made the book more interesting and would exhibit the close connection that topological dynamics has with other branches of Mathematics.

Another subject which is almost entirely missing is the strong tie, formal as well as actual, between ergodic theory and the theory of minimal sets. However perhaps this is a subject for another book.

An obvious disadvantage of the book is the regretable lack of index.

BIBLIOGRAPHY

[G-H] W. H. Gottschalk and G. H. Hedlund, *Topological dynamics*, Amer. Math. Soc. Colloq. Publ., vol. 36, Amer. Math. Soc., Providence, R.I., 1955.

[E,1] R. Ellis, *A semigroup associated with a transformation group*, Trans. Amer. Math. Soc. **94** (1960), 272-281.

[E,2] ——, *The Veechs Structure Theorem*, Trans. Amer. Math. Soc. **186** (1973), 203-218.

[E,3] ——, *Lectures on topological dynamics*, Benjamin, New York, 1969.

[A-G-H] L. Auslander, L. Green and F. Hahn, *Flows on homogeneous spaces*, Ann. of Math. Studies no. 53, Princeton Univ. Press, Princeton, N.J., 1963.

[F] H. Furstenberg, *The structure of distal flows*, Amer. J. Math. **85** (1963), 477-515.

[D] F. M. Dekking, *Combinatorial and statistical properties of sequences generated by substitutions*, Ph.D. Thesis, Katholicke Universiteit, Nijmegen, 1980.

[V,1] W. A. Veech, *Point distal flows*, Amer. J. Math. **92** (1970), 205-242.

[V,2] ——, *Topological dynamics*, Bull. Amer. Math. Soc. **83** (1977), 775-830.

[E-G-S] R. Ellis, S. Glasner and L. Shapiro, *PI-flows*, Adv. in Math. **17** (1975), 213-260.

[G] S. Glasner, *Proximal flows*, Lecture Notes in Math., vol. 517, Springer-Verlag, Berlin and New York, 1976.

[B] I. V. Bronstein, *Extensions of minimal transformation groups*, Sitjthoff and Noordhoff, 1979. (Russian edition, 1975)

[M] J. C. Martin, *Substitution minimal flows*, Amer. J. Math. **93** (1971), 503-526.

ELI GLASNER
TEL-AVIV UNIVERSITY

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 21, Number 2, October 1989

0273-0979/89 $1.00 + $.25 per page

Obstacle problems in mathematical physics, by J.-F. Rodrigues. North-Holland Mathematics Studies, vol. 134, North-Holland, Amsterdam, 1987, xvi + 352 pp., $71.00. ISBN 0-444-70187-7

In the beginning, the study of variational problems was very simple. The typical problem was to minimize a convex functional over a simple subset of a standard Banach space (or Hilbert space). For example, if Ω is a planar domain with smooth boundary Σ and g is a function in $W^{1,2}$, the space of functions with square integrable derivatives, we can look for functions minimizing the functional

$$I(v) = \int_\Omega |Dv|^2\,dx$$

over the subset $\mathbf{K}$ of $W^{1,2}$ consisting of all functions which agree with g on Σ. (Because of the form of the functional, $W^{1,2}$ is a natural space to work in.) It is not hard to show that I does indeed have a unique minimizer u in **K**, which is a weak solution of the boundary value problem

$$\Delta u = 0 \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega.$$

From standard P.D.E. theory, u is analytic inside Ω and if g is continuous, etc., then so is u.

More general functions I can be considered, for example,

$$I(v) = \int_\Omega F(x, v, Dv)\, dx$$

for a convex C^2 function F with appropriate growth properties and **K** as before in a suitable underlying Banach space; the additional technicalities are not important for the present discussion.

Other boundary conditions can be introduced as well. The functional

$$J(v) = \int_\Omega (|Dv|^2 + \tfrac{1}{2}v^2)\, dx + \beta \int_\Sigma v\, ds$$

over $W^{1,2}$ has a unique minimum u satisfying

$$\Delta u = u \quad \text{in } \Omega, \qquad D_n u = \beta \quad \text{on } \Sigma,$$

where D_n denotes the inner normal derivative.

The conversion of minimization problem to boundary value problem becomes more complicated if the subset **K** is modified slightly. A case in point is the "simple" question: What is the height of liquid in a tube of constant cross-section Ω with a curved bottom? Assuming that the height u does not oscillate much, it is found by minimizing the functional J over the subset **K** of $W^{1,2}$ defined by the inequality $u \geq \psi$ in Ω for a function ψ known as an obstacle, which in this case describes the shape of the bottom of the tube. (A more exact functional can be written down, but the important new element is the introduction of the obstacle.) Now it is easy to show that u satisfies the conditions

$$(1)\quad \Delta u = u \quad \text{in } \Omega^* = \Omega \cap \{u > \psi\}, \qquad D_n u = \beta \quad \text{on } \Sigma^* = \Sigma \cap \{u > \psi\};$$

however, the structure of the set $\{u > \psi\}$ is not so clear. (If $D_n\psi \geq \beta$ on Σ, then $D_n u = \beta$ on Σ.) The study of this set is basic in obstacle problems. For future reference we use the standard notation: $I = \{x \in \Omega | u(x) = \psi(x)\}$ is the coincidence set, $\Lambda = \Omega \backslash I$ is the noncoincidence set, and $\Phi = \partial I \cap \Omega$ is the free boundary. When the free boundary, the solution, and the obstacle are smooth, then

$$(2) \qquad u = \psi \quad \text{and} \quad Du = D\psi \quad \text{on } \Phi.$$

If Φ, Ω^*, and Σ^* are known, then (1), (2) form an overdetermined boundary value problem for u, but since Φ is unknown and $u \geq \psi$, we get a problem which turns out to be solvable.

The study of obstacle problems has several basic pieces:

(A) existence and uniqueness of solutions,

(B) regularity of u, depending on the regularity of ψ,

(C) regularity of Φ.

As it turns out, verification of (A) is usually easy by abstract methods because the functional is usually convex and lower semicontinuous on a convex subset **K** of some Banach space. To see how to get a handle on (B) we consider the problem of minimizing J over $W^{1,2}$ assuming that $D_n\psi \geq \beta$ on Σ. The minimum u of this problem is also the solution of

$$\min\{-\Delta u + u, u - \psi\} = 0 \quad \text{in } \Omega, \qquad D_n u = \beta \quad \text{on } \Sigma,$$

and this boundary value problem can be studied via the method of penalization: for a suitable sequence of functions μ_j which vanish for positive arguments and tend to $+\infty$ for negative arguments, we introduce the family of classical boundary value problems

$$-\Delta u_j + u_j + \mu_j(u_j - \psi) = 0 \quad \text{in } \Omega, \qquad D_n u_j = \beta \quad \text{on } \Sigma.$$

For a suitable choice of μ_j's, the u_j's converge uniformly to u, and uniform estimates on the u_j's and their derivatives lead to corresponding estimates on u, which imply regularity of the solution of the obstacle problem. For example if ψ has bounded second derivatives and if Σ is smooth enough then u has bounded second derivatives. Simple examples show that additional smoothness of the data of the obstacle problem do not imply any more regularity of u. (There are also other approaches to this regularity question, but this limiting smoothness is a property of the obstacle problem and not of the penalization method.)

Unfortunately, the methods for studying regularity of solutions of obstacle problems do not give any information on the free boundary. Results in this direction are known, though. As important early theorem on regularity of the free boundary is due to Lewy and Stampacchia: if the domain is two-dimensional and strictly convex with analytic boundary and if the obstacle is analytic and strictly convex with $\psi < 0$ on Σ, $\psi > 0$ somewhere in Ω, then the free boundary Φ for the minimizer of I over the subset of $W^{1,2}$ with zero boundary data is analytic. More precise relations between the regularity of the data and the regularity of the free boundary are known today, including some limiting smoothness results.

Other types of obstacle problems can also be considered. For example, one can assume that the obstacle ψ is only defined on some lower dimensional set Ψ (which may be on the boundary) and that the inequality $u \geq \psi$ only holds on Ψ in the definition of **K**. Alternatively one can impose constraints on the gradient of u. These problems have received much attention lately.

Several books are available on the study of obstacle problems and the closely related topic of variational inequalities. Two classics are **[1]** by Duvaut and Lions, and **[2]** by Kinderlehrer and Stampacchia. The first spends a great deal of time on the physical processes which motivate the mathematical study while the second deals more with the mathematical foundations of the theory. The biggest drawback to these books today is that they are fairly old (they were written in 1974 and 1980, respectively), and a lot of progress has been made since they were written. More recent books are by Friedman **[3]**, Troianiello **[4]**, and Rodrigues **[5]**. Friedman's book is certainly the most encyclopedic of the three, with something for everybody although with an emphasis on problems involving jets and cavities. Troianiello, on the other hand, has a more focused point of view in a very different direction. He develops the classical theory of elliptic differential equations and regularity of their solutions in the framework of variational inequalities. A lot of high powered technical machinery is used, and this book is the most theoretical of those listed here. Rodrigues points out in the preface that he is trying for a modern version of Duvaut and Lions; he is more concerned with physical motivation than with development of theory. Theorems on elliptic differential equations are quoted as needed. The book succeeds at emphasizing the physical point of view without disregarding mathematical rigor. Some of the models are described rather sketchily, though.

In addition to the applications, Rodrigues spends more time on the study of stability of free boundaries then the other authors listed.

REFERENCES

1. G. Duvaut and J.-L. Lions, *Inequalities in mechanics and physics*, Springer-Verlag, Berlin and New York, 1976.

2. D. Kinderlehrer and G. Stampacchia, *An introduction to variational inequalities and their applications*, Academic Press, New York, 1980.

3. A. Friedman, *Variational principles and free-boundary problems*, Wiley, 1982 (reprinted Krieger, Malabar, FL, 1988).

4. G. M. Troianiello, *Elliptic differential equations and obstacle problems*, Plenum, New York, 1987.

5. J.-F. Rodrigues, *Obstacle problems im mathematical physics*, North-Holland, Amsterdam, 1987.

GARY M. LIEBERMAN
IOWA STATE UNIVERSITY

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 21, Number 2, October 1989

0273-0979/89 $1.00 + $.25 per page

Nest algebras, by Kenneth R. Davidson. Pitman Research Notes in Mathematical Sciences, vol. 191, Longman Scientific and Technical, Harlow, 1988, 411 pp., $74.95. ISBN 0-582-01993-1

Nest algebras were introduced by J. R. Ringrose in 1965 shortly after studies by R. V. Kadison and I. M. Singer of a related class of operator algebras, and in the last twenty-five years the subject has matured to the extent that they form a moderately well-understood class in the category of non-self-adjoint operator algebras. Most notably Ringrose's similarity problem has been resolved, finally, in a curious way requiring a deep and unexpected excursion into the analysis of quasitriangular algebras. Moreover there are now multiple points of contact with other areas of operator theory and many intriguing basic problems remain unsettled.

"Non-self-adjoint." This is, unfortunately, a rather inelegant adjective, a kind of apologetic antidefinition, but it may be seen less in the coming