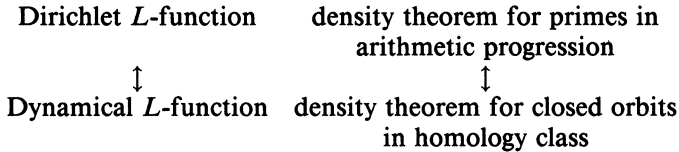


DYNAMICAL L -FUNCTIONS AND HOMOLOGY OF CLOSED ORBITS

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The phenomenon which we shall present in this note may be illustrated, in short, by the following diagram.



In the dynamical case, however, the “*ideal class group*” (= the first integral homology group) might have infinite order, so that some extra phenomena will be seen.

To fix our terminology, we let $\{\phi_t\}$ be a smooth, transitive Anosov flow [4] on a closed manifold X . We assume that ϕ_t has the weak-mixing property [17]. We denote by h the *topological entropy* of ϕ_t , and by μ a measure of *maximal entropy* on X , that is, an invariant probability measure whose metric entropy h_μ equals h . It is known that there exists exactly one measure with $h_\mu = h$ [20]. The *canonical winding cycle* Φ , which measures the average of “homological” direction in which the orbits of the flow are traveling, is defined by

$$\Phi(\omega) = \int_X \langle \omega, Z \rangle d\mu,$$

where ω is a closed 1-form, and Z is the vector field generating the flow. Since $\Phi(\text{exact forms}) = 0$, the linear map Φ yields actually a homology class in $H_1(X, \mathbf{R}) = \text{Hom}(H^1(X, \mathbf{R}), \mathbf{R})$ (see [16]).

We now classify closed orbits of the flow by means of the homology classes, and count the number of them. More generally, given a surjective homomorphism ψ of $H_1(X, \mathbf{Z})$ onto an abelian group H , we set for each $\alpha \in H$ and positive number x ,

$$\pi(x, \alpha) = \{\mathfrak{p}; \text{closed orbits with } \psi[\mathfrak{p}] = \alpha \text{ and } l(\mathfrak{p}) < x\},$$

$$\Pi(x, \alpha) = \#\pi(x, \alpha),$$

where $[\mathfrak{p}]$ denotes the homology class of \mathfrak{p} and $l(\mathfrak{p})$ the period. From now on, we shall identify the dual $H^\dagger = \text{Hom}(H, \mathbf{Z})$ with a subgroup in $H^1(X, \mathbf{Z})$ via the transpose of ψ . Set $b = \text{rank } H$.

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THEOREM 1 (DENSITY THEOREM). *Let ϕ_t be a transitive, weak-mixing Anosov flow and h the topological entropy. If the winding cycle Φ vanishes on H^\dagger , then the counting function $\Pi(x, \alpha)$ has the following asymptotic behavior,*

$$(1) \quad \Pi(x, \alpha) \sim C \frac{e^{hx}}{x^{1+(b/2)}}$$

as x goes to infinity. Here the constant C is positive, not depending on α , and described in terms of some dynamical quantities (see (3)).

The following shows that the condition on the winding cycle above is necessary to guarantee asymptotic behavior as in (1).

THEOREM 2. *If Φ does not vanish on H^\dagger , then for every positive integer N ,*

$$\frac{\Pi(x, \alpha)}{e^{hx}} = o(x^{-N}) \quad (x \uparrow \infty).$$

The density theorem says that closed orbits are equidistributed in a homological sense. The following theorem asserts that closed orbits in each homology class are equidistributed spatially.

THEOREM 3 (EQUIDISTRIBUTION THEOREM). *Let $g \in C^\infty(X)$. Under the same conditions as in Theorem 1, we have*

$$(2) \quad \lim_{x \rightarrow \infty} \Pi(x, \alpha)^{-1} \sum_{\mathfrak{p} \in \pi(x, \alpha)} l(\mathfrak{p})^{-1} \int_{\mathfrak{p}} g = \int_X g d\mu.$$

REMARK. (2) holds only if Φ vanishes on H^\dagger . In fact, the right-hand side of (2) equals $\Phi(\omega)$ if we put $g = \langle \omega, Z \rangle$. On the other hand, if $\omega \in H^\dagger$, then $\int_{\mathfrak{p}} \langle \omega, Z \rangle = \int_{\mathfrak{p}} \omega$ does not depend on $\mathfrak{p} \in \pi(x, \alpha)$, so that the right-hand side of (2) equals

$$\left(\int_{\mathfrak{p}} \omega \right) \Pi(x, \alpha)^{-1} \sum_{\mathfrak{p} \in \pi(x, \alpha)} l(\mathfrak{p})^{-1},$$

which goes to zero as $x \uparrow \infty$.

For a finite group H , the condition in Theorem 1 is always valid since $H^\dagger = (0)$. In this special case, the density and equidistribution theorem have been established by Parry-Pollicott [12, 13], Parry [11], and Adachi and Sunada [3].

A typical example of Anosov flows with vanishing winding cycle (on the full cohomology group) is the geodesic flow on the unit tangent sphere bundle on a negatively curved Riemannian manifold. Thus Theorem 1 generalizes the density theorem for constant negatively curved spaces given by Phillips and Sarnak [14] and Katsuda and Sunada [8].

The proof of the above theorems relies heavily on some analytic properties of the dynamical L -functions, which are defined by

$$L(s, \chi) = \prod_{\mathfrak{p}} (1 - \chi[\mathfrak{p}]e^{-s l(\mathfrak{p})})^{-1},$$

where $\chi \in \text{Hom}(H_1(X, \mathbf{Z}), U(1))$. It is known that $L(s, \chi)$ converges absolutely in $\text{Re } s > h$, and that there exists an open domain D containing $\{\text{Re } s \geq h\}$ such that $L(s, \chi)$ is extended to D as a nowhere vanishing meromorphic function. Furthermore we observe that there exists a smooth function $s = s(\chi)$ defined on an open neighborhood of $\mathbf{1}$, the trivial character, such that $s(\mathbf{1}) = h$ and $s(\chi)$ is a unique (simple) pole of $L(s, \chi)$ near $s = h$. Since $L(s, \chi)$ is holomorphic in $\text{Re } s > h$, we find that $\text{Re } s(\chi) \leq h$. Therefore the function $\text{Re } s(\chi)$ attains its maximum at $\chi = \mathbf{1}$, so that $\nabla \text{Re } s(\chi) = 0$ at $\chi = \mathbf{1}$ and $\text{Hess}_{\chi=\mathbf{1}} \text{Re } s(\chi)$ is a negative semidefinite quadratic form on $H^1(X, \mathbf{R})$.

To express $\text{Hess } \text{Re } s(\chi)$ in terms of dynamical quantities, we introduce the covariance form δ on $H^1(X, \mathbf{R})$ defined by

$$\delta(\omega, \omega) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_X d\mu(x) \left(\int_0^t \langle \omega, Z \rangle (\varphi_\tau x) dt - t\Phi(\omega) \right)^2,$$

(see Denker and Philipp [5] for existence of the limit).

THEOREM 4. $\text{Hess}_{\chi=\mathbf{1}} \text{Re } s(\chi) = -4\pi^2 \delta$, and $\nabla \text{Im } s(\chi) = \Phi$ at $\chi = \mathbf{1}$.

REMARK. In case of geodesic flows, $s(\chi)$ is real.

The null space of δ is characterized in

THEOREM 5. $\delta(\omega, \omega) = 0$ if and only if $\int_p \omega = l(p)\Phi(\omega)$ for every closed orbit p .

In view of this theorem, it is easy to deduce that the following three conditions are equivalent.

- (1) δ is degenerate,
- (2) there exists a nontrivial character χ such that $L(s, \chi)$ has a pole on the line $\text{Re } s = h$,
- (3) $E^s \oplus E^u$ is integrable (see [15] for the notation and meaning). Therefore the flow is topologically conjugate to the suspension of an Anosov diffeomorphism, and each $\alpha \in H_1(X, \mathbf{Z})$ contains only finitely many closed orbits.

An obvious but important remark is that $\text{Ker } \Phi \cap (\text{Null space of } \delta) = (0)$, so that if Φ vanishes on H^\dagger , then δ is nondegenerate on the subspace $H^\dagger \otimes \mathbf{R} \subset H^1(X, \mathbf{R})$. An important point of the proof of Theorems 1 and 2 is to examine the singularities of the function

$$\int_{\hat{H}} \chi(-\alpha) \left(-\frac{d}{ds} \right)^g \frac{L'(s, \chi)}{L(s, \chi)} d\chi = \sum_{\substack{k=1 \\ k\psi[p]=\alpha}}^{\infty} \sum_p k^g l(p)^{g+1} e^{-skl(p)},$$

where $d\chi$ denotes the normalized Haar measure on the character group $\hat{H} \subset \text{Hom}(H_1(X, \mathbf{Z}), U(1))$. In the course of the argument, it turns out that the constant C in Theorem 1 is equal to

$$(3) \quad (2\pi)^{-b/2} \text{vol}(\hat{H})^{-1} h^{-1}.$$

Here $\text{vol}(\hat{H})$ is the volumn of \hat{H} with respect to the Lebesgue measure induced from the metric δ on $T_1 \hat{H} = H^\dagger \otimes \mathbf{R}$.

If M is a locally symmetric space of negative curvature, then the L -function associated to the geodesic flow on the unit tangent sphere bundle UM is connected with the Selberg zeta function, whose singularities are completely described by the spectrum of the twisted Laplacian Δ_χ acting on sections of the flat line bundle associated with the character χ . This leads to the equality

$$s(\chi) = \frac{h}{2} + \left(\frac{h^2}{4} - \lambda_0(\chi) \right)^{1/2},$$

where $\lambda_0(\chi)$ denotes the first eigenvalue of Δ_χ . Employing a perturbation technique for $\lambda_0(\chi)$ (see [8]), we get

PROPOSITION. *Let η be a harmonic 1-form on M . Then*

$$\delta(\Pi^*\eta, \Pi^*\eta) = \frac{1}{\frac{h}{2} \text{vol}(M)} \int_M |\eta|^2,$$

where $\Pi : UM \rightarrow M$ is the projection map.

It should be noted that Π induces an isomorphism

$$\Pi^* : H^1(M, \mathbf{R}) \simeq H^1(UM, \mathbf{R}),$$

and that the measure μ coincides with the canonical smooth measure on UM .

Details will appear elsewhere.

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REFERENCES

1. T. Adachi, *Meromorphic extension of dynamical L-functions*, Kumamoto J. Math. **1** (1988), 9–24.
2. T. Adachi and T. Sunada, *Homology of closed geodesics in a negatively curved manifold*, J. Differential Geom. **26** (1987), 81–99.
3. ———, *Twisted Perron-Frobenius theorem and L-functions*, J. Funct. Anal. **71** (1987), 1–46.
4. R. Bowen, *Symbolic dynamics for hyperbolic flows*, Amer. J. Math. **95** (1973), 429–459.
5. M. Denker and W. Philipp, *Approximation by Brownian motion for Gibbs measure and flows under a function*, Ergodic Theory Dynamical Systems **4** (1984), 541–552.
6. C. Epstein, *Asymptotics for closed geodesics in a homology class-finite volume case*, Duke Math. J. **55** (1987), 717–757.
7. I. M. Gel'fand and I. I. Pyateckii-Shapiro, *A theorem of Poincaré*, Dokl. Akad. Nauk **127** (1959), 490–493.
8. A. Katsuda and T. Sunada, *Homology and closed geodesics in a compact Riemann surface*, Amer. J. Math. **109** (1987), 145–156.
9. A. Katsuda, *Density theorems for closed orbits*, Proc. Taniguchi Sympos. Geometry and Analysis on Manifolds 1987, Lecture Notes in Math., vol. 1339, Springer-Verlag, Berlin and New York, 1988, pp. 182–202.

10. S. Manabe and Y. Ochi, *The central limit theorem for current-valued processes induced by geodesic flows*, preprint.
11. W. Parry, *Bowen's equidistribution theory and the Dirichlet density theorem*, Ergodic Theory Dynamical Systems **4** (1984), 117–134.
12. W. Parry and M. Pollicott, *An analogue of the prime number theorem for closed orbits of Axiom A flows*, Ann. of Math. (2) **118** (1983), 573–591.
13. —, *The Chebotarev theorem for Galois coverings of Axiom A flows*, Ergodic Theory Dynamical Systems **6** (1986), 133–148.
14. R. Phillips and P. Sarnak, *Geodesics in homology classes*, Duke Math. J. **55** (1987), 287–297.
15. J. Plante, *Anosov flows*, Amer. J. Math. **94** (1972), 729–758.
16. —, *Homology of closed orbits of Anosov flows*, Proc. Amer. Math. Soc. **37** (1973), 297–300.
17. M. Pollicott, *On the rate of mixing of Axiom A flows*, Invent. Math. **81** (1985), 413–426.
18. D. Ruelle, *Thermodynamic formalism*, Addison-Wesley, Mass., 1978.
19. A. Selberg, *Harmonic analysis and discontinuous subgroups in weakly symmetric Riemannian spaces with applications to Dirichlet series*, J. Indian Math. Soc. **20** (1956), 47–87.
20. Ya. G. Sinai, *Gibbs measure in ergodic theory*, Russian Math. Surveys **27** (1972), 21–69.
21. T. Sunada, *L-functions in geometry and some applications*, Proc. Taniguchi Sympos. 1985, Curvature and Topology of Riemannian Manifolds, Lecture Notes in Math., vol. 1201, Springer-Verlag, Berlin and New York, 1986, pp. 266–284.
22. S. Zelditch, *Geodesics in homology classes and periods of automorphic forms*, preprint.

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