community has yet to give a fair reading and assessment to the work of Errett Bishop in constructive mathematics. In this review I have tried to portray a mathematician who was often capable of envisioning what others could not. Perhaps some day our science will approach his constructive mathematics as the same kind of thinking and will work at understanding what Errett is trying to tell us.

REFERENCES

[C] L. Carleson, *Mergelyan's theorem on uniform polynomial approximation*, Math. Scand. **15** (1964), 167–175.

[COCA] Proceedings of the Conference on Complex Analysis, Minneapolis 1964, Springer-Verlag, 1965.

[D] A. Douady, *Le problème des modules pour les sous-espaces analytiques d'un espace analytique donné*, Ann. Inst. Fourier (Grenoble) **16** (1966), 1–95.

[EB] Errett Bishop, *Reflections on him and his research*, Contemp. Math., vol. 39, Amer. Math. Soc., Providence, R. I., 1985.

[G] A. Gleason, *Analytic structure in a maximal ideal space.*

[GR] R. C. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Englewood Cliffs, N. J., 1965.

[K] N. Kerzman, *Holder and L^p-estimates for solutions of $u = f$ in strongly pseudoconvex domains*, Comm. Pure Appl. Math. **24** (1971), 301–379.

[KL] V. L. Klee, *Extremal structure of convex sets.* II, Math. Z. **69** (1958), 90–104.

[L1] H. Lewy, *On the local character of the solutions of an atypical linear differential equation in three variables and a related theorem for regular functions of two complex variables*, Ann. of Math. (2) **64** (1956), 514–522.

[L2] ——, *On hulls of holomorphy*, Comm. Pure Appl. Math. **13** (1960), 587–591.

[O] K. Oka, *Domaines finis sans point critique intérieur*, Japanese J. Math. **27** (1953), 97–155.

[R] H. Rossi, *Holomorphically convex sets in several complex variables*, Ann. of Math. (2) **74** (1961), 470–493.

[S1] G. Stolzenberg, *Uniform approximation on smooth curves*, Acta Math. **115** (1966).

[S2] ——, *Volumes, limits and extensions of analytic varieties*, Springer-Verlag, Berlin and New York, 1966.

[SI] Y. T. Siu, *Techniques of extension of analytic objects*, Marcel Dekker, 1974.

[W1] J. Wermer, *Function rings and Riemann surfaces*, Ann. of Math. (2) **67** (1958), 45–77.

[W2] ——, *Rings of analytic functions*, Ann. of Math. (2) **67** (1958), 497–516.

[W3] ——, *An example concerning polynomial convexity*, Math. Ann. **139** (1959), 147–150.

HUGO ROSSI
UNIVERSITY OF UTAH

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 19, Number 2, October 1988

0273-0979/88 $1.00 + $.25 per page

Symmetries of Maxwell's equations, by W. I. Fushchich and A. G. Nikitin. Translated by John R. Schulenberger. Mathematics and its Applications. D. Reidel Publishing Company, Dordrecht, 1987, xiv + 214 pp., $74.00. ISBN 90-277-2320-6

The mathematical formalization of symmetry into the all-important abstract concept of a group had its origins in Galois' study of the solutions of polynomial equations. Similarly, continuous groups trace their lineage back to Sophus Lie's attempts to extend Galois theory to the solutions of differential equations.[1] Lie began with the wonderful observation that all of the known special methods for integrating ordinary differential equations (many of which we still teach in a first course on the subject) were but manifestations of the single underlying concept of the invariance of the equation under a one-parameter group of transformations. This seminal idea has proved to be extraordinarily fertile, branching far from its origins; Lie groups now appear in every branch of mathematics and its applications, from number theory to particle physics, from topology to elasticity. However, in contrast to later reworkings of the subject, Lie's approach was quite practical—his groups were always realized concretely as explicit transformation groups acting on some (usually Euclidean) space. Any reader of his collected works cannot but be impressed by the vast number of explicit examples, results, and computations. Extensions to partial differential equations, differential geometry, contact transformations, etc. were but a few of the prodigious outpourings of Lie's fertile imagination. A crucial ingredient in Lie's approach is his computational algorithm for determining the symmetry group of a differential equation, which can readily be applied in practice to equations of physical and mathematical interest.

Perhaps the high water mark of the classical phase of the romance between Lie groups and differential equations was the celebrated theorem of Emmy Noether, [**8**], proving the one-to-one corrrespondence between symmetry groups of variational problems and conservation laws of their associated Euler-Lagrange equations. Her one paper in mathematical physics, before moving on (up?) to the more rarified atmosphere of pure algebra, is a touchstone, and has become, arguably, the most quoted theorem in all the mathematical physics literature. With this one masterful display of her applied mathematical talents, she succeeded in seducing Lie's more practically-minded theory, for it soon forsook its coarser origins in applications to differential equations and mathematical physics, and, lured by the promises of the French (E. Cartan, Chevalley, Bourbaki) attained mathematical respectibility, globalization and abstraction, and, above all, fashionability. The problem was, of course, that the groups Lie had introduced, those that arise naturally from a study of differential equations, are (i) local transformation groups acting concretely on an open subset of some Euclidean space (the space of independent and dependent variables for the differential equations in question) and (worse yet) (ii) often fail to be particularly aesthetically pleasing Lie groups, e.g., semisimple, solvable, etc. The French theorists succeeded in divorcing the Lie group from its transformation space, globalizing the resulting abstraction, and thereby endowing it with full mathematical respectability. (An interesting sidelight: the corresponding infinite dimensional Lie "psedudogroups" were never wholly convinced that this turn of events was really beneficial, and, to this day, cause untold headaches for the unrepentant Bourbakist, who still can't find the right universal abstract object to represent them!) Its humble, practical origins in the mire of differential equations effectively forgotten, Lie

[1] In point of fact, despite his unquestioned originality, Lie was not successful in this particular quest, which had to wait until the more refined theory of Picard and Vessiot.

group theory entered the canon of the pure mathematician. Except for the important, but essentially unrelated developments in representation theory, physicists, engineers and applied mathematicians were left waiting for almost half a century before they could once again lay proper claim to its marvelous power.

In retrospect, from the applied viewpoint, the history of Lie group theory borders on the absurd. In 1918, with the results of Lie and Noether in hand, applied researchers had in their possession an extensive arsenal of concrete, practical algroithms for conducting a systematic and complete investigation into the symmetry properties of the important equations of mathematical physics and engineering, an enterprise that has now been seen to lead to profound and far-reaching consequences. However, for almost 40 years, *nothing happened*! I'm not sure that we will ever fully comprehend the true reason for this all-encompassing inaction, although it would make a fascinating study in the sociology of mathematics and physics in this century. However, some of the principal reasons, in my opinion, are that (a) Lie himself was curiously uninterested in physical applications of his work, whereas (b) Noether never made it clear that Lie's constructive methods could be combined with her theorem to efficate a completely algorithmic derivation of symmetry groups and conservation laws in mathematical physics. The subject just fell into a complete eclipse, mentioned, if at all, as an interesting, but completely developed topic in an ordinary differential equations text, e.g., [**5**], or, in the case of Noether's theorem, in watered down versions palatable to physicists of the time.

The first glimmer of hope in the darkness was Garrett Birkhoff's book on hydrodynamics, [**2**], which was the first to explicitly champion the use of the group concept in practical applications. However, the resulting small stirring of research activity in the West in the early 50s quickly died away, and it was not until L. V. Ovsiannikov began trumpeting Lie's cause from Siberia in the late 50s that the Soviets woke up to the possibilities, and, a decade later, applied mathematicians in the West began to take notice. The 70s witnessed a great explosion of research in the subject, further spurred on by interactions with soliton equations and Hamiltonian systems. The entire subject of Lie groups and differential equations is now very much alive, and has produced numerous important applications to contemporary applied mathematics, physics, chemistry and engineering.

The slightly apochryphal division of the realm of groups between the continuous (Lie) and the discrete (Klein) certainly left Klein with the raw end of the deal, since Lie's far greater success in developing his subject, as anyone who has seriously looked at "discrete math" realizes, stems from the fact that one can "do calculus" on Lie groups. In particular, the infinitesimal generators (Lie algebra) of the group linearly encode all the (local) information about the group itself. For example, the *linear* infinitesimal symmetry conditions translate immediately into genuine symmetry conditions. To illustrate, suppose we are given a system of partial differential equations

$$\Delta(x, u^{(n)}) = 0.$$

Let G be a local group of transformations acting on the space $X \times U$ of independent and dependent variables, consisting of transformations of the form

$$(\tilde{x}, \tilde{u}) = g \cdot (x, u) = (\varphi(x, u), \psi(x, u)), \qquad g \in G.$$

The transformations in G act on functions $u = f(x)$ by point-wise transformation of their graphs. We call G a symmetry group of the system $\Delta = 0$ if it transforms solutions to solutions. The infinitesimal generators of the group are vector fields on $X \times U$, which take the general form

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi_\alpha(x, u) \frac{\partial}{\partial u^\alpha}.$$

The group transformations are recovered by integrating the autonomous system of ordinary differential equations governing the flow of these vector fields.

Since the group acts on functions, it also acts on their derivatives (jets) by "prolongation". While the formula for the prolonged group transformations is quite complicated, there is a simple explicit formula for the corresponding prolonged infinitesimal generators, which are vector fields pr $\mathbf{v}$ on the space of derivatives $(x, u^{(n)})$. Lie proved the fundamental result that, under some very mild nondegeneracy conditions (almost always satisfied in practice), the connected transformation group G is a symmetry group of the system of differential equations $\Delta = 0$ if and only if the infinitesimal invariance condition

$$(*) \qquad \operatorname{pr} \mathbf{v}[\Delta] = 0 \quad \text{whenever } \Delta = 0$$

holds for every infinitesimal generator $\mathbf{v}$ of G. An important point is that one can use $(*)$ to explicitly determine the most general vector field $\mathbf{v}$ leaving $\Delta = 0$ invariant, and thereby completely determine the most general (connected) symmetry group of any given system of differential equations! Indeed, $(*)$ constitutes a large, overdetermined system of elementary partial differential equations for the coefficient functions ξ^i, φ_α of the infinitesimal generator $\mathbf{v}$, which can almost always be explicitly solved for examples of practical importance. There are now several symbolic manipulation computer programs available which will do this for you—given a system of partial differential equations, it will explicitly calculate the symmetry group, cf. [**11**].

Once the symmetry group of a differential equation has been determined, there are many further applications available, which are of use not only for linear equations, but, even more importantly, for nonlinear differential equations as well. One can (a) construct new solutions from old ones using group elements, (b) in the case of ordinary differential equations, use the groups to reduce the order, or even solve them explicitly by quadrature, (c) in the case of partial differential equations, determine explicit "group-invariant" solutions, generalizing the classical similarity solutions of such fundamental importance in applications, (d) in the case of linear or Hamilton-Jacobi equations, use the groups to determine coordinate systems in which one can separate variables, (e) in the case of equations admitting a variational structure, either as the Euler-Lagrange equations for some variational principle, or as equations in Hamiltonian form, use the symmetry groups to explicitly determine conservation laws. Other applications include solution of boundary value problems, bifurcation theory, linearization of differential equations, numerical integration schemes, complete integrability, scattering theory, etc., etc. In all cases, the constructions are explicit, and can be readily implemented in practice.

Turning to the particular topic of the book under review, the first remark is that the study of the symmetry of Maxwell's equations has a long, and, until recently, separate history quite apart from the mainstream of the standard applications of Lie groups to differential equations. In 1909, apparently unaware of the straightforward infinitesimal methods of Lie, H. Bateman, [**1**], and E. Cunningham, [**3**], first derived the conformal invariance of Maxwell's equations in vacuum. Along with the "Larmor-Rainich transformations", which rotate the electric and magnetic fields, these constitute the full symmetry group of point transformations for Maxwell's equations. The contributions of the authors have been in generalizing these transformations to include a number of nonlocal, "hidden" symmetries. The basic method used is to work in Fourier transform space; the original inspiration can be traced back to a remarkable observation of Fock, [**4**], relating the Schrödinger equation for the hydrogen atom in momentum space to the equation for spherical functions on the 4-sphere. In particular, this observation produces an $O(4)$ symmetry group of nonlocal transformations for the Schrödinger equation for the hydrogen atom, [**7**]. Fushchich and Nikitin, in a long series of papers, have pushed this idea in many directions, leading to many previously undetected symmetries of systems connected with Maxwell's equations and the Dirac equation.

The book itself is primarily a summary of the authors' papers on this subject over the last decade. It provides an exhaustive (and, at times, exhausting) treatment of this particular niche in a much larger subject, and, as such, is certainly of interest to specialists. Many of the above-mentioned applications of Lie groups are treated in this particular context, although separation of variables, a subject still in its infancy in the case of linear systems of partial differential equations, [**6**], is not mentioned. The reader will see how the general methods, along with some interesting generalizations, are applied to one specific example (and its many variants), although the treatment is such that the uninitiated may get lost in an unavoidable maze of notation. A reader who is just beginning in the subject is well advised to spend the initial effort assimilating a more general introduction to the entire range of applications of Lie group theory to differential equations, e.g., [**9**, **10**], before attempting to specialize to this one particular topic. Nevertheless, the book represents an important contribution to the study of the role of symmetry in electrodynamics and relativistic physics.

REFERENCES

1. H. Bateman, *The conformal transformations of a space of four dimensions and their applications to geometrical optics*, Proc. London Math. Soc. **7** (1909), 70–89.

2. G. Birkhoff, *Hydrodynamics–A study in logic, fact and similitude*, 1st ed., Princeton Univ. Press, Princeton, N.J., 1950.

3. E. Cunningham, *The principle of relativity in electrodynamics and an extension thereof*, Proc. London Math. Soc. **8** (1909), 77–98.

4. V. Fock, *Zur Theorie des Wasserstoffatoms*, Z. Physik **98** (1935), 145–154.

5. E. L. Ince, *Ordinary differential equations*, Dover, New York, 1956.

6. E. G. Kalnins, W. Miller, Jr. and G. C. Williams, *Matrix operator symmetries of the Dirac equation and separation of variables*, J. Math. Phys. **27** (1986), 1893–1900.

7. I. A. Malkin and V. I. Man′ko, *Symmetry of the hydrogen atom*, Soviet J. Nuclear Phys. **3** (1966), 267–274.

8. E. Noether, *Invariante Variationsprobleme*, Nachr. Konig. Gesell. Wissen. Gottingen, Math.-Phys. Kl. (1918), 235–257 (see Transport Theory and Stat. Phys. **1** (1971), 186–207 for an English translation).

9. P. J. Olver, *Applications of Lie groups to differential equations*, Graduate Texts in Math., vol. 107, Springer-Verlag, New York, 1986.

10. L. V. Ovsiannikov, *Group analysis of differential equations*, Academic Press, New York, 1982.

11. F. Schwarz, *Automatically determining symmetries of partial differential equations*, Computing **34** (1985), 91–106.

PETER J. OLVER
UNIVERSITY OF MINNESOTA

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 19, Number 2, October 1988

0273-0979/88 $1.00 + $.25 per page

Knots, by Gerhard Burde and Heiner Zieschang. DeGruyter Studies in Mathematics, vol. 5, Walter DeGruyter, Berlin, New York, 1985, x+399 pp., $49.95. ISBN 0-89925-014-9

On knots, by Louis H. Kauffman. Annals of Mathematics Studies, vol. 115, Princeton University Press, Princeton, N.J., 1987, xv+480 pp., $50.00 ($18.95 paperback). ISBN 0-691-08434-3

The central problem in knot and link theory is to distinguish link types via computable invariants. Figure 1 shows an example. for 75 years the two knots in Figure 1 were thought to represent distinct knot types, until in 1974 it was discovered that a totally unmotivated but very simple change in the projection takes the left picture to the right [**P**]. If we cannot find such a change, how can we be sure that two knots are distinct?

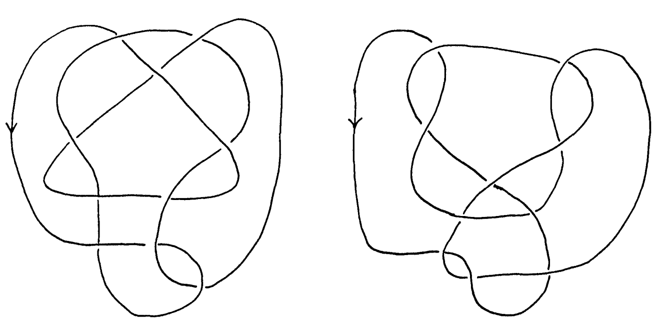

FIGURE 1

This review was written when the author was visiting the University of Paris VII. Partial support and the hospitality of the Mathematics Department during that visit are gratefull acknowledged.