addressed, not least for its remarkably complete bibliography of the whole subject of reproducing kernels.

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Geometric inequalities, by Yu. D. Burago and V. A. Zalgaller. (Translated by A. B. Sossinsky), Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1988, xiv + 331 pp., $97.00. ISBN 3-540-13615-0

This volume presents us with a masterful treatment of a subject that is not so easily treated. The basic difficulty is that "geometric inequalities" is not so much a subject as a collection of topics drawing from diverse fields and using a wide variety of methods. One can therefore not expect the kind of cohesiveness or of structural development that is possible in a single-topic book. At most one hopes for a broadly representative selection of theorems organized by approach or content, with a good accounting of each and ample references for following up in any given direction; and that is just what we get.

All the classical topics are found here: the isoperimetric inequality in its many guises, the Brunn-Minkowski inequality with its various consequences, area and volume bounds of different kinds. There are also many inequalities involving curvatures: Gauss, mean, Ricci, etc. The methods include those of differential geometry, geometric measure theory, and convex sets. In each of these areas, the book is right up to date, including the latest results to the time of writing.

In addition to these classical topics, there are some more modern ones. Chapter 3 includes an extended and illuminating discussion of various notions of area and measure, including the newer approaches dating from the 1960s: the *perimeter* of Caccioppoli and de Giorgi, *integral currents* of Federer and Fleming, Almgren's *varifolds*. Their relative merits and disadvantages are carefully and even-handedly pointed out. Chapter 6, on Riemannian manifolds, provides a complete proof of Margulis' Theorem giving a lower bound for the volume of a compact negatively curved manifold in terms of a lower bound on the curvature.

A few post-modern results are at least alluded to, in particular some of those from Gromov's paper on "Filling Riemannian manifolds" which appeared after the original Russian edition of this book, but before the English version was written.

Another feature of the English edition of this book, not present in the original, is an extended (25-page) Addendum to Chapter 4 written by A. G. Khovanskiĭ. He provides a fairly detailed description of one of the prettiest recent developments in the subject: some unexpected links between geometric inequalities and algebraic geometry. The key notion is that of "mixed volumes" introduced by Minkowski. On first acquaintance the idea of "mixed volume" seems a bit strained, without obvious geometric content except in special cases. However, it arises in a natural fashion in computations of Minkowski sums of domains, and has proved a useful concept in the field of geometric inequalities, although without must influence on the wider world of mathematics. Then in 1975, D. M. Bernstein found that it provided a geometric interpretation of an old formula of Minding concerning the number of solutions to a system of polynomial equations. Specifically, here is what Bernstein proved. To each polynomial P in n variables, associate a polyhedron in $\mathbf{R}^n$—the *Newton polyhedron* of P—defined to be the convex hull of the set of lattice points $(m_1, \ldots, m_n)$ such that P contains a term $cz_1^{m_1} \cdots z_n^{m_n}$, $c \neq 0$. One can also allow negative exponents, in which case P is called a *Laurent polynomial*. Note that a nonzero constant term in P corresponds to the origin in $\mathbf{R}^n$. What Bernstein proved is that *for a general system of n polynomial equations in n complex variables, $P_k = 0$, $k = 1, \ldots, n$, where each P_k has a nonzero constant term, the number of complex roots is equal to the mixed volume of the corresponding Newton polyhedra multiplied by $n!$*. (A suitable modification works if the constant term is missing.)

Three years later, algebraic geometry repaid its debt to Minkowski when independently, Khovanskiĭ in Moscow and Tessier in Paris showed how one can use the Hodge index theorem to prove a famous inequality on mixed volumes that had originally been obtained independently by A. D. Alexandrov and Werner Fenchel in 1936. (The Alexandrov-Fenchel inequality in turn implies the isoperimetric inequality and a series of other classical inequalities.) Khovanskiĭ gives a very clear outline of the argument, with a number of explicit examples to illustrate the concepts introduced. The one fault I would find with this section is the cursory treatment of Bézout's Theorem, which for most readers would be the best point of entry into the subject, as the result on intersections most easily stated and most likely to be familiar. It says that the number of

solutions of n polynomial equations in n dimensions is just the product of the degrees. Of course, in order to get such a simple statement, one has first of all to define and use the notion of multiplicity of intersection, and second, projectivize everything in order to count intersections at infinity. That is done by introducing homogeneous coordinates, making the polynomials homogeneous, so that the corresponding Newton polyhedra have a special form.

It would have been helpful to have some of that explained, preferably at the beginning of the section.

On the whole, this book is to be commended for choice of topics, clear exposition, and relatively few typographical errors. The translation is generally good and very readable. It has a bit the flavor of a nonnative speaker who knows the language well, but is tripped up by subtle distinctions—the kind that gives away the underground agent in spy stories—"conclusive" for "definitive," "far going" for "far reaching." Some of them can add charm: "wave" for "tilde," but when it comes to mathematical terminology, I would have thought the editors might find someone to go through and standardize the terms, such as "integer current" for "integral current," and "unit" for "identity" element of a group. A reader being introduced to the terminology for the first time should be aware of possible discrepancies.

One of the attractive, but also perilous features of this subject is that it is a game anybody can play, with little preparation. As a consequence, the same result is often proved, published, vanishes from sight, and then reproved, republished, sometimes repeatedly. For those of us working in the area at the time that André Weil's collected works were published a few years back, it was a surprise, that would have been a pleasant one, to discover that his very first paper was on the isoperimetric inequality, except that the theorem he proved is universally referred to as the Beckenbach-Radó theorem. It states that the standard isoperimetric inequality in the plane continues to hold for simply-connected domains on surfaces with nonpositive Gauss curvature. André Weil obtained the result seven years before Radó, but his paper was consigned to oblivion. During a year of fairly intensive and extensive library research on the subject of isoperimetric inequalities, I did not see a single reference to Weil's paper. The present book misses a chance to rectify that, but in other respects the bibliography is excellent and comprehensive.

Of course the subject marches on. Two references that appeared in 1988, too late to make it into the present book, are the book *Convex bodies and Algebraic geometry* by Tadao Oda, and two long papers on isoperimetric inequalities in the journal *Astérisque*

by Sylvestre Gallot. But this book will clearly be *the* reference for some time to come.

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Amenable Banach algebras, by J.-P. Pier. Pitman Research Notes in Mathematics Series, vol. 172, Longman Scientific and Technical, Harlow and New York, 1988, 161 pp., $47.95. ISBN 0-582-01480-8

The concept of amenability was first defined for locally compact groups having evolved from the idea of a translation invariant mean or average on the bounded L^∞-functions on the real line used by von Neumann. If G is a locally compact group, then (left) Haar measure m induces a left translation invariant continuous positive linear functional on $L^1(G)$, the space of m integrable functions. There is no such translation invariant linear functional on $L^\infty(G)$, or on several other large spaces of bounded functions, for most locally compact groups G. The groups for which there is such a positive invariant mean were called amenable by M. M. Day (1950). The transition of amenability from groups to Banach algebras arose from the transfer of Hochschild cohomology into this setting.

If X is a Banach module over a Banach algebra A, then the first (continuous Hochschild) cohomology group $H^1(A, X)$ is the quotient of the linear space of (continuous) derivations by the space of inner derivations. A derivation D from A into X is a linear operator from A into X such that $D(ab) = aD(b) + D(a)b$ for all a, b in A, and D is inner if there is an x in X such that $D(a) = ax - xa$ for all a in X. B. E. Johnson [7] showed that the amenability of a locally compact group G is equivalent to the first cohomology group $H^1(L^1(G), X)$ being zero for each dual $L^1(G)$-module X. One direction of the proof uses the in-