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THE CLASSIFICATION OF NONLINEAR SIMILARITIES OVER $\mathbf{Z}_{2^r}$

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The real representations ρ_1 and ρ_2 of a finite group G are *topologically similar* (written $\rho_1 \sim_t \rho_2$) if there is a homeomorphism $h: V(\rho_1) \to V(\rho_2)$, where $V(\rho_i)$ denotes the vector space of the representation ρ_i, such that $h(\rho_1(g)\cdot v) = \rho_2(g)\cdot h(v)$ for $v \in V(\rho_1)$ and $g \in G$ (i.e. the representation spaces are equivariantly homeomorphic).

De Rham [dR] conjectured that topological similarity implies the linear equivalence of the two representations, and proved some results in this direction. De Rham's Conjecture has been shown to be true in numerous special cases, e.g. G of exponent 2 or 4 (easy), of exponent p^r or $2p^r$ with p an odd prime [Sch], or more generally if the exponent of G is odd or two times an odd number [HP or MR]. (A complete determination of those groups for which de Rham's Conjecture is always true is given in [CS_4].)

The first counterexamples to de Rham's Conjecture were given by the first two authors, for $G = \mathbf{Z}_{4q}$ for any $q > 1$, between representations of dimension ≥ 9. The first three authors, together with the last, then give counterexamples over $\mathbf{Z}_{4q}$ in all dimensions ≥ 6 for all $q > 2$ [CSSW]. (We show here that $q > 2$ is necessary for counterexamples in dimension < 9.) Thus, by [CS_2],

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six is the minimal dimension for *nonlinear similarity* (i.e. topological similarity between linearly inequivalent representations).

Here, we restrict attention to cyclic 2-groups, $G = \mathbf{Z}_{2^r}$, representing the opposite extreme from the odd-order theorem of [HP or MR]. We give a complete set of necessary and sufficient conditions for topological similarity over such G. This gives the first complete classification of topological similarity over a group which admits nonlinear similarities. We also give the first construction of nonlinear similarities which systematically handles isotropy groups of arbitrarily high index in the representation spaces. And in fact, such isotropy groups are necessary for many of the examples we give below. (All previously known examples of nonlinear similarities were obtained from those of [CS_1 or CSSW] for cyclic groups or of [M] for quaternionic groups by either stabilization or taking induced representations. In all three cases, the basic constructions involve only isotropy of index ≤ 2 outside the free part.)

As an application of our main theorem we prove a conjecture of the first two authors [CS_5]:

Theorem 1. *Let* $G = \mathbf{Z}_{2^r}$ *act smoothly on a* (mod 2) *homology sphere. Then the tangent representations of* G *at any two fixed-points are topologically similar.*

As another application of our main theorem, we give some results on the group $R_{\mathrm{Top}}(G)$, which is the quotient of the real representation ring $RO(G)$ by the subgroup generated by all differences of topologically similar representations. Thus, $R_{\mathrm{Top}}(G)$ is the Grothendieck group of topological equivalence classes of linear representations under direct sum. In particular, its calculation is equivalent to the stable topological classification of representations. The rank of $R_{\mathrm{Top}}(G)$ was calculated in [CS_4] for any finite G; on the other hand, the torsion subgroup is always a 2-group, but rather difficult to compute. The first complete calculation of a group $R_{\mathrm{Top}}(G)$ with nontrivial torsion is given in [CSSW] for the groups $G = \mathbf{Z}_{4q}$ with q odd. In this case, the relations all arose from the six-dimensional similarities constructed there, and the resultant 2-torsion elements all have order 2. Here, we analyse the situation for $G = \mathbf{Z}_{2^r}$ and compute the (large) exponent of the 2-torsion there.

Write $\mathbf{t}^k$ for the 2-dimensional representation defined by letting the generator of the group rotate $\mathbf{R}^2$ through $2\pi k/2^r$ radians.

Theorem 2. *For* $r > 3$, *the torsion in* $R_{\mathrm{Top}}(\mathbf{Z}_{2^r})$ *has exponent* 2^{r-2}. *This exponent is realized as the order of* $\mathbf{t} - \mathbf{t}^5$.

However, for $\mathbf{Z}_8$ the exponent of the group, and the order of $\mathbf{t}-\mathbf{t}^5$ are both 4. In Theorem 2, the lower bounds for the exponents come out of homotopy theory. The (equal) upper bound comes out of the main theorem below.

1. The main theorem

In describing when representations ρ_1 and ρ_2 are topologically similar, it is convenient to decompose them as $\rho_i = \theta_i + \eta_i$, where θ_i is *free* (i.e. G acts freely on its unit sphere) and η_i is *singular* (the action is nowhere free). Moreover, elementary induction arguments reduce the general classification (for G cyclic) to the case where $\eta_1 = \eta_2$, which will now be considered.

The main theorem will give necessary and sufficient conditions for topological similarity of representations of $\mathbf{Z}_{2^r}$. We state each of them in a topological version and an equivalent computational version, denoted by a prime on the label of the former. The topological versions are in terms of the equality of certain invariants of the lens spaces, L_{θ_i}, obtained as the orbit space of the free action of G on the unit sphere in the representation space $V(\theta_i)$. These invariants concern homotopy type, tangential data recorded by normal invariants, Reidemeister or Whitehead torsion, and a generalization of the Atiyah-Bott-Singer ρ-invariant that comes out of the G-signature theorem. The translation of these criteria into the given computational formulae uses special facts about the group ring $\mathbf{Z}[\mathbf{Z}_{2^r}]$ that come from the number theory of the cyclotomic rings of integers $\mathbf{Z}(\zeta_{2^s})$, $s \le r$, where ζ_{2^s} is a primitive 2^sth root of unity.

Theorem 3. *Let θ_1 and θ_2 be free representations of $G = \mathbf{Z}_{2^r}$ and let η be arbitrary. Then $\theta_1 + \eta \sim_t \theta_2 + \eta$ if and only if conditions A-D (or equivalently* A$'$ – D$'$*) below hold.*

Using equivariant topological engulfing [SW$_1$] it is easily seen [SW$_2$] that $\rho_1 \sim_t \rho_2$ if and only if the unit spheres $S(\rho_i)$ are $G-h$-cobordant in the category of locally linear actions. This, together with the join structure on a linear sphere and the ability to stabilize a topological similarity by direct sum with identity maps shows the necessity of the following:

(A) *There is a homotopy equivalence of lens spaces (respecting the canonical identifications on π_1), $f : L_{\theta_1} \to L_{\theta_2}$ whose stable normal invariant (in the topological category) vanishes (i.e. for any free representation θ, the map $L_{\theta_1+\theta} \to L_{\theta_2+\theta}$ induced by f is normally cobordant to the identity map of $L_{\theta_2+\theta}$).*

This condition may be verified via an algorithm extending the method used in [CS_1] to compute normal invariants in terms of the Pontrjagin classes of lens spaces covered by the given ones. The algorithm is given as follows. Let $\theta_1 = \mathbf{t}^{r_1} + \cdots + \mathbf{t}^{r_j}$ and let $\theta_2 = \mathbf{t}^{s_1} + \cdots + \mathbf{t}^{s_j}$. Since the θ_i are free, and since $\mathbf{t}^i = \mathbf{t}^{-i}$ as real representations, we may assume that $r_i \equiv s_i \equiv 1 \bmod 4$ for all i. Let $\sigma_i(x_i, \dots, x_j)$ be the ith elementary symmetric function of $x_1, \dots, x_j$. Then Condition A is equivalent to the following:

(A′) For $1 \le i \le j$, we have

$$\prod r_k \equiv \prod s_k \mod 2^r,$$

and

$$\begin{aligned}\sigma_i(r_1^2, \dots, r_j^2) &- \sigma_i(s_1^2, \dots, s_j^2)\\ &\equiv 2\left(\prod r_k - \prod s_k\right)\binom{j-1}{i-1} \mod 2^{r+3}.\end{aligned}$$

The necessity of the following may be shown either by the locally linear G-surgery theory of Madsen, Rothenberg and the third author [MRS] and the computations of equivariant topological Whitehead groups of the third author [S], or by a careful analysis of the discriminant map using Condition C below. Both approaches make use of the algebra of [CS_3].

(B) *The Whitehead torsion, $\tau(f)$, of the above homotopy equivalence of lens spaces transfers to a square in the Whitehead group of the index two subgroup of G.*

Computationally, we get

(B′) *When we restrict the action to $\mathbf{Z}_{2^{r-1}} \subset \mathbf{Z}_{2^r}$, we obtain the following*

$$\theta_i|_{\mathbf{Z}_{2^{r-1}}} = 2\phi_i + \theta,$$

with $\phi_1 = \mathbf{t}^{r_1} + \cdots + \mathbf{t}^{r_{2k}}$ and $\phi_2 = \mathbf{t}^{s_1} + \cdots + \mathbf{t}^{s_{2k}}$, and

$$\prod r_{2i} \equiv \prod s_{2i} \mod 2^{r-1}.$$

This puts severe restrictions on $\tau(f)$, as seen via the calculations of [CS_3]. In fact, the permissible torsions are, mod squares, precisely the torsion elements which are considered in [CSSW]. In particular, the algebra needed for the calculation of condition D below is contained in [CSSW and CS_3].

Using work of [Roth W, Rose W] on Lipschitz analysis on topological manifolds and equivariant higher signature theorems, or the later purely topological generalization of [CSW] on higher signature theorems we obtain

(C) *The generalized ρ-invariants of the unit spheres $S(\rho_i)$ agree.*

Here, the generalized ρ-invariant is a function from $G-\{1\}$ to the complex numbers defined as follows. First, identify the trivial and nontrivial one-dimensional real representations with $\mathbf{t}^0$ and $\mathbf{t}^{2^{r-1}}$, respectively, for the purposes of the following formula. Then if $\alpha = \sum n_i \mathbf{t}^i$ is a representation, and if T generates G, then

$$\rho(S(\alpha))(T^k) = \prod_i \left(\frac{\zeta_{2^r}^{ik} + 1}{\zeta_{2^r}^{ik} - 1} \right)^{n_i}.$$

Here, the product is taken over all i such that $\zeta_{2^r}^{ik} \neq 1$.

This formula is precisely the usual Atiyah-Bott-Singer ρ-invariant of L_α if α is free. Thus, for $\rho_i = \theta_i + \eta$ as above, Condition C says that multiplication by $\rho(S(\eta))$ kills the difference between the usual ρ-invariants of L_{θ_1} and L_{θ_2}. Since the restriction of the usual (or generalized) ρ-invariant to a subgroup is the ρ-invariant for the restricted action, and since the restrictions of the θ_i to $\mathbf{Z}_4$ are linearly equivalent (both are free), we need only consider the value of the ρ-invariant on elements outside $\mathbf{Z}_4$. An inspection of the formula above shows that Condition C is equivalent to:

(C$'$) *For $3 \leq s \leq r$, if $\rho(L_{\theta_1}) - \rho(L_{\theta_2})$ doesn't vanish on generators of $\mathbf{Z}_{2^s} \subset \mathbf{Z}_{2^r}$, then $\mathbf{Z}_{2^{s-1}}$ appears as an isotropy group in η.*

The generalized ρ-invariant vanishes entirely in the presence of sufficiently many isotropy subgroups in η. For whatever choice of θ_1 and θ_2, it will vanish if each H with $\mathbf{Z}_4 \subset H \subset \mathbf{Z}_{2^{r-1}}$ is present. Thus, Condition C is automatically satisfied stably, and does not affect the calculation of R_{Top}.

In the presence of conditions A–C above, we construct an equivariant normal cobordism between $S(\theta_1+\eta')$ and $S(\theta_2+\eta')$, for a suitable $\eta' \subset \eta$. There is a final surgery obstruction, the vanishing of which is necessary by [MRS]:

(D) *If η contains no trivial representations, then $\tau(f)$ is a square in the Whitehead group of G. Otherwise, let $R_2(G)$ be the subquotient of the complex representation ring $R(G)$ given by*

$$R_2(G) = \frac{\{2(\rho+\overline{\rho})\}}{\{4(\rho+\overline{\rho})\}},$$

where p varies over all the complex representations of G, and let

$$\delta : \widehat{H}^0(\mathbf{Z}_2 ; \mathrm{Wh}(G^+)) \to R_2(G)$$

be the map on Tate cohomology induced by the multisignature and the "inverse" of the discriminant map [W]. *Then $\delta(\tau(f))$ is the* *image in* R_2 *of the multisignature of an element of* $L_0^p(H, +)$, *where* H *is the largest proper subgroup of* G *which occurs as an isotropy group in* $S(\eta)$.

When η contains the trivial representation, the computational version is only slightly simpler. We may eliminate reference to L^p-groups in favor of a purely representation-theoretic statement about the $\delta(\tau(f))$.

(D′) *If* η *contains no trivial representations, then* $\theta_i = 2\phi_i + \theta$, *with* $\phi_i = \mathbf{t}^{r_1} + \cdots + \mathbf{t}^{r_k}$, $\phi_2 = \mathbf{t}^{s_1} + \cdots + \mathbf{t}^{s_k}$, *and* $\prod r_i \equiv \prod s_i \bmod 2^r$.

If η *does contain the trivial representation, let* H *be the largest proper subgroup of* G *which appears as an isotropy subgroup in* η, *and let* $\imath\colon R_2(H) \to R_2(G)$ *be induced by the induction map of representation rings. Then* $\delta(\tau(f)) = \imath(x)$ *for* $x = \sum n_i \mathbf{t}^i$, *with* $\sum_{i \equiv 1 \bmod 4} n_i \equiv 0 \bmod 4$.

If the H in condition D is the index two subgroup, then the condition on δ is equivalent (via condition B and the algebra of [CS$_3$] and [CSW]) to the statement that the class represented by $\tau(f)$ in $H^1(\mathbf{Z}_2; \mathrm{Wh}(G^-)) \cong L_1^h(G, -)$ [W] vanishes in $L_1^p(G, -)$. The determination of which torsions vanish there is given explicitly in [CSSW].

Conditions A and B depend only on the arithmetic of the eigenvalues of the θ_i, while conditions C and D depend only on this together with the isotropy subgroups which occur in η. This yields the following stability theorem.

Theorem 4. *Let* ρ_1, ρ_2 *and* α *be arbitrary representations of* $G = \mathbf{Z}_{2^r}$. *Suppose that any* H *with* $\mathbf{Z}_4 \subset H \subset G$ *which appears as an isotropy subgroup in* $S(\alpha)$ *also appears in the* $S(\rho_i)$. *Then* $\rho_1 + \alpha \sim_t \rho_2$ *if and only if* $\rho_1 \sim_t \rho_2$. *Thus, if each* H *with* $\mathbf{Z}_4 \subset H \subset G$ *appears as an isotropy group in the* $S(\rho_i)$, *then* $\rho_1 \sim_t \rho_2$ *if and only if* $\rho_1 = \rho_2$ *in* $R_{\mathrm{Top}}(G)$.

The full range of isotropy groups specified in Theorem 4 is required to obtain the similarity $2^{r-2}\mathbf{t}+\eta \sim_t 2^{r-2}\mathbf{t}^5+\eta$, $r > 3$, which is given in Theorem 2. However, often fewer isotropy groups are necessary for stability, depending on the generalized ρ-invariants of the representations and on the equivariant torsion of the G-homotopy equivalence between their unit spheres.

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