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Divisors, by Richard R. Hall and Gérald Tenenbaum. Cambridge Tracts in Mathematics, Vol. 90. Cambridge University Press, Cambridge, New York, New Rochelle, Melbourne, and Sydney, 1988, xvi + 167 pp., $39.50. ISBN 0-521-34056-x

Number theory has its foundation in the Fundamental Theorem of Arithmetic which states that every integer $n > 1$ can be written uniquely in the form

$$(1) \qquad n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}, \qquad p_1 < p_2 < \cdots < p_r,$$

where the p_i's are primes and the α_i's positive integers. From an algebraic point of view, this result describes completely the set of positive integers as a free semigroup generated by the primes. However, for many problems in analytic number theory one would like to have more information about the structure of the prime factorization (1) and, in particular, the number and the size of the prime factors involved.

The prime factorization of an integer can of course take quite different shapes. On the one hand, if n itself is a prime, then its prime factorization consists of a single prime raised to the first power. On the other hand, if n is of the form $k!$, say, then it has a prime factorization with many small primes and relatively large exponents α_i. However, these are extreme cases that apply only to a relatively sparse set of integers n, and one might ask if it is possible to describe more precisely the prime factorization of a "typical," or "random," integer. This turns out to be the case; in fact, the study of such questions has led to the development of a new branch of number theory, called probablilistic number theory.

The first result in this direction, obtained in 1917 by Hardy and Ramanujan [HR], showed that a "random" integer n has about $\log\log n$ prime factors in the following sense: Let $\omega(n)$ denote the number of distinct prime factors of n, so that $\omega(n) = r$ in the representation (1). Let $\psi(n)$ be a function of n tending to infinity arbitrarily slowly, as $n \to \infty$. Then the inequality

$$(2) \qquad |\omega(n) - \log\log(n)| \le \psi(n)\sqrt{\log\log n}$$

holds for "almost all" positive integers n in the sense that the proportion of positive integers $n \le N$ for which (2) holds tends to one, as $N \to \infty$.

Some twenty years later, Erdös and Kac [EK] proved that for any real number z the relation

$$\lim_{N\to\infty} \frac{1}{N}\#\{n \le N\colon \omega(n) \le \log\log n + z\sqrt{\log\log n}\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-x^2/2}\,dx.$$

holds. This result refines the Hardy-Ramanujan theorem, for it shows not only that the number of prime factors of a randomly chosen integer n lies "with probability one" in an interval of the form (2), but more precisely that this number behaves like a normally distributed random variable with mean $\log\log n$ and standard deviation $\sqrt{\log\log n}$.

Results of this type, it turned out, depend very little on the particular definition of the function $\omega(n)$, and similar "limit theorems" can be proved for rather wide classes of "arithmetic functions," i.e., functions defined on the set of positive integers. This is one of the main subjects of probabilistic number theory; for a survey of this area see, for example, Elliott [El].

The theorems of Hardy-Ramanujan and Erdös-Kac describe the number of prime factors in the facorization of a random integer in a very precise, and in a sense best-possible, way. The next problem is to determine, as precisely as possible, the size of the prime factors p_i in (1). Since by the Hardy-Ramanujan theorem a typical integer n has about $\log\log n$ such prime factors, it is appropriate to rescale and consider the numbers $\log\log p_i$ rather than p_i. These numbers are all contained in the interval $[\log\log 2, \log\log n]$, so that, assuming a reasonably uniform distribution over this interval, one might expect that $\log\log p_i \approx i$. This is indeed the case, as was first shown by Erdös [Er] with the following result: Let $\varepsilon > 0$ and let $\xi(n)$ be a function tending to infinity with n. Then for almost all positive integers n with prime factorization (1) the inequalities

$$(3) \qquad |\log\log p_i - i| \le (1+\varepsilon)\sqrt{2i\log\log i} \qquad (\xi(n) \le i \le r)$$

hold. (Here "almost all" has the same meaning as in the theorem of Hardy-Ramanujan.) The result is best-possible in the sense that it becomes false if, in (3), $1+\varepsilon$ is replaced by $1-\varepsilon$.

There is a striking analogy between this result and the law of iterated logarithm in probability theory which states that if $X_1, X_2, \ldots$ are independent, identically distributed random variables with finite mean μ and variance σ^2, then the partial sums $S_n = \sum_{i=1}^n X_i$ satisfy

$$\limsup_{n\to\infty} \frac{|S_n - n\mu|}{\sigma\sqrt{2n\log\log n}} = 1 \tag{4}$$

with probability one. In fact, a comparison of (3) and (4) shows that the numbers $\log\log p_i$ behave like the i th partial sums of a sequence of independent, identically distributed random variables with mean 1 and variance 1.

The above results show that the behavior of the sequence of prime divisors of a random integer is rather regular and can be explained by relatively simple probabilistic models. One might ask now whether similar results hold for the sequence of all divisors of a random integer. This question, which forms the main subject of the book under review, turns out to be much more difficult. Our present knowledge here is far less complete than in the case of prime divisors, and many open problems remain. Nevertheless, as will be described below, there has been significant progress over the past few years which has yielded a number of results, most notably the resolution of a 50 year old conjecture of Erdös.

In their paper [HR] referred to above, Hardy and Ramanujan also considered the number of divisors of an integer n, denoted by $\tau(n)$, and showed that for any $\varepsilon > 0$ the bounds

$$(\log n)^{\log 2-\varepsilon} \le \tau(n) \le (\log n)^{\log 2+\varepsilon} \tag{5}$$

hold for almost all positive integers n. In fact, they proved (5) with the exponent ε replaced by $\psi(n)/\sqrt{\log\log n}$ for any function $\psi(n)$ tending to infinity.

The upper and lower bounds in (5) have quite different orders of magnitude, and one might ask whether it is possible to localize $\tau(n)$ for almost all n within a factor $1\pm\varepsilon$ of a "smooth" function $f(n)$, as has been the case with the function $\omega(n)$ (see (2)). It turns out that this is not possible under reasonable smoothness assumptions (for example, monotonicity) on the approximating function $f(n)$. This difficulty in getting a hold on the number of divisors of a random integer is symptomatic of the problems one encounters in trying to determine the distribution of the divisors of a random integer.

As another example of this phenomenon, consider the function

$$F_n(u) = \frac{1}{\tau(n)}\#\{d|n : d \le n^u\}. \tag{6}$$

For each n, $F_n(u)$ is a distribution function in the usual probabilistic sense. Deshouillers, Dress, and Tenenbaum [DDT] showed that, on average over n, this distribution behaves like an arc sine law; namely, one has for every u,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F_n(u) = \frac{2}{\pi} \arcsin \sqrt{u}.$$

This might lead one to expect that for almost all integers n, $F_n(u)$ is close to $(2/\pi) \arcsin \sqrt{u}$. However, this turns out to be false; in fact, as Tenenbaum [Te 1] had shown, there is no subset $A \subset \mathbf{N}$ containing a positive proportion of all integers such that, as $n \to \infty$ through the set A, the distributions $F_n(u)$ converge (in the usual probabilistic sense) to a limit distribution $F(u)$.

Despite these negative results there are some recognizable patterns in the distribution of the divisors of a random integer. It was realized early on that the divisors, after a logarithmic rescaling, were far from being uniformly distributed over the interval $[0, \log n]$ but rather seemed to appear in clusters. In order to investigate the amount of "clustering", Hall and Tenenbaum [HT 1] considered functions like

$$U(n, \alpha) = \#\{(d, d') : d, d'|n, (d, d') = 1, \\ |\log(d/d')| \le (\log n)^{\alpha}\}.$$

Since by (5) the average distance between the $\tau(n)$ numbers $\log d$, $d|n$, is $(\log n)^{1-\log 2+o(1)}$ for a random integer n, one should have $U(n, \alpha) = 1$ for all $\alpha \le 1 - \log 2 - \varepsilon$ and almost all n, if these numbers were uniformly distributed. However, Hall and Tenenbaum showed that this is not the case if $\alpha > 0$.

The motivation for this research came to a large degree from a conjecture of Erdös, formulated more than 50 years ago, according to which almost all positive integers n have two divisors d and d' satisfying $d < d' \le 2d$. This simple but deep conjecture has for a long time resisted attempts at its solution. Its difficulty lies in the fact that it expresses the irregularity, rather than the regularity, of the distribution of the divisors of a typical integer. Indeed, if the numbers $\log d$, $d|n$, were uniformly distributed over the interval $[0, \log n]$, then by (5) the distance between any two such numbers would be at least $(\log n)^{1-\log 2+o(1)}$ for an integer with the "normal" number of divisors, whereas Erdös' conjecture predicts that, for almost all n, at least two of these numbers are less than $\log 2$ apart.

In 1984, Maier and Tenenbaum [MT 1] finally proved the conjecture (and in doing so earned the $650 in prize money which had been offered by Erdös for its solution). In fact, they proved a much stronger result, namely that the function $U(n, \alpha)$ defined above satisfies $U(n, \alpha) > 1$ for any fixed $\alpha > 1 - \log 3$ and almost all n. In the book under review, the authors go even further by showing that for any fixed $\alpha \leq 1$ and almost all n,

$$U(n, \alpha) = 1 + (\log n)^{\log 3 - 1 + \alpha + o(1)}. \tag{7}$$

Thus, the number of coprime pairs d, d' of divisors of n that satisfy $|\log(d/d')| \leq t$ is roughly proportional to t. In other words, while the distribution of the numbers $\log d$, $d|n$, over the interval $[0, \log n]$ can be quite irregular, the differences of these numbers are approximately uniformly distributed for almost all n. The proofs of these results make an ingenious use of Fourier techniques, elementary arguments and probabilistic ideas. The same ideas have already yielded other results (see, for example, [Ma]), and could lead to further applications in the future.

A different measure for the "clustering" of divisors is provided by the function

$$\Delta(n) = \max_{u \geq 0} \#\{d|n : u < \log d \leq u + 1\}.$$

Maier and Tenenbaum [MT 1] showed that there exists a constant $\alpha > 0$ such that $\Delta(n) \geq (\log\log n)^{\alpha}$ holds for almost all n. This result implies Erdös' conjecture, and its proof is based on the same method. In the other direction, Maier and Tenenbaum [MT 2] proved that if $\psi(n)$ tends to infinity as $n \to \infty$, then $\Delta(n) \leq \psi(n) \log\log n$ holds for almost all n. It is conjectured that the "correct" lower bound, valid for almost all n, is $\Delta(n) \geq (1/\psi(n)) \log\log n$, for any function $\psi(n)$ tending to infinity.

The function $\Delta(n)$ is known as Hooley's Δ-function. It was studied by Hooley in a fundamental paper [Ho] where he showed that upper bounds for the averages

$$\frac{1}{N} \sum_{n \leq N} \Delta(n) \tag{8}$$

of $\Delta(n)$ had unexpected applications to various seemingly unconnected problems in additive number theory and diophantine approximation. His argument made use of the bound $\ll (\log N)^{4/\pi - 1}$ for (8) which he established by Fourier techniques. Hall and Tenenbaum [Te 2, HT 2] subsequently improved Hooley's bounds, showing in particular that (8) is bounded by $\ll (\log N)^{\varepsilon}$ for any $\varepsilon > 0$, and their results led to corresponding improvements in the applications considered by Hooley. More recently, Vaughan [Va] has used these results in work on Waring's problem.

The results and topics discussed above form the main themes of the book by Hall and Tenenbaum. Bounds for averages of Hooley's Δ-function are established in Chapters 6 and 7, the final two chapters of the book. The results in this part are the most difficult of the book, but also, in view of Hooley's work [Ho], the most useful for applications. (Unfortunately, the authors did not give any examples for such applications; the reader interested in seeing how these results are applied will have to consult Hooley's original paper.) In Chapters 4 and 5, Erdös' conjecture, the estimate (7) for $U(n, \alpha)$, as well as a number of related results, are proved. In Chapter 2, estimates for the proportion of positive integers $n \leq x$ having a divisor in a given interval $[y, z]$ are derived and applied, in particular, to prove the above-mentioned "negative" result on the behavior of the distribution functions (6). The remaining chapters are devoted to the distribution of the prime divisors of an integer, and to the derivation of auxiliary results needed in the latter part of the book. Each chapter concludes with a section of notes, and a set of exercises many of which are at an advanced level and provide useful insights even to the expert in the field.

Many of the results described in this book are due to the authors themselves. Some results, like the lower bound in (7), have never appeared before in the literature, while in other cases, as with Erdös' "law of iterated logarithm" (3), a detailed proof is given here for the first time. Due to the nature of the subject, the proofs are often very technical, but the exposition is clear and the authors have provided ample motivation throughout the book. Underneath the surface the reader will recognize a few recurring general ideas and principles, such as "Rankin's method," which the authors have very skillfully exploited here and which are useful in other areas as well.

All in all, this book represents a well-written treatment of an attractive though little known subject that has seen some remarkable breakthroughs in recent years. It should be of value to researchers in analytic or probabilistic number theory, whether they are interested in the subject of the distribution of divisors for its own sake, in the techniques displayed in the proofs which they might find useful in their own research, or in applications of the results of this book to other parts of number theory. It could also serve as a text for a seminar or a reading text for graduate students with some background in analytic number theory.

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