BOOK REVIEWS

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Foundations of infinitesimal stochastic analysis, by K. D. Stroyan and J. M. Bayod. Studies in Logic and the Foundations of Mathematics, vol. 119, North-Holland, Amsterdam, 1986, xii + 478 pp., $47.50. ISBN 0-444-87927-7

Stochastic analysis is currently one of the most active areas of application of nonstandard analysis. This is attested by the copious literature surveyed by Cutland [C] and Anderson [A2]. Most current work treats probability theory via the Loeb construction [L], which obtains standard probability spaces and stochastic processes from nonstandard ones by a rounding off operation. The probability spaces obtained are called Loeb spaces. The book under review is a general account of the measure theory of Loeb spaces and an introduction to the treatment of stochastic processes via Loeb spaces.

Nonstandard analysis is a modern approach to using infinitesimals in analysis, or in mathematics in general, to express limits and notions deriving from limits. For example, in the infinitesimal calculus on the reals,

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=4$$

can be expressed in the intuitive way, as

$$\text{for all } x\approx 2,\quad \frac{x^2-4}{x-2}\approx 4.$$

The symbol $\approx$ means "infinitely close," i.e. differing by an infinitesimal. Nonstandard analysis was originated by Abraham Robinson and modeled on Leibnitz's theory of infinitesimals. The advantages of the modern theory over that of Leibnitz are that it has precise, rigorously justified rules for manipulating infinitesimals, and that the notion of "infinitely close" can be used to represent the concept of limit, not only in connection with the reals, but in any topological context, and even in some contexts where the notion of limit is not exactly topological. For instance, a Loeb space can be thought of as a generalized limit of probability spaces, and there is a similar notion for Banach spaces, the nonstandard hull [HM].

To make use of infinitesimals, imagine a universe of mathematical discourse (such as some form of set theory) which contains, besides the usual objects in mathematics, called "standard," additional "nonstandard" elements, which one may think of as ideal objects. All the theorems of ordinary, standard mathematics apply in this enriched universe, provided they are interpreted in the sense appropriate to it. This property is called the Transfer Principle, or sometimes Leibnitz's Principle, because Leibnitz said that infinitesimals and infinite numbers should have the same properties as reals of more ordinary size. The theory of the enriched, nonstandard, universe is richer than that of the standard universe, because it has the distinction between standard and nonstandard objects. For instance, an infinitesimal real number is a nonstandard real which is smaller in absolute value than any nonzero standard real. It is desirable that notions such as being infinitesimal be nonvacuous in all interesting cases. To this end the nonstandard universe is constructed so as to satisfy a Saturation Principle, which implies this. The construction is carried out using tools from mathematical logic, the theory of mathematical languages and theories, in particular from its department for manipulating mathematical universes, model theory.

The aim of most work in nonstandard analysis has been to use it to obtain results expressed in the language of standard mathematics. This can be done because the nonstandard universe is connected to the standard one in two ways. The first is the Transfer Principle already mentioned. The second is that a nonstandard object can be rounded off to a standard object to which it is infinitely close, in some appropriate sense. This rounding off is called forming the *standard part*. The simplest kind of standard part is the one on the reals, which maps a nonstandard real to an infinitely close standard real, which is unique, if it exists. The Loeb construction is a standard part operation which rounds off a nonstandard probability space and any random processes on it to standard ones. The nonstandard hull construction is an analogous standard part construction for Banach spaces. In either case, the saturation of the nonstandard space ensures that the resulting standard probability space or Banach space is very rich. In the Loeb space case, this implies, in particular, that any distribution which can be approximately realized by a random process on the Loeb space can in fact be realized there. A Loeb space is a kind of universal probability space. Nonstandard hulls have similar universality properties.

The Transfer Principle implies that a standard part operation always corresponds to some kind of limit, understood in a broad sense. The ordinary standard part on the reals corresponds to taking a limit point of bounded sequence of reals. Applying the Loeb construction to a nonstandard probability space with a stochastic process corresponds to constructing a standard probability space with a process whose distribution law is a weak limit point of a certain tight family of distributions. The claim is that the connection between a limit object and its approximation is more transparent if it is represented by a standard part operation. It is also technically simpler because it is unnecessary to take subsequences.

A simple and typical application of the Loeb construction is Anderson's **[A1]** construction of Brownian motion as the standard part of a random walk with infinitesimal time steps and space steps of magnitude root of the time step. The reader who knows probability theory will recognize that Anderson's construction corresponds to Donsker's invariance principle, which says that suitable random walks converge in distribution to a Brownian motion. Anderson's approach amounts to a construction of Brownian motion and a proof of the invariance principle rolled into one. This is generally the case with existence proofs for stochastic processes using the Loeb construction.

Note that Anderson's construction finds a continuous standard object by forming the standard part of a discrete nonstandard process. This is a typical pattern in nonstandard analysis—the nonstandard constructions are themselves finitary, and the machinery of the nonstandard model gives them the power of infinitary mathematics. Because one can obtain continuous standard objects naturally using finitary nonstandard constructions, it is particularly easy to derive existence results using nonstandard analysis. This is the practical side of the universality of Loeb spaces.

The Anderson construction represents half of a common method for using nonstandard analysis to solve problems expressed in the language of standard mathematics. This is illustrated by the diagram in Figure 1. Given a standard problem, one finds a nonstandard problem of which it is the standard part. This is called a *lifting* of the original problem. Then one solves the nonstandard problem. Usually it is a finitary problem, and the existence of a solution is unproblematical. Then, one shows that this nonstandard solution has a standard part, and that the standard part solves the standard problem. This method is exemplified in **[K]** for the case of finding weak solutions of stochastic differential equations.

$$\begin{tikzcd}
\text{Nonstandard problem} \arrow[r, "\text{Nonstandard construction}"] & \text{Nonstandard solution} \arrow[d, "\text{Standard part}"] \\
\text{Standard problem} \arrow[u, "\text{Lifting}", "= (\text{Standard part})^{-1}"'] \arrow[r] & \text{Standard solution}
\end{tikzcd}$$

FIGURE 1

The problem with this method is that it requires two translations, from standard to nonstandard and back again. Implicit in this is the notational nuisance of having to deal with and distinguish two parallel languages and sets of mathematical objects, and the effort required to learn a second mathematical language. Now, the advantage of nonstandard analysis is a more intuitive approach to the notion of limit. But most mathematicians would say that they understand limits rather well already, well enough that it is not worth this much trouble to understand them from a slightly better perspective. These considerations incline us to believe that nonstandard analysis will remain marginal as a technique for solving problems of standard mathematics.

This leaves two alternatives. The first is to forget about nonstandard analysis, except for special uses, such as showing that there are probability spaces and Banach spaces with certain universality properties, namely Loeb spaces and nonstandard hulls. The theory left over would be the model theory of structures occurring in analysis.

The second alternative is to develop mathematics from the nonstandard point of view, rather than merely using nonstandard analysis as a technical tool in developing standard mathematics. If nonstandard analysis has as much mathematical content as standard mathematics, this must be possible. If the nonstandard approach to such a basic notion as limit really is more intuitive, then this must be desirable.

This program has been carried out for a significant part of probability theory in Nelson's recent book [N]. Nelson's premise is that the measure-theoretic notion of a probability space, with a σ-algebra of measurable subsets, and probability as a set function on that σ-algebra, is more abstract and set-theoretically sophisticated than is needed to express the basic notions of probability theory—random variables, stochastic processes, independence, conditional expectation. He shows that, in the context of nonstandard analysis, discrete probability spaces, where probability is a weight function on elements, admit an elegant theory with the same probabilistic content as the classical theory. Now this is in fact the burden of most work in nonstandard probability theory, such as that developed by Stroyan and Bayod, since most work in nonstandard probability uses only discrete nonstandard spaces to develop stochastic analysis. (Nonstandard probability is not, however, limited to discrete spaces.) The new idea is that it is unnecessary to relate the nonstandard theory to the standard one. The mainstream work in nonstandard probability would be simplified and improved by this approach, not invalidated.

Why has not more mathematics already been developed from a purely nonstandard point of view? The reason seems to be simply that rigorous standard analysis developed first. Consequently, nonstandard analysis has developed as an extension of standard analysis, and has tended to use standard concepts as a point of reference and a criterion for whether nonstandard results are meaningful. For instance, a nonstandard object is considered interesting if it is nearstandard in some relevant sense, i.e. if it approximates some standard object, its standard part. But often the standard object was invented to be the limit of something—essentially, to be the standard part of that nonstandard object.

Let us consider real numbers as a concrete example. In nonstandard analysis one can base a theory with the same content as the infinitesimal calculus on the rationals. It is not apparent that the reals would be much missed. Would it then be considered important to construct such a set as the reals? If one did want to construct the set of reals, it would be defined as a set which contained standard parts for all finite rationals. What sort of set theory would arise from set constructions such as this?

The aim of the purely nonstandard approach is to lighten the burden of modern mathematical rigor. The set theoretic foundation has indeed made mathematics into a well-defined deductive science, but it has made it less accessible to the nonspecialists who should be actually applying it, has encumbered mathematicians with details that do not seem to increase mathematical content, and is suspected of spawning problems which are irrelevant to the problems which mathematics is made to solve. For example, applied probabilists commonly regard the measure theoretic foundations of their discipline as something to be avoided and ignored as far as possible, which is pretty far. Why not, then, use discrete spaces in the rigorous nonstandard setting, and ignore measure theory altogether? Why not use finite dimensional linear spaces in a nonstandard setting, and dispense with Banach spaces? And so on.

The book of Stroyan and Bayod is a thorough treatment of the Loeb space approach to probability theory. It begins at the beginning, with a brief introduction to nonstandard analysis. I found the earlier parts the most pleasing. They give a thorough treatment of basic Loeb space measure theory, including key notions such as lifting and projection (standard part), the relation of conditional expectation to these operations, aspects of products of Loeb spaces, and distributions on Loeb spaces. The latter includes the pleasant result that if two random variables on a hyperfinite Loeb space with the counting probability measure have the same distribution, then there is an internal bijection of the space which maps one approximately to the other. To appreciate the nonstandard approach, the reader should try to prove the equivalent standard result: given a large finite probability space with counting probability measure, and two random variables on that space having almost the same distribution, there is a bijection of the space which maps one random variable approximately (in probability) to the other.

The one undesirable aspect in this part is the development of Loeb measure in excessive generality, including infinite measures. These cannot be σ-finite, so their theory is rather complicated. This complexity is for nought, since they are never used in the book. The authors do provide a map to help the reader navigate around the infinite measures, but a clear road needing no map would have been better.

The remainder of the book deals with matters relating specifically to stochastic processes, such as filtrations, liftings and projections for processes with various sample path properties (continuous, jumps, measurable) and for progressively measurable and predictable processes. I found this part of the theory less pleasing, because it seems more complicated than it ought to be. This is not entirely the fault of the nonstandard analysis or of the authors, since the standard theory has a rather formidable array of topologies, which are not all easy to use. Nevertheless, it seems to me that a there must be a more abstract approach which could simplify matters, as could a more general treatment of distribution. The book goes on to introduce stochastic analysis on Loeb spaces, but much of the existing nonstandard work in this area is rather technical, and the authors often refer to the original papers for details.

I would say that this is the best book to read to get a broad feel for Loeb spaces. It has all the basic material, and a lot of examples which show just what sort of things can be done with Loeb spaces, and which cannot. There is helpful advice about which analogies between standard and nonstandard concepts are helpful and which are misleading. Those who wish to continue study of nonstandard probability theory, or who prefer less general Loeb space theory oriented specifically to continuous sample path processes, should read Keisler's monograph **[K]**, which is a development of Brownian stochastic integration and the associated differential equations in a Loeb space setting, or **[AFH-KL]**, which surveys a wide variety of applications of nonstandard analysis, with emphasis on probability theory. Of course, we warmly recommend the different departure, [N]. A good, current general introduction to nonstandard analysis is **[HL]**.

References

[AFH-KL] Albeverio, J. E. Fenstad, R. Høegh-Krohn, and T. Lindstrøm, *Nonstandard methods in stochastic analysis and mathematical physics*, Academic Press, New York, 1986.

[A1] R. M. Anderson, *A nonstandard representation for Brownian motion and Itô integration*, Israel J. Math. **25** (1976), 15–46.

[A2] R. M. Anderson, Review of **[HL]**, Bull. Amer. Math. Soc. (N.S.) **16** (1987), 298–306.

[C] N. J. Cutland, *Nonstandard measure theory and its applications*, J. London Math. Soc. **15** (1983), 529–589.

[HL] A. E. Hurd and P. A. Loeb, *An Introduction to nonstandard real analysis*, Academic Press, New York, 1985.

[HM] C. W. Henson and L. C. Moore, *Nonstandard analysis and the theory of Banach spaces*, (A. E. Hurd, ed.), Nonstandard Analysis: Recent Developments, Lecture Notes in Math., vol. 983, Springer-Verlag, Berlin and New York, 1983.

[K] H. J. Keisler, *An infinitesimal approach to stochastic analysis*, Mem. Amer. Math. Soc. **48**, (1984), no. 297..

[L] P. A. Loeb, *Conversion from standard to standard measure spaces and applications to probability theory*, Trans. Amer. Math. Soc. **211** (1975), 113–122.

[N] E. Nelson, *Radically elementary probability theory*, Ann. of Math. Studies no. 117, Princeton Univ. Press, Princeton, N. J., 1987.

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General theory of Markov processes, by Michael Sharpe. Academic Press, San Diego, 1988, xi + 419 pp., $49.50. ISBN 0-12-639060-6

Probability theory is concerned with *random variables* and their distributions, and a family of random variables (X_t) indexed by a parameter is called a *stochastic process*. Among stochastic processes, one can roughly distinguish two main categories, whose study uses widely different methods: true stochastic processes are those for which there is indeed some