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Triangulated categories in the representation theory of finite dimensional algebras, by Dieter Happel. Cambridge University Press, Cambridge, New York, New Rochelle, Melbourne, Sydney, 1988. (London Mathematical Society Lecture Notes Series, vol. 119), ix + 208 pp., $24.95. ISBN 0-521-33922-7

The concept of a triangulated category seems to have appeared in the sixties, and to go back to suggestions of Grothendieck. The basic reference is the article of Verdier [26]. The most famous example is the derived category $D^b(\mathscr{A})$ of bounded complexes over an abelian category $\mathscr{A}$. It is constructed as follows. Let $C^b(\mathscr{A})$ denote the (abelian) category of bounded complexes over $\mathscr{A}$. In many situations, occurring for instance in algebraic geometry, complexes are determined only up to quasi-isomorphisms, that is, morphisms of complexes which induce an isomorphism in cohomology. We would then like to replace $C^b(\mathscr{A})$ by a new category in which such complexes could be declared isomorphic. The first step in the construction comes from the observation that in such situations, morphisms of complexes are only determined up to homotopy. It is thus reasonable to replace $C^b(\mathscr{A})$ by the homotopy category $K^b(\mathscr{A})$, whose objects are the same as those of $C^b(\mathscr{A})$, but whose morphisms are the homotopy classes of morphisms of complexes. The homotopy category $K^b(\mathscr{A})$ is not abelian in general. It enjoys a weaker structure, that of a triangulated category. In this structure, short exact sequences are replaced by what are called distinguished triangles. Finally, one obtains the derived category $D^b(\mathscr{A})$ from $K^b(\mathscr{A})$ by declaring all the quasi-isomorphisms invertible, or, more formally, by localising $K^b(\mathscr{A})$ relative to the multiplicative system of quasi-isomorphisms (using the calculus of fractions developed in [11]). The category $D^b(\mathscr{A})$ inherits from $K^b(\mathscr{A})$ the structure of a triangulated category. Since its introduction, this concept was used in many problems of algebraic geometry and homological algebra, most notably in duality theory (see, for instance, [18, 6 or 20]).

Recently, it was shown that the derived categories of certain categories of coherent sheaves are equivalent to the derived category of bounded complexes of finitely generated modules over certain finite dimensional algebras (see, for instance, [7, 5, 21] or, more recently, [12]). There was therefore a need to bridge the gap between the theory of triangulated categories and the representation theory of finite dimensional algebras which has been the subject of intense research over the last twenty years. As the title indicates, Happel's research monograph *Triangulated categories in the representation theory of finite dimensional algebras* does just that. Given an algebra A (by which is always meant an associative algebra, with 1, finite dimensional over an algebraically closed field k), he considers the derived category D^b (mod A), briefly denoted by $D^b(A)$, of bounded complexes over the category mod A of finitely generated left A-modules. On the one hand, Happel's book contains a concrete description of $D^b(A)$, on the other hand, it shows how one can use the concept of triangulated category to study the category mod A.

The first two chapters of the book are mainly concerned with the concrete description of $D^b(A)$. The author starts by defining and studying a notion of Auslander-Reiten triangles in $D^b(A)$, inspired by the notion of Auslander-Reiten sequences in mod A (see [4]). If A has finite global dimension, then $D^b(A)$ has Auslander-Reiten triangles. It is then possible to define the quiver of $D^b(A)$ as one defines the Auslander-Reiten quiver of an algebra, and to give a complete description of this former quiver, for A basic and hereditary. Next, the author considers the repetitive algebra $\hat{A}$ of an algebra A, see [19]. This is defined as follows. Let ${}_AQ_A$ be the minimal injective cogenerator bimodule, that is, let ${}_AQ_A$ be the k-vector space $Q = \operatorname{Hom}_k(A, k)$ endowed with the A-A-bimodule structure defined by

$$(afb)(x) = f(bxa)$$

for $a, b, x \in A$ and $f \in Q$. The underlying k-vector space of $\hat{A}$ is defined by

$$\widehat{A} = \left(\bigoplus_{i \in \mathbf{Z}} A_i\right) \oplus \left(\bigoplus_{i \in \mathbf{Z}} Q_i\right)$$

where $A_i = A$ and $Q_i = Q$ for all $i \in \mathbf{Z}$. Let $(a_i, f_i)_{i\in\mathbf{Z}}$ and $(b_i, g_i)_{i\in\mathbf{Z}}$ be two elements of $\widehat{A}$, then their product is defined by

$$(a_i, f_i)_{i\in\mathbf{Z}} \cdot (b_i, g_i)_{i\in\mathbf{Z}} = (a_ib_i, a_{i+1}g_i + f_ib_i)_{i\in\mathbf{Z}}.$$

One thus obtains an infinite dimensional associative self-injective algebra without identity. More suggestively, one can look at $\widehat{A}$ as being the doubly infinite matrix algebra

$$\widehat{A} = \begin{bmatrix} \ddots & & & & 0 \\ \ddots & A_{i-1} & & & \\ & Q_{i-1} & A_i & & \\ & & Q_i & A_{i+1} & \\ 0 & & & Q_{i+1} & \ddots \\ & & & & \ddots \end{bmatrix}$$

where matrices have only finitely many nonzero coefficients, $A_i = A$ and $Q_i = Q$ for all $i \in \mathbf{Z}$, all the other coefficients are zero, and the multiplication is induced from the A-A-bimodule structure of Q, and the zero map $Q \otimes_A Q \to 0$. The author exhibits a structure of triangulated category on the stable module category $\underline{\text{mod}}\ \widehat{A}$, whose objects are the same as those of mod $\widehat{A}$ (thus, finite dimensional left $\widehat{A}$-modules), but the set of morphisms from ${}_{\widehat{A}}M$ to ${}_{\widehat{A}}N$ equals the quotient of $\text{Hom}_{\widehat{A}}(M, N)$ modulo the subset of those morphisms which factor through a projective $\widehat{A}$-module. The main theorem states that there exists a full and faithful exact functor $D^b(A) \to \underline{\text{mod}}\ \widehat{A}$. If A has finite global dimension, this functor is even an equivalence of triangulated categories. This result, originally published in [13], has had since numerous applications. For instance, it allowed Happel and Ringel to describe in [17] the derived category of a tubular algebra (in the sense of [25]).

The third chapter is devoted to tilting theory, which is now a highly developed branch of representation theory. The tilting process can be understood as follows. For an algebra A, some (finite dimensional) A-modules M have the property that the endomorphism algebra $B = \text{End}({}_AM)$ retains many of the properties of A. Thus, it is possible to understand a lot about the category mod B using our knowledge of mod A. Such a module ${}_AM$ is called a tilting module (intuitively, a tilting module is thus a module which is "close to," but generally not equal to, a Morita progenerator in mod A). Tilting theory has its origin in the use by Bernstein, Gelfand and Ponomarev of reflection functors in order to prove Gabriel's theorem [8]. The first axiomatic definition of tilting modules is due to Brenner and Butler [9], and the one most commonly used nowadays to Happel and Ringel [16]. Further generalisations of this notion are due to Miyashita ([22], see also [13]) and Wakamatsu [27]. It is shown in the third chapter of Happel's monograph that the derived category is invariant under tilting: let

A be an algebra of finite global dimension and ${}_AM$ an A-module such that $\operatorname{Ext}^i_A(M, M) = 0$ (for all $i > 0$) and there exists an exact sequence

$$0 \to {}_AA \to M^0 \to M^1 \to \cdots \to M^s \to 0$$

where all the M^j $(0 \leq j \leq s)$ are direct sums of indecomposable direct summands of M, then, if $B = \operatorname{End}({}_AM)$, there exists an equivalence $D^b(A) \xrightarrow{\sim} D^b(B)$ of triangulated categories. This is in particular the case if M is a tilting module, that is, if M is as above, with $s \leq 1$ and $\operatorname{pd} M \leq 1$. This invariance of the derived category was shown in [13], and was since generalised by Cline, Parshall and Scott [10] to the case where $\operatorname{pd} M < \infty$, but gl. dim. A is arbitrary. They also asked the question to find necessary and sufficient conditions on two algebras A and B so that $D^b(A) \xrightarrow{\sim} D^b(B)$, as triangulated categories, a problem solved by J. Rickard [23].

Using the invariance of the derived category under tilting, Happel proves most of the classical results of tilting theory, as well as some new results. For instance, he shows that, if A is a hereditary algebra and ${}_AM$ a module such that $\operatorname{Ext}^1_A(M, M) = 0$, then $B = \operatorname{End}({}_AM)$ is a tilted algebra (that is, there exists a module ${}_BU$ such that $\operatorname{End}({}_BU)$ is hereditary).

The fourth chapter studies properties of the piecewise hereditary algebras. Since the derived category of a hereditary algebra H is completely described, it is reasonable to consider the algebras A such that $D^b(A) \xrightarrow{\sim} D^b(H)$, as triangulated categories. Such an algebra A is called piecewise hereditary of type H. The main theorem (taken from [15]) states that an algebra A is piecewise hereditary of type H if and only if A is tiltable to H, that is, there exists a sequence of triples $(A_i, {}_{A_i}M^i, A_{i+1} = \operatorname{End}({}_{A_i}M^i))_{0\leq i<m}$ such that $A_0 = A$, $A_m = H$ and M^i is a tilting A_i-module (so that the class of piecewise hereditary algebras coincides with the class of iterated tilted algebras [1]).

The last, short, chapter of this book is about trivial extension algebras. Let again ${}_AQ_A = \operatorname{Hom}_k({}_AA_A, k)$ denote the minimal injective cogenerator bimodule. The trivial extension $T(A)$ of A by Q has as underlying k-vector space $A \oplus Q$ with the multiplication defined by

$$(a, f)(b, g) = (ab, ag + fb)$$

(for $a, b \in A$ and $f, g \in Q$). Then $T(A)$ is a self-injective and even a symmetric algebra. It is $\mathbf{Z}$-graded, with the elements of

$A \oplus 0$ of degree 0 and those of $0 \oplus Q$ of degree 1. The category $\text{mod}^{\mathbf{Z}} T(A)$ of finitely generated $\mathbf{Z}$-graded left $T(A)$-modules with morphisms of degree 0 is equivalent to $\text{mod}\hat{A}$, so that, if $\text{gl. dim. } A < \infty$, then $D^b(A) \xrightarrow{\sim} \underline{\text{mod}}^{\mathbf{Z}} T(A)$, as triangulated categories. Happel gives a short proof of the statement that $T(A)$ is representation-finite (that is, has only finitely many nonisomorphic indecomposable modules) if and only if A is tiltable to a representation-finite hereditary algebra [2].

This monograph is in my opinion a very valuable introduction to the use of triangulated categories in representation theory. Since it was written, and partly thanks to it, the theory has advanced very quickly (let us just mention the results in [3, 10, 14, 17, 23, 24]). It should be easily readable for the expert in representation theory, the graduate student and the algebraist, with only the definition of the derived category assumed. One of its most attractive features is the large number of examples explained in detail throughout the book. Finally, since most of the results exposed have never before appeared in a book, the nonexpert and the research student may find Happel's monograph a useful introduction to many of the branches of modern representation theory.

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IBRAHIM ASSEM

UNIVERSITÉ DE SHERBROOKE