aspects of his theory. The first point that the reader should be aware of is that predicative arithmetic is not a theory at all in the usual sense of the predicate calculus. Rather, it consists in building up a stock of sentences (in the usual language of arithmetic discussed above) according to the following procedure. Firstly, call a formula $\phi(x)$ a *number system* if

$$Q \vdash \phi(0) \wedge \forall x(\phi(x) \to \phi(Sx)),$$

where Q consists of the first seven axioms of PA set out above. (Q is known as Robinson's Arithmetic and is regarded by Nelson as a minimal axiomatization of arithmetic.) Now suppose we have predicatively established the sentences $\Phi_1, \cdots, \Phi_n$. Then a sentence Φ_{n+1} may be added to the list provided that for every number system $\phi(x)$, there is some number system $\psi(x)$ such that

$$Q \vdash \forall x(\psi(x) \to \phi(x)) \quad \text{and} \quad Q \vdash (\Phi_1 \wedge \cdots \wedge \Phi_{n+1})^{\psi},$$

where, for any sentence Φ, Φ^{ψ} denotes the result of restricting all quantifiers in Φ to the number system ψ.

Thus many of the chapters in this book begin with the phrase "let T be the current theory... " and proceed by showing that new sentences may be predicatively added to T (although they may not actually be derivable from T) to obtain a stronger current theory. In this way, then, predicative arithmetic is built up.

Of course the spirit of the program is that a sentence, A, is to be regarded as predicatively established if $Q \cup \{A\}$ can be (explicitly) interpreted in the minimal theory Q. However, this cannot be taken as the definition because, by a result of Solovay, there are sentences A, B such that $Q \cup \{A\}$ and $Q \cup \{B\}$ are both interpretable in Q but $Q \cup \{A \wedge B\}$ is not. (This result was unknown to Nelson when he wrote the book, but he remarks on the problem.) The construction process above clearly avoids this difficulty and indeed any logical consequence of the "current theory" may be predicatively added to it. Of course Solovay's result still has the worrying consequence that there is no unique way to develop predicative arithmetic but such a dichotomy never arises in the book. The author's first aim in fact is to direct the "current theory" towards a technical justification of his philosophical comments concerning the two examples mentioned above. Namely he shows that any instance, $I(\Phi)$, of an induction axiom with $\phi(x)$ a *bounded* formula (e.g. "x is even or odd") is predicatively derivable, but that no function of (at least) exponential growth (e.g. $f(x)$ =the least y such that y is divisible by every number between 1 and x) is predicatively interpretable. (A function is predicatively interpretable if there is a natural bounded formula, $\psi(x, y)$ say, defining its graph such that the sentence $\forall x \exists! y\ \psi(x, y)$ is predicatively derivable.) The latter result (originally obtained by Paris and Dimitracopoulus) provokes much discussion on whether exponentiation should be regarded as a total function or not. It is clear where Nelson's sympathies lie but I shall not go into his arguments here except to point out that he has to address himself to the following apparent paradox:–although the statement "$\forall x\ \exists y\ 2^x = y$" is not predicatively derivable, the statement "$\forall x\ \exists p$ (p is a predicative proof of the statement "$\exists y\ 2^x = y$")" is so! The second proposition here is formalised via a predicative arithmetization of syntax and logic and is in fact closely related to the consistency problem for predicative arithmetic (and for Q itself)—subjects which occupy the later chapters of this book.

Nelson has done a good and careful job at presenting the huge number of formal proofs necessary for the development of his theory. Even so, I would not recommend anyone to read this book unless he or she had already acquired some intuition on weak subsystems of Peano Arithmetic. Suitable references for this are provided by Nelson and indeed most of his technical material has received more conventional treatments elsewhere. However, I think that the book is a valuable addition to the literature both for mathematical logicians because of the systematic and exhaustive account of interpretation in weak systems of arithmetic, and for philosophers of mathematics for the way in which conclusions compatible with strict finitism are deduced from assumptions based purely on a formalist viewpoint.

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Commutative rings with zero divisors, by James A. Huckaba. Marcel Dekker, New York and Basel, 1989, x+216 pp., $79.75. ISBN 0-8247-7844-8

This excellent monograph on the titled subject covers a huge amount of research over the past thirty years. The author manages in just over 200 pages (not densely printed—more about this later) to include works from over 200 papers. The Index of Main Results lists 120 theorems, and the remarkably complete end-of-chapter notes tell where each and every one comes from! The work is estimably enriched by more than 20 mostly difficult examples (and counterexamples) worked out in the last chapter, which the motivated reader reads appropriately alongside the foregoing. (No it-can-be-showns for Professor Huckaba!)

References in the sequel, especially to the chapter notes may be found in the text.

Chapter I (Total Quotient Rings) introduces various properties of the commutative ring R, its total quotient ring denoted

$$T(R) = \{a/b | a \in R, b \in R^*\},$$

where R^* is the set $R \backslash Z(R)$, and $Z(R)$ is the set of zero divisors of R. Also frequently used is the so-called complete (or maximal) ring $Q(R)$ of quotients, for which the author refers to the classic book of Lambek *Lectures on Rings and Modules* (currently reprinted by Chelsea).