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# A GENERALIZATION OF SELBERG'S BETA INTEGRAL

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ABSTRACT. We evaluate several infinite families of multi-dimensional integrals which are generalizations or analogs of Euler's classical beta integral. We first evaluate a $q$-analog of Selberg's beta integral. This integral is then used to prove the Macdonald-Morris conjectures for the affine root systems of types $S(C_l)$ and $S(C_l)^\vee$ and to give a new proof of these conjectures for $S(BC_l)$, $S(B_l)$, $S(B_l)^\vee$ and $S(D_l)$ .

## 1. INTRODUCTION

In 1944, A. Selberg [23] evaluated the following integral (see also Aomoto [1]):

$$(1)\quad \int_0^1 \cdots \int_0^1 \prod_{1\le i<j\le n} |t_i - t_j|^{2z} \prod_{j=1}^n t_j^{x-1}(1-t_j)^{y-1} dt_j = \prod_{j=1}^n \frac{\Gamma(x+(j-1)z)\Gamma(y+(j-1)z)\Gamma(jz+1)}{\Gamma(x+y+(n+j-2)z)\Gamma(z+1)},$$

where $n$ is a positive integer, $x, y, z \in \mathbf{C}$ and $\mathrm{Re}(x)$, $\mathrm{Re}(y) > 0$ and $\mathrm{Re}(z) > -\max\{\frac{1}{n}, \mathrm{Re}(x)/(n-1), \mathrm{Re}(y)/(n-1)\}$. For $n = 1$, the integral (1) reduces to Euler's classical beta integral.

Now let $n \ge 1$ and $a_1, a_2, a_3, a_4, b, q \in \mathbf{C}$ with

$$\max\{|a_1|, \ldots, |a_4|, |b|, |q|\} < 1.$$

For $c \in \mathbf{C}$ define

$$[c; q]_\infty = [c]_\infty = \prod_{k=0}^{\infty}(1 - cq^k).$$

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If $T^n$ is the $n$-fold direct product of the unit circle $\{t \in \mathbf{C} \mid |t| = 1\}$ traversed in the positive direction, then we can evaluate the integral

$$(2)\quad \frac{1}{(2\pi i)^n}\int_{T^n}\prod_{1\le j<k\le n}\frac{[t_jt_k^{-1}]_\infty[t_j^{-1}t_k]_\infty[t_jt_k]_\infty[t_j^{-1}t_k^{-1}]_\infty}{[bt_jt_k^{-1}]_\infty[bt_j^{-1}t_k]_\infty[bt_jt_k]_\infty[bt_j^{-1}t_k^{-1}]_\infty}$$
$$\cdot\prod_{j=1}^{n}\frac{[t_j^2]_\infty[t_j^{-2}]_\infty dt_j}{\prod_{k=1}^{4}\{[a_kt_j]_\infty[a_kt_j^{-1}]_\infty\}t_j}$$
$$= 2^n n!\prod_{j=1}^{n}\frac{[b]_\infty\left[b^{n+j-2}\prod_{k=1}^{4}a_k\right]_\infty}{[b^j]_\infty[q]_\infty\prod_{1\le k<l\le 4}[a_ka_lb^{j-1}]_\infty}.$$

Then $n = 1$ case of integral (2) is due to Askey and Wilson [4]. The integral (2) is a $q$-analog of (1) in the sense that after a change of variables and an appropriate specialization of (2) and limit as $q \to 1$, then (1) can be deduced from (2).

Selberg's integral (1) has had diverse applications in fields ranging from number theory, physics, statistics, combinatorics, algebra and analysis. Two particular applications were a use by Bombieri to prove Mehta's conjecture [18] and by Macdonald [17] to prove some of his conjectures ($q = 1$ case) for the affine root systems (for definition and properties see [15]) of types $S(BC_l)$, $S(B_l)$, $S(B_l)^\vee$, $S(C_l)$, $S(C_l)^\vee$ and $S(D_l)$ for all $l \ge 1$ (when defined). Just as Macdonald used integral (1) to prove some of his ($q = 1$) conjectures, we will use integral (2) to prove for the same set of affine root systems the corresponding Macdonald-Morris conjectures with arbitrary parameter $q$.

Macdonald's root system conjectures in [17] were motivated partly by a conjecture of Dyson [7] related to the root system $A_n$, a $q$-analog of Dyson's conjecture made by Andrews [2] and some conjectures of Morris [19] for the root system of type $G_2$. Dyson's conjecture was proved by Gunson [10] and Wilson [25]. The Andrews-Dyson conjecture was proved by Zeilberger and Bressoud [28].

Morris' Conjecture A in [19] for arbitrary parameter $q$ and any reduced irreducible affine root system $S$ extends Macdonald's Conjectures 2.3 and 3.1 in [17]. In the simplest case of these Macdonald-Morris conjectures, let $R$ be a reduced finite (*not affine*) root system of rank $l$ with basis $\{\alpha_1, \dots, \alpha_l\}$. For each $\alpha \in R$, let $e^\alpha$ be the formal exponential, which is an element of the group ring of the lattice generated by $R$. Let $d_1, \dots, d_l$

be the degrees of the fundamental invariants of the Weyl group $W(R)$.

**Conjecture (Macdonald** [17, Conjecture 3.1]**).** *With the above notation, the constant term (i.e. involving $q$ but no exponential $e^{\alpha}$) in*

$$\prod_{\alpha>0}\prod_{i=1}^{k}(1-q^{i-1}e^{-\alpha})(1-q^{i}e^{\alpha})$$

*where $k$ is a positive integer or $+\infty$ is*

$$\prod_{i=1}^{l}\begin{bmatrix} kd_i \\ k \end{bmatrix}$$

*where*

$$\begin{bmatrix} n \\ r \end{bmatrix}$$

*is the "q-binomial coefficient"*

$$\frac{(1-q^{n})(1-q^{n-1})\cdots(1-q^{n-r+1})}{(1-q)(1-q^{2})\cdots(1-q^{r})}.$$

We will actually prove the more general Morris' Conjecture A [19] for the affine root systems $S$ of types $S(BC_l)$, $S(B_l)$, $S(B_l)^{\vee}$, $S(C_l)$, $S(C_l)^{\vee}$ and $S(D_l)$ for all $l \geq 1$ (when defined) and for arbitrary parameter $q$. Macdonald's Conjecture 3.1 stated above, where $R$ is a finite root system of type $B_l$, $C_l$ or $D_l$, then follows as a special case of Morris' Conjecture A for $S(B_l)$, $S(C_l)$ and $S(D_l)$. Kadell [14] has previously proved these conjectures for all affine root systems of type $S(BC_l)$ and hence $S(B_l)$, $S(B_l)^{\vee}$ and $S(D_l)$. The Macdonald-Morris conjectures for $R = G_2$ have been proved by Habsieger [13] and Zeilberger [26]. See Garvan [8] for $F_4$, Garvan and Gonnet [9] for $S(F_4)^{\vee}$, Zeilberger [27] for $S(G_2)^{\vee}$ and Opdam [20] for the $q = 1$ conjectures. There is also the conjecture of Rahman [21] which seems related to the special case of integral (2) where $a_2 = q^{1/2}a_1$ and $a_4 = q^{1/2}a_3$.

## 2. Proof of integral (2)

Since the $n = 1$ case of (2) is proved in [4], we may assume that $n \geq 2$. Denote the integral on the left-hand side of (2) by $I_n(a_1, a_2, a_3, a_4; b; q)$. Let $c_j \in \mathbf{C}$, $|c_j| < 1$, for $1 \leq j \leq 2n+2$

with $q$ and $T$ as above. In [11] we have evaluated the integral

$$(3)\quad \frac{1}{(2\pi i)^n}\int_{T^n}\frac{\prod_{1\le j<k\le n}\{[t_jt_k^{-1}]_\infty[t_j^{-1}t_k]_\infty[t_jt_k]_\infty[t_j^{-1}t_k^{-1}]_\infty\}}{\prod_{j=1}^{2n+2}\prod_{k=1}^{n}[c_jt_k]_\infty[c_jt_k^{-1}]_\infty}\cdot\prod_{j=1}^{n}\frac{[t_j^2]_\infty[t_j^{-2}]_\infty\,dt_j}{t_j}$$

$$=\frac{2^n n!\left[\prod_{j=1}^{2n+2}c_j\right]_\infty}{[q]_\infty^n\prod_{1\le j<k\le 2n+2}[c_jc_k]_\infty}.$$

With notation as above, consider the integral

$$(4)\quad \frac{1}{(2\pi i)^{2n-1}}\int_{T^n}\int_{T^{n-1}}\frac{\prod_{1\le j<k\le n}\{[t_jt_k^{-1}]_\infty[t_j^{-1}t_k]_\infty[t_jt_k]_\infty[t_j^{-1}t_k^{-1}]_\infty\}}{\prod_{j=1}^{n}\prod_{k=1}^{4}[a_kt_j]_\infty[a_kt_j^{-1}]_\infty}$$

$$\cdot\frac{\prod_{j=1}^{n}[t_j^2]_\infty[t_j^{-2}]_\infty\prod_{1\le j<k\le n-1}\{[s_js_k^{-1}]_\infty[s_j^{-1}s_k]_\infty[s_js_k]_\infty[s_j^{-1}s_k^{-1}]_\infty\}}{\prod_{j=1}^{n}\prod_{k=1}^{n-1}\{[b^{1/2}s_kt_j]_\infty[b^{1/2}s_k^{-1}t_j]_\infty[b^{1/2}s_kt_j^{-1}]_\infty[b^{1/2}s_k^{-1}t_j^{-1}]_\infty\}}$$

$$\cdot\prod_{k=1}^{n-1}\frac{[s_k^2]_\infty[s_k^{-2}]_\infty\,ds_k}{s_k}\prod_{j=1}^{n}\frac{dt_j}{t_j},$$

where $b^{1/2}$ is any fixed square root of $b$. In the integral (4) we may use identity (3) to evaluate the interior integral either with respect to the set of variables $\{s_1,\dots,s_{n-1}\}$ or, by changing the order of integration, with respect to the set of variables $\{t_1,\dots,t_n\}$. Equating the resulting integrals we obtain

$$(5)\quad \frac{2^{n-1}(n-1)![b^n]_\infty}{[q]_\infty^{n-1}[b]_\infty^n}I_n(a_1,a_2,a_3,a_4;b;q)$$

$$\frac{2^n n![b^{n-1}\prod_{j=1}^{4}a_j]_\infty}{[q]_\infty^n[b]_\infty^{n-1}\prod_{1\le j<k\le 4}[a_ja_k]_\infty}I_{n-1}(a_1b^{1/2},\dots,a_4b^{1/2};b;q).$$

We finish the proof of identity (2) by doing induction on $n$, using identity (5) and the Askey-Wilson integral for the case $n = 1$.

## 3. Morris' Conjecture A

We sketch a proof of Morris' Conjecture A [19] for the affine root systems $S$ of types $S(BC_l)$, $S(B_l)$, $S(B_l)^\vee$, $S(C_l)$, $S(C_l)^\vee$ and $S(D_l)$ where $l \geq 1$ (when defined) and for arbitrary parameter $q$. The proof consists of specializing the parameters in identity (2) and making use of the identity found in Theorem 2.8 of [16]. As an illustration of this method of proof of Morris' Conjecture A, consider the case $S = S(C_l)$ where $l \geq 2$. Consider the integral $I_l(a^{1/2}, -a^{1/2}, q^{1/2}a^{1/2}, -q^{1/2}a^{1/2}; b; q)$ where $|a|, |b| < 1$. Multiply the integrand in this integral by

$$\prod_{1\leq j<k\leq l} \frac{(1 - bw(t_j^{-1}t_k))(1 - bw(t_j^{-1}t_k^{-1}))}{(1 - w(t_j^{-1}t_k))(1 - w(t_j^{-1}t_k^{-1}))} \prod_{j=1}^{l} \frac{(1 - aw(t_j^{-2}))}{(1 - w(t_j^{-2}))},$$

where $w$ is an element of the Weyl group $W$ of $C_l$, i.e. a permutation of the variables $t_1, \ldots t_l$ together with inversions $t_j \to t_j^{-1}$ and the corresponding action on $t_1^{-1}, \ldots, t_l^{-1}$. The resulting integral is independent of $w \in W$. Now summing over $w \in W$ and using the identity [16, Theorem 2.8] for $C_l$ we obtain

$$(6) \quad \frac{1}{(2\pi i)^l} \int_{T^l} \prod_{1\leq j<k\leq l} \frac{[t_jt_k^{-1}]_\infty [qt_j^{-1}t_k]_\infty [t_jt_k]_\infty [qt_j^{-1}t_k^{-1}]_\infty}{[bt_jt_k^{-1}]_\infty [qbt_j^{-1}t_k]_\infty [bt_jt_k]_\infty [qbt_j^{-1}t_k^{-1}]_\infty}$$

$$\cdot \prod_{j=1}^{l} \frac{[t_j^2]_\infty [qt_j^{-2}]_\infty \, dt_j}{[at_j^2]_\infty [qat_j^{-2}]_\infty t_j}$$

$$= \prod_{j=1}^{l} \frac{[qb]_\infty [qa^2b^{l+j-2}]_\infty [qab^{j-1}]_\infty^2}{[q]_\infty [qb^j]_\infty [qa^2b^{2(j-1)}]_\infty^2},$$

which is equivalent to Morris' Conjecture A for $S(C_l)$ [19, p. 131]. Setting $a = b$ in (6), this also proves Macdonald's Conjecture 3.1 for $R = C_l$ as stated above.

## 4. Some integral evaluations

We state some integral identities whose proofs are similar to that of (2), making use of integral identities from [11 and 12].

Details of the proofs of these and realted integral identities should be given elsewhere.

Let $n \geq 1$ and $z_1, \dots, z_n, \alpha_1, \dots, \alpha_4, a_1, \dots, a_4, \beta_1, \beta_2, b, \delta \in \mathbf{C}$ and $m_1, \dots, m_n \in \mathbf{Z}$. Choose $z_1, \dots, z_n$ so that the integrands in the integrals (9) and (10) below have no poles. Then

$$(7)\quad \frac{1}{(2\pi i)^n} \int_{-i\infty}^{i\infty} \cdots \int_{-i\infty}^{i\infty} \prod_{1\leq j<k\leq n} \left\{ \frac{\Gamma(\delta + t_j - t_k)\Gamma(\delta + t_k - t_j)}{\Gamma(t_j - t_k)\Gamma(t_k - t_j)} \cdot \frac{\Gamma(\delta + t_j + t_k)\Gamma(\delta - t_j - t_k)}{\Gamma(t_j + t_k)\Gamma(-t_j - t_k)} \right\} \prod_{j=1}^{n} \frac{\prod_{k=1}^{4} \{\Gamma(\alpha_k + t_j)\Gamma(\alpha_k - t_j)\}\, dt_j}{\Gamma(2t_j)\Gamma(-2t_j)}$$
$$= 2^n n! \prod_{j=1}^{n} \frac{\Gamma(j\delta) \prod_{1\leq k<l\leq 4} \Gamma(\alpha_k + \alpha_l + (j-1)\delta)}{\Gamma(\delta)\Gamma\left((n+j-2)\delta + \sum_{k=1}^{4} \alpha_k\right)},$$

where the contours of integration are the imaginary axis and

$$\min\{\mathrm{Re}(\delta), \mathrm{Re}(\alpha_1), \dots, \mathrm{Re}(\alpha_4)\} > 0;$$

$$(8)\quad \frac{1}{(2\pi i)^n} \int_{-i\infty}^{i\infty} \cdots \int_{-i\infty}^{i\infty} \prod_{\substack{1\leq j,k\leq n \\ j\neq k}} \frac{\Gamma(\delta + t_j - t_k)}{\Gamma(t_j - t_k)} \cdot \prod_{j=1}^{n} \left\{ \prod_{k=1}^{2} [(\Gamma(\alpha_k + t_j)\Gamma(\beta_k - t_j)]\, dt_j \right\}$$
$$= n! \prod_{j=1}^{n} \frac{\Gamma(j\delta) \prod_{k,l=1}^{2} \Gamma(\alpha_k + \beta_l + (j-1)\delta)}{\Gamma(\delta)\Gamma\left((n+j-2)\delta + \sum_{k=1}^{2} (\alpha_k + \beta_k)\right)},$$

where the contours of integration are the imaginary axis and

$$\min\{\mathrm{Re}(\delta), \mathrm{Re}(\alpha_1), \mathrm{Re}(\alpha_2), \mathrm{Re}(\beta_1), \mathrm{Re}(\beta_2)\} > 0;$$

(9)

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{\substack{1\le j,k\le n\\ j\ne k}}\frac{\Gamma(1+z_j+t_j-z_k-t_k)}{\Gamma(1+\delta+z_j+t_j-z_k-t_k)}$$

$$\cdot\prod_{j=1}^{n}\frac{e^{2\pi i m_j t_j}dt_j}{\prod_{k=1}^{2}\Gamma(1+\alpha_k+z_j+t_j)\Gamma(1+\beta_k-z_j-t_j)}$$

$$=\begin{cases}\displaystyle\prod_{j=1}^{n}\frac{\Gamma(1+\delta)\Gamma\left(1+(n+j-2)\delta+\sum_{k=1}^{2}(\alpha_\kappa+\beta_\kappa)\right)}{\Gamma(1+j\delta)\prod_{k,l=1}^{2}\Gamma(1+\alpha_k+\beta_l+(j-1)\delta)}, & \\ \qquad\qquad\text{if } m_1=\cdots=m_n=0 & \\ 0, \quad \text{otherwise} & \end{cases}$$

where

$$\min\left\{\operatorname{Re}\left((n-1)\delta+\sum_{k=1}^{2}(\alpha_k+\beta_k)\right),\ \operatorname{Re}\left(2(n-1)\delta+\sum_{k=1}^{2}(\alpha_k+\beta_k)\right)\right\}>-1;$$

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\prod_{1\le j<k\le n}\left\{\frac{[bq^{1+z_j+t_j-z_k}]_\infty[bq^{1-z_j-t_j+z_k+t_k}]_\infty}{[q^{1+z_j+t_j-z_k}]_\infty[q^{1-z_j-t_j+z_k+t_k}]_\infty}\cdot\frac{[bq^{1+z_j+t_j+z_k+t_k}]_\infty[bq^{1-z_j-t_j-z_k-t_k}]_\infty}{[q^{1+z_j+t_j+z_j+t_k}]_\infty[q^{1-z_j-t_j-z_k-t_k}]_\infty}\right\}$$

$$\cdot\prod_{j=1}^{n}\frac{\prod_{k=1}^{4}\{a_kq^{1+z_j+t_j}]_\infty[a_kq^{1-z_j-t_j}]_\infty\}\cdot e^{2\pi i m_j t_j}dt_j}{[q^{1+2z_j+2t_j}]_\infty[q^{1-2z_j-2t_j}]_\infty}$$

(10)

$$=\begin{cases}\displaystyle\prod_{j=1}^{n}\frac{[q]_\infty[qb^j]_\infty\prod_{1\le k<l\le 4}[qa_ka_lb^{j-1}]_\infty}{[qb]_\infty\left[qb^{n+j-2}\prod_{k=1}^{4}a_k\right]_\infty} & \\ \qquad\qquad\text{if } m_1=\cdots=m_n=0 & \\ 0, \quad \text{otherwise} & \end{cases}$$

where

$$\max\left\{\left|qb^{n-1}\prod_{k=1}^{4}a_k\right|, \left|qb^{2(n-1)}\prod_{k=1}^{4}a_k\right|\right\} < 1$$

and for simplicity we assume that $q \in \mathbf{R}$, $0 < q < 1$. The $n = 1$ case of (7) is due to de Branges [6] and Wilson [24], of (8) to Barnes [5], of (9) to Ramanujan [22] and (10) essentially to Askey [3].

*Remarks.* The integrals (9) and (10) are equivalent to multiple series summation theorems which generalize classical bilateral hypergeometric series summation theorems: Dougall's ${}_2H_2$ sum and Bailey's ${}_6\psi_6$ sum. A similar connection between some related integral evaluations and the corresponding multiple series identities is explained in [12]. As we plan to describe elsewhere, we are led to conjecture a family of multiple series summation identities which are equivalent to the Macdonald-Morris conjectures and contain the Macdonald identities [15] as special cases.

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