be called the feasibility problem of polyhedra (i.e., deciding if a point satisfies a set of linear inequalities). This chapter also includes the longest increasing subsequence problem and other miscellaneous topics.

The book contains many figures, numerical examples, exercises and answers. It is a very nice textbook as well as a good reference book. Although the book is published in 1988, it contains recent references some from 1987 and 1988.

The writing is formal and rigorous as many books in mathematics are. It is a book which should be on the bookshelf of all people interested in combinatorics.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 21, Number 2, October 1989

0273-0979/89 $1.00 + $.25 per page

Minimal flows and their extensions, by Joseph Auslander. North-Holland Mathematical Studies, vol. 153, North-Holland, Amsterdam, New York, Oxford and Tokyo, 1988, xi + 265 pp., $86.75. ISBN 0-444-70453-1

Abstract topological dynamics deals with actions of groups on compact spaces. Such an action is called a *flow*. At present the emphasis of research is on *minimal flows*, i.e., actions for which no proper closed invariant subsets exist.

The abstract axiomatic approach to the subject began with Gottschalk and Hedlund's pioneering book of 1955 **[G-H]**, where the search for a suitable framework for the (then) new subject is apparent.

The introduction of the enveloping semigroup of a flow by R. Ellis (1960) **[E,1]** and the proof that it actually is a group when the flow is distal, were the next important achievements.

The action of the group T on the compact space X is *distal* if (assuming the existence of a metric on X) $\operatorname{Inf}\{d(tx,ty): t \in T\} > 0$ for every pair of distinct points $x, y \in X$. This notion which was first introduced by D. Hilbert, was very central to the development of the subject. A stronger condition on a flow is that the group T acts *isometrically* on X. For a while it was not clear whether these two conditions are not equivalent. Then in 1963 H. Furstenberg realized that Kakutani-Anzai's constructions of skew products on the torus yield examples of minimal distal but not isometric flows. A simple example of this type is the transformation $(x,y) \mapsto (x + \alpha, x+y) x, y \in \mathbb{R}/\mathbb{Z}$, α irrational. (At the same time and independently another example was found in **[A-G-H]**.) H. Furstenberg then took the next major step in the theory of topological dynamics, a step which gave the subject its present character. Recognizing the above example to be an *isometric extension* of an isometric flow, (the rotation by α on the circle) and observing that such extensions preserve distallity, he defined the class of Quasi-Isometric flows to be the class of minimal flows which can be represented as towers of isometric extensions starting with an isometric flow. Using R. Ellis' enveloping group he then showed that *every* distal flow is Quasi-Isometric. **[F]**

A pair of points $x, y \in X$ is called *proximal* if $\operatorname{Inf}\{d(tx, ty): t \in T\} = 0$. A *homomorphism* (or an *extension*) of minimal flows $X \xrightarrow{\pi} Y$ is a continuous map (necessarily onto) which is equivariant: $\pi(tx) = t\pi(x)$ $(\forall t \in T, x \in X)$. An extension π is *proximal* if whenever $\pi(x) = \pi(y)$ then x and y are proximal. In order to better understand the next stages along which the general theory developed, let us look closely at the following example.

The Morse Thue sequence $\omega(n)$:

$$0110100110010110\cdots$$

is a curious object, it can be described by each of the following algorithms

1. $\omega(0) = 0$, $\omega(2n) = \omega(n)$, $\omega(2n+1) = 1 - \omega(n)$ $(n \in \mathbb{N})$

2. Let a_n be the number of ones in the binary expansion of $n \in \mathbb{N}$. Then $\omega(n) = 0$ or 1 according to whether a_n is even or odd.

3. Let $W_n = \omega(0)\omega(1)\cdots\omega(2^n - 1)$ and let $\overline{A}$ be the word obtained from A by interchanging 0's and 1's. Let $W_0 = 0$ and $W_{n+1} = W_n\overline{W}_n$.

4. Let $S = \{0, 1\}$; define $\theta: S \to S^2$ by $\theta(0) = 01$, $\theta(1) = 10$. Put $\theta(s_1 \cdots s_k) = \theta(s_1)\cdots\theta(s_k) \in S^k$ and define $\theta^1 = \theta$, $\theta^{n+1} = \theta \circ \theta^n$ for $n \geq 1$. Then $\theta^n(0) = \omega(0)\cdots\omega(2^n - 1)$.

5. Let $M(z)$ be the unique solution to the functional equation

$$M(z^2) = M(z)(1-z)^{-1} - z(1-z)^{-1}(1-z^2)^{-1} \qquad |z| < 1,\ M(0) = 0.$$

Then M is analytic and $M(z) = \sum_{n \geq 0} \omega(n) z^n$. (See **[D]** for these and a few more.) The sequence became famous when, in one attempt, it was used to disprove Burnside's conjecture that a bounded group is finite.

Considering ω as an element of $\Omega = \{0, 1\}^{\mathbb{Z}}$, where $\omega(-n) = \omega(n-1)$, let $X \subset \Omega$ be its orbit closure under the shift $\sigma(\sigma\xi(n) = \xi(n+1))$. Then (X, σ) is a minimal flow called the Morse minimal set. The homeomorphism $\varphi: \xi \to \overline{\xi}$ where $\overline{\xi}(n) = \overline{\xi(n)}$ (and $\overline{0} = 1, \overline{1} = 0$) preserves X and commutes with σ. The quotient space Y, of X modulo the group $\{\varphi, \varphi^2 = \mathrm{id}\}$ is a factor of (X, σ) in the sense that the natural projection $X \xrightarrow{\pi_1} Y$ satisfies $\pi_1\sigma = \sigma\pi_1$. For every $\xi \in X$ there exists a sequence k_i such that $\sigma^{k_i}\omega \to \xi$ and we can associate with ξ the dyadic sequence $\{a_n\}$, $0 \leq a_n \leq 2^n - 1$, according to the rule $a_n = \lim\{k_i (\operatorname{mod} 2^n)\}$. It is easy to check that this limit exists and is independent of the particular choice of the sequence $\{k_i\}$. Clearly also the dyadic sequences corresponding to ξ and $\overline{\xi}$ coincide, so that we can consider the map $Y \xrightarrow{\pi_2} G$, where G is the compact group of sequences $\{\{a_n\}: 0 \leq a_n \leq 2^n - 1, a_{n+1} \equiv a_n \mod 2^n\}$. Moreover $\pi_2\sigma y = (\pi_2 y) + 1$ where $1 = (1, 0, 0, \ldots,) \in G$. In fact it is not hard to describe π_2 explicitly. If $\eta \in \Omega$ is defined by $\eta(n) = \omega(n)$ for $n \geq 0$ and $\eta(n) = \overline{\omega(n)}$ for $n < 0$ then $\eta \in X$ and denoting $y_1 = \pi_1(\omega)$, $y_2 = \pi_1(\eta)$ we have for all $n \in \mathbb{Z}$, $\pi_2^{-1}(n \cdot 1) = \{\sigma^n y_1, \sigma^n y_2\}$ while $\pi_2^{-1}(g)$ is a singleton for every $g \in G \backslash \{n \cdot 1: n \in \mathbb{Z}\}$. The map π_2 is therefore "almost one to one" hence proximal.

This description of X as a "tower" $X \xrightarrow{\pi_1} Y \xrightarrow{\pi_2} G \to$ point, consisting of alternating isometric and proximal homomorphisms, is very satisfying and in a sense gives a complete description of (X, σ). For example we can immediately deduce from it that X carries a unique σ-invariant measure, that it has zero topological entropy, that most of its points are distal points and many other dynamical properties which are of interest in both topological dynamics and ergodic theory.

I have described this example in great detail because it serves as a simple model for a tower given by structure theorems for minimal sets. These structure theorems are the main theme of the book under review.

The Furstenberg structure theorem which characterizes minimal distal flows as towers consisting of isometric extensions served as a prototype for the Veech-Ellis' theorem for point-distal flows [**V**,1 **E**,2], and then for the work of Ellis-Glasner-Shapiro on PI-flows [**E-G-S**]. The latter was finally improved in [**V**,2] to give a general structure theorem for minimal flows representing them, up to a proximal extension, as weakly mixing extensions of PI-flows (those flows possessing a tower of proximal and isometric extensions.) The last chapter of the book under review is devoted to the presentation of a streamlined proof of this theorem.

The book can be divided into two parts. The first (Chapters 1–7), introduces the reader to the basic notions, and using methods of I. U. Bronstein, [**B**], culminates in an elementary proof of Furstenberg's theorem. A nice feature of this part is J. P. Troallic's proof of R. Ellis joint continuity theorem.

The second part of the book (Chapters 8–14) is slightly more advanced and uses the algebraic theory of minimal flows developed mainly by R. Ellis [**E**,3]. Alongside the development of the structure theorems which follows [**G**], one can find in this part an ingenious treatment of the equicontinuous structure relation due to the late D. McMahon, a chapter on invariant measures and a chapter on embeddings of flows in the Bebutov shift. This chapter is taken from an undeservedly forgotten Ph.D. thesis of A. Jaworski.

The author does not mention to whom this book is addressed. The leisurely pace, clarity of presentation and the self consistency which is carefully preserved throughout makes it ideal for graduate students seeking an introduction to this branch of topological dynamics.

My main criticism of the book is its lack of examples. A detailed analysis of Toeplitz and Sturmian sequences would have been helpful in understanding almost automorphic flows and their reach measure theoretical behaviour. Distal skew products on the n-torus are intimately connected with number theoretical results.

Substitution minimal sets (one of which is Morse's example) and J. C. Martin's work on substitutions of constant length, [**M**], can serve as a beautiful illustration of the Veech-Ellis Theorem on point distal flows. Chacon's flow and the various types of dynamical behaviour of horocycle flows are of great value in understanding weak mixing and disjointness properties. All these are just a few well-known examples which would have made the book more interesting and would exhibit the close connection that topological dynamics has with other branches of Mathematics.

Another subject which is almost entirely missing is the strong tie, formal as well as actual, between ergodic theory and the theory of minimal sets. However perhaps this is a subject for another book.

An obvious disadvantage of the book is the regretable lack of index.

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BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 21, Number 2, October 1989

0273-0979/89 $1.00 + $.25 per page

Obstacle problems in mathematical physics, by J.-F. Rodrigues. North-Holland Mathematics Studies, vol. 134, North-Holland, Amsterdam, 1987, xvi + 352 pp., $71.00. ISBN 0-444-70187-7

In the beginning, the study of variational problems was very simple. The typical problem was to minimize a convex functional over a simple subset of a standard Banach space (or Hilbert space). For example, if Ω is a planar domain with smooth boundary Σ and g is a function in $W^{1,2}$, the space of functions with square integrable derivatives, we can look for functions minimizing the functional

$$I(v) = \int_\Omega |Dv|^2\, dx$$

over the subset $\mathbf{K}$ of $W^{1,2}$ consisting of all functions which agree with g on Σ. (Because of the form of the functional, $W^{1,2}$ is a natural space to work